THOM ISOTOPY THEOREM FOR NON PROPER MAPS AND COMPUTATION OF SETS OF STRATIFIED GENERALIZED CRITICAL VALUES

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Abstract. Let $X \subset \mathbb{C}^n$ be an affine variety and $f : X \to \mathbb{C}^m$ be the restriction to X of a polynomial map $\mathbb{C}^n \to \mathbb{C}^m$. In this paper, we construct an affine Whitney stratification of X. The set $K(f)$ of stratified generalized critical values of f can be also computed. We show that $K(f)$ is a nowhere dense subset of $\mathbb{C}^m$, which contains the set $B(f)$ of bifurcation values of f by proving a version of the isotopy lemma for non-proper polynomial maps on singular varieties.

1. Introduction

Ehresmann's fibration theorem [3] states that a proper smooth surjective submersion $f : X \to N$ between smooth manifolds is a locally trivial fibration. With some extra assumptions, this result has been considered in different contexts.

Firstly, if we remove the assumption of properness or smoothness, in general, Ehresmann's fibration theorem does not hold since f might have "local singularities" or "singularities at infinity". The set of points in N where f fails to be trivial, denoted by $B(f)$, is called the **bifurcation set** of f, which is the union of the set $K_0(f)$ of **critical values** and the set $B_\infty(f)$ of **bifurcation values at infinity** of f. So far, characterizing $B_\infty(f)$ is still an open problem. In general, people use a bigger set (but easier to describe), the **set of asymptotic critical values of** f, denoted by $K_\infty(f)$, to control $B_\infty(f)$. The set $K_\infty(f)$ is always a nowhere dense subset of $\mathbb{C}^m$ and it is a good aproximation of the set $B_\infty(f)$. For dominant maps on smooth complex affine varieties, the computation of $K_\infty(f)$, and hence of the **set of generalized critical values**, $K(f) := K_0(f) \cup K_\infty(f)$, is given in [8].

Now if X is singular, we need to partition X into disjoint smooth manifolds and then apply Ehresmann's fibration theorem on each part. However, if we do not require any extra assumption, then the trivialization on the parts may not match. This obstacle can be overcome by introducing the Whitney conditions [17, 18]. Indeed, if f is proper and X admits a Whitney stratification, then f is locally trivial if it is a submersions on stratas [14, 10, 16]. Moreover, if f is non proper and non smooth, we can also define the bifurcation set of f such that f is locally trivial outside

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$B(f)$. However, so far, to our knowledge, no connection between $B(f)$ and the **set of stratified generalized critical values of** f, defined by $K(f) := \bigcup_{X_\alpha \in \mathcal{S}} K(f|_{X_\alpha})$, for a Whitney stratification $\mathcal{S}$ of X, has been established.

Let $X \subset \mathbb{C}^n$ be a singular algebraic set of dimension $n - r$ with $I(X) = \{g_1, \ldots, g_\omega\}$ and let $f := (f_1, \ldots, f_m) : X \to \mathbb{C}^m$ be a polynomial dominant map. Now restricting ourselves to the cases of dominant polynomial maps on singular affine varieties, the main goals of this paper are the following:

- Construct an affine Whitney stratification $\mathcal{S}$ of X.
- Establish some version of the Thom isotopy lemma for f which yield the inclusion $B(f) \subset \bigcup_{X_\alpha \in \mathcal{S}} K(f|_{X_\alpha})$.
- Calculate the set of stratified generalized critical values of f given by $K(f) := \bigcup_{X_\alpha \in \mathcal{S}} K(f|_{X_\alpha})$.

The paper is organized as follows. In Section 2, we recall the definitions of Whitney regularity and Whitney stratification, then we construct an affine stratification from a filtration of X by means of some hypersurfaces, and refine it to get an affine Whitney stratification. Some versions of the Thom isotopy lemma for non-proper polynomial maps (Theorem 3.1 and Corollary 3.1) will be given in Section 3. Then we compute the set of stratified generalized critical values of f, which contains the bifurcation values of f, where $f := (f_1, \ldots, f_m) : X \to \mathbb{C}^m$ is a polynomial dominant map, in the last Section 4.

For the remainder of the paper, the differential of f at a point x is identified with its (row) matrix, so we write $d_x f = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)$. Let

$$\nabla f(x) := \begin{bmatrix} \overline{\frac{\partial f}{\partial x_1}(x)} \\ \vdots \\ \overline{\frac{\partial f}{\partial x_n}(x)} \end{bmatrix},$$

the Hermitian transpose of $d_x f$. For $v, w \in \mathbb{C}^n$, denote by $\langle v, w \rangle = \sum_{i=1}^n \overline{v}_i w_i$ the Hermitian product, and let $v \cdot w = \sum_{i=1}^n v_i w_i$. For the set $A \subset \mathbb{C}^n$, set $\overline{A} := \{x : \overline{x} \in A\}$ and let $\overline{A}^{\mathcal{Z}}$ be the Zariski closure of A. For an algebraic variety X, the singular part and the regular part of X are denoted respectively by $\operatorname{sing}(X)$ and $\operatorname{reg}(X)$.

2. Affine Whitney stratifications

2.1. **Preliminaries.** For any two different points $x, y \in \mathbb{C}^n$, define the secant $\overline{xy}$ to be the line passing through the origin which is parallel to the line through x and y.

A **stratification** $\mathcal{S}$ of X is a decomposition of X into a locally finite disjoint union $X = \bigsqcup_{\alpha \in I} X_\alpha$ of non-empty, non-singular, connected, locally closed subvarieties, called strata, such that the boundary ∂X_α of any stratum X_α is a union of strata. If, in addition, for each α, the closure $\overline{X}_\alpha$ and the boundary $\partial X_\alpha := \overline{X}_\alpha \setminus X_\alpha$ are affine varieties, then we call $\mathcal{S}$ an **affine stratification**. It is obvious that any affine stratification is finite.

For linear subspaces $F, G \subseteq \mathbb{C}^n$, let

$$\delta(F,G) := \sup_{\substack{x\in F\\ \|x\|=1}} \operatorname{dist}(x,G),$$

where $\operatorname{dist}(x,G)$ is the Hermitian distance.

Let (X_α, X_β) be a pair of strata of $\mathcal{S}$ such that $X_\beta \subset \overline{X}_\alpha$ and let $x \in X_\beta$. We recall some regularity conditions:

(a) The pair (X_α, X_β) is said to be **(a) Whitney regular at** $x \in X_\beta$ if it satisfies the following Whitney condition (a) at x: if $x^k \in X_\alpha$ is any sequence such that $x^k \to x$ and $T_{x^k}X_\alpha \to T$, then $T \supset T_x X_\beta$.

(w) The pair (X_α, X_β) is said to be **(w) regular** at $x \in X_\beta$ (or **(a) strict Whitney regular at** x **with exponent** 1) if it satisfies the following condition (w) at x: there exist a neighborhood U of x in $\mathbb{C}^n$ and a constant $c > 0$ such that for any $y \in X_\alpha \cap U$ and $x' \in X_\beta \cap U$, we have $\delta(T_{x'}X_\beta, T_y X_\alpha) \leqslant c\|y - x'\|$.

(b) The pair (X_α, X_β) is said to be **Whitney regular at** $x \in X_\beta$ if it satisfies the following Whitney condition (b) at x: for any sequence $x^k \in X_\alpha$ and $y^k \in X_\beta$, $y^k \neq x^k$, such that $x^k \to x$, $y^k \to x$, $T_{x^k}X_\alpha \to T$ and $\overline{x^k y^k}$ converges to a line ℓ in the projective space $\mathbb{P}^{n-1}$, we have $\ell \subset T$.

We say that the pair (X_α, X_β) is **(a) Whitney regular** (resp. **Whitney regular**) if it is (a) Whitney regular (resp. Whitney regular) at every point of X_β. We say that $\mathcal{S}$ is an **(a) Whitney stratification** (resp. a **Whitney stratification**) if any pair of strata (X_α, X_β) of $\mathcal{S}$ with $X_\beta \subset \overline{X}_\alpha$ is (a) Whitney regular (resp. Whitney regular). It is well-known that Whitney regularity implies (a) Whitney regularity [17, 18]. Moreover, in light of [13], the Whitney condition (b) is equivalent to the condition (w) for the category of complex analytic sets, so to check the Whitney regularity, we can verify either the condition (w) or the condition (b).

For the purpose of this paper, we also need the following notion of Whitney (resp. (a) Whitney) regularity along a stratum. Let X_β be a stratum of $\mathcal{S}$ and let $x \in X_\beta$. We say that X_β is **Whitney regular** (resp. **(a) Whitney regular**) at x if for any stratum X_α such that $X_\beta \subset \overline{X}_\alpha$, the pair (X_α, X_β) is Whitney (resp. (a) Whitney) regular at x. The stratum X_β is **Whitney regular** (resp. **(a) Whitney regular**) if it is Whitney (resp. (a) Whitney) regular at every point of X_β. It is clear that $\mathcal{S}$ is a Whitney (resp. an (a) Whitney) stratification if and only if each stratum of $\mathcal{S}$ is Whitney (resp. (a) Whitney) regular.

2.2. **Construction of affine stratifications.** Let us, first of all, fix an affine stratification of X whose construction is based on the following proposition.

Proposition 2.1. *Let $X \subset \mathbb{C}^n$ be an affine subvariety of pure codimension r. Assume that $I(X) = \{g_1, \ldots, g_\omega\}$, where $\deg g_i \leq D$. Let W be an affine subvariety of positive codimension in X with $I(W) = \{g_1, \ldots, g_\omega, u_1, \ldots, u_\tau\}$ where $u_i \notin I(X)$ and $\deg u_i \leqslant D'$. Then there exist a polynomial $p_{X,W}$ on $\mathbb{C}^n$ of degree less than or equal to $r(D-1) + D'$ such that $W \subseteq V(p_{X,W}) := \{x \in \mathbb{C}^n :$*

$p_{X,W}(x) = 0\}$ *and* $X \setminus V(p_{X,W})$ *is a smooth, dense subset of* X. *Moreover, the polynomial* $p_{X,W}$ *can be constructed effectively.*

Proof. Let $X = \bigcup_{i=1}^m X_i$, where X_i are irreducible (hence r-codimensional) components of X. Take sufficiently generic (random) numbers $\alpha_{ij} \in \mathbb{C}$, $i = 1, \dots, r$, $j = 1, \dots, \omega$ and set

$$G_i = \sum_{j=1}^{\omega} \alpha_{ij} g_j, \ i = 1, \dots, r.$$

Note that the set $Z := V(G_1, \dots, G_r)$ has pure codimension r and $X \subset Z$. Let $\gamma_1, \dots, \gamma_\tau$ be some (random) generic numbers and set

$$H := \begin{cases} 1 & \text{if } \ W = \emptyset, \\ \sum_{i=1}^{\tau} \gamma_i u_i & \text{otherwise.} \end{cases}$$

Clearly $\dim \big(X \cap V(H)\big) < \dim X$. Moreover, for a sufficiently general linear r-dimensional subspace $L^r \subset \mathbb{C}^n$ the intersection $L^r \cap Z$ has only isolated smooth points and $L^r \cap X_i \neq \emptyset$ for every $i = 1, \dots, m$. We can assume that L^r is determined by the linear forms $l_i = \sum_{j=1}^n \beta_{ij} x_j$, $i = 1, \dots, n-r$, where β_{ij} sufficiently generic (random) numbers. Now take

$$p_{X,W} = |Jac(G_1, \dots, G_r, l_1, \dots, l_{n-r})| \cdot H,$$

where $Jac(.)$ denotes the Jacobian matrix. Then $p_{X,W}$ is a polynomial with the required properties.

The polynomial $p_{X,W}$ can be find by using a probabilistic algorithm. First recall the following.

Theorem 2.1 ([2])**.** *Let* I *be an ideal in* $k[x_1, \dots, x_n]$ *and let* $G = \{g_1, \dots, g_s\}$ *be a Gröbner basis for* I *with respect to a graded monomial order in* $k[x_1, \dots, x_n]$. *Then* $G^h = \{g_1^h, \dots, g_s^h\}$ *is a basis for* $I^h \subset k[x_0, x_1, \dots, x_n]$.

This theorem allows us to compute the set of points at infinity of an affine variety given by the ideal I, to this aim it is enough to compute the Groebner basis $\{g_1, \dots, g_s\}$ of the ideal I and then to consider the ideal $I_\infty = \{x_0, g_1^h, \dots, g_s^h\}$. Now we sketch the algorithm to compute the polynomial $p_{X,W}$. Note that for a given ideal I we can compute $\dim V(I)$ by [15].

INPUT: The ideal $I = I(X) = \{g_1, \dots, g_\omega\}$ and the ideal $J = I(W) = \{g_1, \dots, g_\omega, u_1, \dots, u_\tau\}$
1) **repeat**
choose random numbers $\alpha_{i1}, \dots, \alpha_{i\omega}$, $i = 1, \dots, r$;
put $G_i := \sum_{k=1}^{\omega} \alpha_{ik} g_k$, $i = 1, \dots, r$;
put $I = \{G_1, \dots, G_r\}$;
compute the ideal $I_\infty = \{H_1, \dots, H_m\} \subset k[x_0, \dots, x_n]$
until $\dim V(I_\infty) = n - r$.
2) **repeat**
choose random numbers $\beta_{i1}, \dots, \beta_{in}$, $i = 1, \dots, n-r$;
put $l_i := \sum_{k=1}^{n} \beta_{ik} x_k$, $i = 1, \dots, n-r$;

put $I = \{G_1, \ldots, G_r, l_1, \ldots, l_{n-r}\}$;

compute the ideal $I_\infty = \{H_1, \ldots, H_m\} \subset k[x_0, \ldots, x_n]$;

if $\dim V(I_\infty) = 0$ **then**

begin

compute $V(G_1, \ldots, G_r, l_1, \ldots, l_r) := \{a_1, \ldots, a_p\}$

end

until $\dim V(I_\infty) = 0$ and $|Jac(G_1, \ldots, G_r, l_1, \ldots, l_{n-r})(a_i)| \neq 0$ for $i = 1, \ldots p$.

3) **repeat**

choose random numbers $\gamma_1, \ldots, \gamma_\tau$;

put $H := \sum_{k=1}^{\tau} \gamma_i u_k$;

put $J = \{G_1, \ldots, G_r, H\}$;

compute the ideal $J_\infty \subset k[x_0, \ldots, x_n]$

until $\dim V(J_\infty) < n - r$.

OUTPUT: $p_{X,W} = |Jac(G_1, \ldots, G_r, l_1, \ldots, l_{n-r})| \cdot H$ $\square$

Remark 2.1. Let us assume that $I(X)$ and $I(W)$ are generated by polynomials from the ring $\mathbb{F}[x_1, ..., x_n]$, where $\mathbb{F}$ is a subfield of $\mathbb{C}$. Then we can choose a polynomial $p_{X,W}$ in this way that $p_{X,W} \in \mathbb{F}[x_1, ..., x_n]$.

Thus with no loss of generality, we can assume that $\text{rank} Jac(g_1, \ldots, g_r) = r$ on some non-empty regular open subset X^0 of X and that $X = \overline{X^0}$. It is clear that $V(p_{X,W})$ contains $\text{sing}(X) \cup W$ and the singular points of the projection $(l_1, \ldots, l_{n-r}) : X \to \mathbb{C}^{n-r}$. Now to construct an affine stratification of X, it is enough to construct an affine filtration $X = X_0 \supset X_1 \supset \cdots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$ by induction with $X_{i+1} := X_i \cap V(p_{X_i,\emptyset})$, $i = 0, \ldots, n-r$. The degree of each X_i can be calculated and depends only on D.

2.3. **Construction of affine Whitney stratifications.** In this section, we will construct an affine Whitney stratification of a given affine variety X, with $I(X) = \{g_1, \ldots, g_\omega\}$ and $\deg g_i \leq D$, by refining the affine stratification given in Subsection 2.2 so that the resulting stratification is still affine and moreover satisfies the Whitney condition.

First of all, inspired by the construction in [5, 13], let us describe the Whitney condition (b) algebraically. Assume that $Y \subset X$ is an affine subvariety of X of dimension $n - p$ with $\dim Y < \dim X$ defined by

$$Y := X \cap \{\widetilde{g}_{r+1} = \cdots = \widetilde{g}_p = 0\}.$$

Set

$$\Gamma_1 := \left\{ \begin{array}{l} (x, y, w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r : \\ g_1(x) = \cdots = g_r(x) = 0 \\ g_1(y) = \cdots = g_r(y) = \widetilde{g}_{r+1}(y) = \cdots = \widetilde{g}_p(y) = 0 \\ w = \gamma(x - y) \\ v = \sum_{i=1}^r \lambda_i d_x g_i \end{array} \right\},$$

and let

$$\pi_1 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r \to \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$$

be the projection on the first $4n$ coordinates. Let $C(X, Y) = \overline{\pi_1(\Gamma_1)}^{\mathcal{Z}} \subset (X \times Y \times \mathbb{C}^n \times \mathbb{C}^n)$, where the closure is taken in the Zariski topology. Of course, $C(X, Y)$ is an affine variety. We have the following.

Lemma 2.1. *For each $(x, x, w, v) \in C(X, Y)$, there are sequences $x^k \in X^0$, $y^k \in Y$, $\gamma^k \in \mathbb{C}$ and $\lambda^k \in \mathbb{C}^r$ such that*

- $x^k \to x$,
- $y^k \to x$,
- $w^k := \gamma^k(x^k - y^k) \to w$,
- $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i \to v$.

Proof. By construction, there are sequences $\bar{x}^k \in X$, $y^k \in Y$, $\bar{\gamma}^k \in \mathbb{C}$ and $\lambda^k \in \mathbb{C}^r$ such that $\bar{x}^k, y^k \to x$, $\bar{w}^k := \bar{\gamma}^k(\bar{x}^k - y^k) \to w$ and $\sum_{i=1}^r \lambda_i^k d_{\bar{x}^k} g_i \to v$. It is clear that by taking subsequences if necessary, we may suppose that:

- either $\bar{x}^k = y^k$ for every k or $\bar{x}^k \neq y^k$ for every k,
- for each i, either $\lambda_i^k \neq 0$ for every k or $\lambda_i^k = 0$ for every k.

Set

$$\gamma^k = \begin{cases} 0 \text{ if } \ \bar{x}^k = y^k \text{ for every } k, \\ \bar{\gamma}^k \text{ if } \ \bar{x}^k \neq y^k \text{ for every } k. \end{cases}$$

Suppose that $\lambda_i^k \neq 0$ for $i = 1, \ldots, r' \leqslant r$, $k \in \mathbb{N}$ and $\lambda_i^k = 0$ for $i = r' + 1, \ldots, r$, $k \in \mathbb{N}$. Since $\bar{x}^k \in \overline{X}^0$, there exists a sequence $x^k \in X^0$ such that

$$\|x^k - \bar{x}^k\| \leqslant \begin{cases} \frac{1}{k} \text{ if } \ \bar{x}^k = y^k \text{ for every } k, \\ \frac{\|\bar{x}^k - y^k\|}{k} \text{ if } \ \bar{x}^k \neq y^k \text{ for every } k, \end{cases}$$

so $x^k \to x$. By continuity, we can also choose x^k so that $\|d_{x^k} g_i - d_{\bar{x}^k} g_i\| < \frac{1}{k\lambda_i^k}$ if $\lambda_i^k \neq 0$. Set $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i$. Then

$$\begin{array}{rcl} \left\|v^k - \sum_{i=1}^r \lambda_i^k d_{\bar{x}^k} g_i\right\| & = & \left\| \sum_{i=1}^{r'} \lambda_i^k \big(d_{x^k} g_i - d_{\bar{x}^k} g_i\big) \right\| \\ & \leqslant & \sum_{i=1}^{r'} |\lambda_i^k| \left\|d_{x^k} g_i - d_{\bar{x}^k} g_i\right\| < \frac{r'}{k} \to 0, \end{array}$$

i.e., $v^k \to v$. Set $w^k := \gamma^k(x^k - y^k)$. Now if $\bar{x}^k = y^k$ for every k, then $\gamma^k = 0$ and $w = \bar{w}^k = 0$, so we have $w^k = 0 = w$. If $\bar{x}^k \neq y^k$ for every k, then

$$\|w^k - \bar{w}^k\| = |\gamma^k| \cdot \|(x^k - \bar{x}^k)\| \leqslant |\gamma^k| \cdot \frac{\|\bar{x}^k - y^k\|}{k} = \frac{\|\bar{w}^k\|}{k} \to 0.$$

Hence $w^k \to w$. The lemma is proved. $\square$

The following algebraic criterion permits us to check the Whitney regularity on $Y^0 = Y \setminus V(p_{Y,W})$, where the notation $V(p_{Y,W})$ is from Proposition 2.1, and the affine set W will be determined later.

Lemma 2.2. *Let $x \in Y^0$. Then the pair (X^0, Y^0) satisfies the Whitney condition (b) at x if and only if for any $(x, x, w, v) \in C(X, Y)$, we have $v \cdot w = 0$.*

Proof. Suppose that (X^0, Y^0) is Whitney regular at x and assume for contradiction that there is $(x, x, w, v) \in C(X, Y)$ such that $v \cdot w \neq 0$. In view of Lemma 2.1, there are sequences $x^k \in X^0,\ y^k \in Y,\ \gamma^k \in \mathbb{C}$ and $\lambda^k \in \mathbb{C}^r$ such that

- $x^k \to x,\ y^k \to x,$
- $w^k := \gamma^k(x^k - y^k) \to w,$
- $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i \to v.$

Note that $w \neq 0$, so w determines the limit of the sequence of secants $\overline{x^k y^k}$ and it follows that $x^k \neq y^k$ for k large enough. By taking a subsequence if necessary, we may assume that $T_{x^k} X^0 \to T$. By assumption, $w \in T$. For each k, let $\{b_1^k, \ldots, b_r^k\}$ be an orthonormal basis of $N_{x^k} X^0$; recall that $N_{x^k} X^0 := \operatorname{span}\{d_{x^k} g_1, \ldots, d_{x^k} g_r\}$ is the conormal space of X^0 at x^k. Obviously $\langle \overline{b_1^k}, \ldots, \overline{b_r^k} \rangle^\perp = T_{x^k} X^0$. By compactness, each sequence b_i^k has an accumulation point b_i. Without loss of generality, suppose that $b_i^k \to b_i$. It is clear that the system $\{b_1, \ldots, b_r\}$ is also orthonormal and $\langle \overline{b}_1, \ldots, \overline{b}_r \rangle^\perp = T$. Let $\widetilde{\lambda}^k = (\widetilde{\lambda}_1^k, \ldots, \widetilde{\lambda}_r^k)$ be such that $v^k := \sum_{i=1}^r \widetilde{\lambda}_i^k b_i^k$. Then $\widetilde{\lambda}^k$ is convergent to a limit $\widetilde{\lambda}$ and it is clear that $v = \sum_{i=1}^r \widetilde{\lambda}_i b_i$. Finally, we have $w \in T = \langle \overline{b_1}, \ldots, \overline{b_r} \rangle^\perp \subset \langle \overline{v} \rangle^\perp$, i.e., $v \cdot w = 0$, which is a contradiction.

Now suppose that $v \cdot w = 0$ for any $(x, x, w, v) \in C(X, Y)$ and assume, that (X^0, Y^0) is not Whitney regular at x. So there are sequences $x^k \in X^0$ and $y^k \in Y^0$ with the following properties:

- $x^k \neq y^k,\ x^k \to x,\ y^k \to y;$
- $T_{x^k} X^0 \to T;$
- the sequence of secants $\overline{x^k y^k}$ tends to a line $\ell \not\subset T$.

For each k, let $\{b_1^k, \ldots, b_r^k\}$ be an orthonormal basis of $N_{x^k} X^0$ so $\langle \overline{b_1^k}, \ldots, \overline{b_r^k} \rangle^\perp = T_{x^k} X^0$. As above, we may assume that $b_i^k \to b_i$. Then the system $\{b_1, \ldots, b_r\}$ is also orthonormal and $\langle \overline{b_1}, \ldots, \overline{b_r} \rangle^\perp = T$. Let $w^k := \frac{x^k - y^k}{\|x^k - y^k\|}$; we can assume that the limit $w := \lim w^k$ exists and clearly w is a direction vector of ℓ. By assumption, $w \notin T = \langle \overline{b}_1, \ldots, \overline{b}_r \rangle^\perp$, i.e., there exists an index j such that $b_j \cdot w \neq 0$. To get a contradiction, it is enough to show that there is a sequence $v^k := \sum_{i=1}^r \lambda_i^k d_{x^k} g_i$ such that $v^k \to b_j$, but this is clear since $b_j \in \operatorname{span}\{d_{x^k} g_1, \ldots, d_{x^k} g_r\}$ so such a sequence always exists. The lemma is proved. $\square$

Now according to [9, 4, 7, 6], it is possible to calculate a basis for the ideal $I(\Gamma_1)$ by calculating the radical of the following ideal in $\mathbb{C}[x, y, w, v, \gamma, \lambda]$:

$$\begin{pmatrix} g_1(x) = \cdots = g_r(x) = 0 \\ g_1(y) = \cdots = g_r(y) = \widetilde{g}_{r+1}(y) = \cdots = \widetilde{g}_p(y) = 0 \\ w = \gamma(x-y) \\ v = \sum_{i=1}^r \lambda_i d_x g_i \end{pmatrix}.$$

Then by Buchberger's algorithm, we can calculate a Gröbner basis of $I(\Gamma_1)$. So in view of [8, Theorem 5.1], [11], we can compute a Gröbner basis of the ideal $I\big(C(X,Y)\big)$. Now we give another criterion for Whitney regularity.

Lemma 2.3. *Let $\{h_1(x,y,w,v), \ldots, h_q(x,y,w,v)\}$ be a Gröbner basis of $I\big(C(X,Y)\big)$ and set*

$$\Gamma_2 := \left\{ \begin{array}{c} (x,x,w,v,\gamma,\lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : \\ h_1(x,x,w,v) = \cdots = h_q(x,x,w,v) = 0 \\ \gamma \sum_{j=1}^n v_j w_j = 1 \\ \lambda p_{Y,\emptyset}(x) = 1 \end{array} \right\},$$

where $p_{Y,\emptyset}(x)$ is the polynomial determined in Proposition 2.1. Then the pair (X^0, Y^0) is not Whitney regular at x if and only if there exists $(w,v,\gamma,\lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ such that $(x,x,w,v,\gamma,\lambda) \in \Gamma_2$.

Proof. Note that $x \in Y^0$ if and only if $p_{Y,\emptyset}(x) \neq 0$, i.e., there exists $\lambda \in \mathbb{C}$ such that $\lambda p_{Y,\emptyset}(x) = 1$. In view of Lemma 2.2, the pair (X^0, Y^0) is not Whitney regular at x if and only if there exist w, v with $v \cdot w \neq 0$ such that $(x,x,w,v) \in C(X,Y)$. The lemma follows easily. $\square$

Now we determine an algebraic set $W = W(X,Y)$ in Y with $\dim W < \dim Y$ and $V(p_{Y,\emptyset}) \subset W$ such that the pair $(X^0, Y \setminus W)$ is Whitney regular. Let

$$\pi_2 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}^n$$

be the projection on the first n coordinates. By Lemma 2.3, $\pi_2(\Gamma_2)$ is the set of points where the Whitney condition (b) fails to be satisfied. By construction, $\overline{\pi_2(\Gamma_2)}^{\mathcal{Z}}$ is affine, where $\overline{\pi_2(\Gamma_2)}^{\mathcal{Z}}$ is the Zariski closure of $\pi_2(\Gamma_2)$. It follows from [17, 18] that $\dim \pi_2(\Gamma_2) < \dim Y$, so $\dim \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}} < \dim Y$. Set

$$W = W(X,Y) := \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}};$$

then obviously $\dim W < \dim Y$. Again, applying [9, 4, 7], [8, Theorem 5.1], [11], we can compute a Gröbner basis of the ideal $I(W)$.

Finally, let

- $X_0 := X$,
- $X_1 := X_0 \cap V(p_{X_0,\emptyset})$,
- $X_2 := X_1 \cap V(p_{X_1,W(X_0,X_1)})$,
- $X_3 := X_2 \cap V(p_{X_2,W(X_0,X_2)\cup W(X_1,X_2)})$, ...,

- $X_i := X_{i-1} \cap V(p_{X_{i-1}, \bigcup_{j=0}^{i-2} W(X_j, X_{i-1})}), \dots$

By induction, we can construct a finite filtration of algebraic sets $X = X_0 \supset X_1 \supset \cdots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$ with $\dim X_i > \dim X_{i+1}$. Let $B_i := X_i \setminus X_{i+1}$. Then $\mathcal{S} := \{B_i\}_{i=1,\dots,q}$ is a Whitney stratification of X. Note that the degree of X_i can be determined explicitly and depends only on D.

3. Thom isotopy lemma for non-proper maps

Let $f : X \to \mathbb{C}^m$ be a polynomial dominant map where X is an algebraic set with $I(X) = \{g_1, \dots, g_\omega\}$. If $\mathcal{S} = \{X_\alpha\}_{\alpha \in I}$ is a stratification of X, then we denote by $\mathrm{sing}(f, \mathcal{S})$ the set of stratified singular points of f, i.e.,

$$\mathrm{sing}(f, \mathcal{S}) = \bigcup_{\alpha \in I} \mathrm{sing}(f, X_\alpha), \tag{1}$$

where $\mathrm{sing}(f, X_\alpha)$ is the set of points where $f|_{X_\alpha}$ is not a submersion. Let $K(f) = K(f, \mathcal{S})$ be the set of stratified generalized critical values of f given by

$$K(f) := \bigcup_{\substack{\alpha \in I, \\ \dim X_\alpha \geqslant m}} K_\infty(f|_{X_\alpha}) \cup \mathrm{sing}(f, \mathcal{S}) \tag{2}$$

Assume that $\mathcal{S}$ is an affine Whitney stratification of X, we prove that $K(f)$ contains the set of bifurcation values of f.

Theorem 3.1 (First isotopy lemma for non-proper maps)**.** *Let $X \subset \mathbb{C}^n$ be an affine variety with an affine Whitney stratification $\mathcal{S}$ and let $f : X \to \mathbb{C}^m$ be a polynomial dominant map. Let $K(f)$ be the set of stratified generalized critical values of f given by (2). Then f is locally trivial outside $K(f)$.*

Before proving Theorem 3.1, recall that the Whitney condition (b) is equivalent to the condition (w) (see [13, V.1.2]), so it is more convenient to use the condition (w) since we will need to construct rugose vector fields in the sense of [16]. In what follows, it is more convenient to work with the underlying real algebraic set of X in $\mathbb{R}^{2n}$, denoted also by X; the affine Whitney stratification $\mathcal{S}$ of X induces a semialgebraic Whitney stratification of the underlying set with the corresponding strata denoted by the same notations X_β. We also identify the polynomial map f with the real polynomial map $(\mathrm{Re} f_1, \dots, \mathrm{Re} f_m, \mathrm{Im} f_1, \dots, \mathrm{Im} f_m) : X \to \mathbb{R}^{2m}$.

Let us recall the definitions pertaining to rugosity. Let $\varphi : X \to \mathbb{R}$ be a real function. We say that φ is a **rugose function** if the following conditions are fulfilled:

- The restriction $\varphi|_{X_\beta}$ to any stratum X_β is a smooth function.
- For any stratum X_β and for any $x \in X_\beta$, there exist a neighborhood U of x in $\mathbb{C}^{2n}$ and a constant $c > 0$ such that for any $y \in X \cap U$ and $x' \in X_\beta \cap U$, we have $|\varphi(y) - \varphi(x')| \leqslant c\|y - x'\|$.

A **rugose map** is a map whose components are rugose functions. A vector field v on X is called a **rugose vector field** if v is a rugose map and $v(x)$ is tangent to the stratum containing x for any $x \in X$.

Proof of Theorem 3.1. Let $z \in \mathbb{C}^m \setminus K(f)$ where we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ and let B be an open box centered at z such that $\overline{B} \cap K(f) = \emptyset$. To prove the theorem, it is enough to prove that f is trivial on B. Without loss of generality, we may suppose that $z = 0$ and $B = (-1,1)^{2m}$. Let $\partial_1, \ldots, \partial_{2m}$ be the restrictions of the coordinate vector fields (on $\mathbb{R}^{2m}$) to $\overline{B}$. Set $U := f^{-1}(\overline{B})$, $U_\beta = U \cap X_\beta$ and

$$I' := \{\beta \in I : \ X_\beta \cap U \neq \emptyset\}.$$

First of all, let us give a sufficient condition for trivializing a rugose vector field.

Lemma 3.1. *For $i = 1, \ldots, 2m$, let v_i be vector fields on X which are rugose in a neighborhood of U. Assume that $df(v_i) = \partial_i$ and there is a positive constant $c > 0$ such that $\|v_i(x)\| \leqslant \frac{\|x\|+1}{c}$ for any $x \in U$. Then f is a topological trivial fibration over $\overline{B}$.*

Proof. It is enough to prove that there is a homeomorphism $\phi : f^{-1}(\overline{B}) \to f^{-1}(0) \times \overline{B}$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(\overline{B}) & \xrightarrow{\phi} & f^{-1}(0) \times \overline{B} \\ f \searrow & & \swarrow \pi \\ & \overline{B} & \end{array}$$

where π denotes the projection on the second factor. We note the following facts:

(i) The flow of v_i preserves the stratification. This is a consequence of rugosity.
(ii) For each i and any $x \in U$, there is a unique integral curve of v_i passing through x (see [16]).

Set $Y^i_t := (y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_n)$ and $Y^i = \{Y^i_t : \ t \in [-1,1]\}$. First of all, we will prove that the flow of v_i induces a homeomorphism $\phi_i : f^{-1}(Y^i) \to f^{-1}(Y^i_0) \times [-1,1]$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(Y^i) & \xrightarrow{\phi_i} & f^{-1}(Y^i_0) \times [-1,1] \\ p_i \circ f \searrow & & \swarrow \pi_i \\ & [-1,1] & \end{array}$$

where π denotes the projection on the second factor and p_i denotes the projection on the i^{th} coordinate. This follows from the following claim which states that there is no trajectory of v_i going to infinity.

Claim 3.1. *For each $x \in f^{-1}(Y^i_0)$, let γ be the integral curve of v_i such that $\gamma(0) = x$. Then γ reaches any level $f^{-1}(Y^i_t)$ at time t for $t \in [-1,1]$.*

Proof. By assumption, $\|\dot\gamma(t)\| \leqslant \frac{\|\gamma(t)\|+1}{c}$. Without loss of generality, suppose that $t>0$. In light of the Gronwall Lemma, by repating the calculation of [1, Theorem 3.5], we get

$$\begin{aligned}\|\gamma(t)\| &\leqslant \|\gamma(0)\| + \int_0^t \frac{\|\gamma(s)\|+1}{c}ds\\ &= \|x\| + \frac{t}{c} + \int_0^t \frac{\|\gamma(s)\|}{c}ds\\ &\leqslant \Big(\|x\| + \frac{t}{c}\Big)\exp\int_0^t \frac{ds}{c} = \Big(\|x\|+\frac{t}{c}\Big)e^{\frac{t}{c}} < +\infty,\end{aligned}$$

which implies that the trajectory γ does not go to infinity at time t. The claim follows. □

For any $x \in f^{-1}(Y_0^i)$, let $h_i(x,t) = x + \int_0^t \dot\gamma\,(s)ds$. Then h_i defines a homeomorphism $f^{-1}(Y_0^i)\times[-1,1] \to f^{-1}(Y^i)$. Then $\phi_i = h_i^{-1}$ is the required homeomorphism.

Now for $x \in f^{-1}(0)$, let $h : f^{-1}(0)\times\overline{B} \to f^{-1}(\overline{B})$ be defined by

$$h(x,t_1,\ldots,t_{2m}) = h_{2m}(\ldots(h_2(h_1(x,t_1),t_2),\ldots,t_{2m}).$$

Then $\phi := h^{-1}$ is a homeomorphism, as required. The lemma is proved. □

For each $\beta \in I'$, it is clear that $f|_{X_\beta}$ is a submersion on $(f|_{X_\beta})^{-1}(\overline{B})$, so for $x \in U_\beta$, the differential $d_x(f|_{X_\beta}) : T_xX_\beta \to \mathbb{R}^{2m}$ is surjective, which induces an isomorphism of vector spaces

$$\widetilde{d_x}(f|_{X_\beta}) : T_xX_\beta/\ker d_x(f|_{X_\beta}) \cong \mathbb{R}^{2m}.$$

Thus for each $i = 1,\ldots,2m$, the vector field ∂_i can be lifted uniquely and smoothly on each stratum X_β with $\beta\in I'$, to the vector field called the horizontal lift of ∂_i and denoted by v_i^β. Clearly, $v_i^\beta(x)$ is the unique vector in T_xX_β which lifts ∂_i and is orthogonal to $\ker d_x(f|_{X_\beta})$. Each v_i^β has the following important properties.

Lemma 3.2. *Let $c>0$ be such that $(\|x\|+1)\nu\big(d_x(f|_{X_\beta})\big) \geqslant c$ for any $\beta\in I'$ and any $x\in U_\beta$; recall that ν is the Rabier function* [12]. *For each $x \in X_\beta$ with $\beta\in I'$, we have*

$$\|v_i^\beta(x)\| \leqslant \frac{\|x\|+1}{c}.$$

Proof. Let $\mathbb{B}_\beta$ be the closed unit ball centered at the origin in T_xX_β. Then $d_x(f|_{X_\beta})(\mathbb{B}_\beta)$ is an ellipsoid in $\mathbb{R}^{2m}$ with $\nu\big(d_x(f|_{X_\beta})\big)$ as the length of shortest semiaxis. Let $\mathbb{B}$ be the closed unit ball centered at the origin in $\mathbb{R}^{2m}$. Then $\big(\widetilde{d}_x(f|_{X_\beta})\big)^{-1}\Big(\nu\big(d_x(f|_{X_\beta})\big)\mathbb{B}\Big)$ is an ellipsoid in T_xX_β with 1 as the lenght of longest semiaxis. Therefore the longest semiaxis of the ellipsoid $\big(\widetilde{d}_x(f|_{X_\beta})\big)^{-1}(\mathbb{B})$ is $1/\nu\big(d_x(f|_{X_\beta})\big)$. Consequently,

$$\|v_i^\beta(x)\| \leqslant \frac{1}{\nu\big(d_x(f|_{X_\beta})\big)} \leqslant \frac{\|x\|+1}{c},$$

which yields the lemma. □

Lemma 3.3. *Let $x \in U$, let X_β be the stratum containing x and let X_α be a stratum such that $X_\beta \subset \overline{X}_\alpha$. Then there exists an open neighborhood W of radius not greater than 1 of x such that for all $y \in W \cap X_\alpha$, we have*

$$\|v_i^\alpha(y)\| < 2\|v_i^\beta(x)\|,$$

for $i = 1, \dots, 2m$.

Proof. Assume for contradiction that there are an index i_0, a stratum X_{α_0} and a sequence $x^k \in X_{\alpha_0}$ such that $x^k \to 0$ and $\|v_i^{\alpha_0}(x^k)\| \geqslant 2\|v_i^\beta(x)\|$. Taking a subsequence if necessary, we may assume that $T_{x^k} X_{\alpha_0} \to T$. Since the stratification is Whitney regular, it is (a) Whitney regular, which yields $T \supset T_x X_\beta$. The following claims are straightforward.

Claim 3.2. *Let $L : H \to \mathbb{R}^m$ be a surjective linear map and let $\widetilde{L} : H/\ker L \cong \mathbb{R}^m$ be the induced linear isomorphism. Let $H' \subset H$ be a linear subspace and assume that $L|_{H'}$ is also surjective. Then for any $w \in \mathbb{R}^m$, we have*

$$\|(\widetilde{L})^{-1}(v)\| \leqslant \|(\widetilde{L|_{H'}})^{-1}(v)\|.$$

Claim 3.3. *The sequence $v_i^{\alpha_0}(x^k)$ is convergent.*

By Claim 3.3, let $w_i := \lim_{k\to\infty} v_i^{\alpha_0}(x^k)$. Then it is clear that $\|w_i\| \geqslant 2\|v_i^\beta(x)\|$ and $w_i = (\widetilde{d_x f|_T})^{-1}(\partial_i)$ where $\widetilde{d_x f|_T}$ is given by the linear isomorphism $T/\ker(d_x f|_T) \cong \mathbb{R}^{2m}$. Since $T \supset T_x X_\beta$, it follows from Claim 3.2 that

$$\|w_i\| \leqslant \|(\widetilde{d_x f|_{T_x X_\beta}})^{-1}(\partial_i)\| = \|v_i^\beta(x)\|.$$

This is a contradiction, which ends the proof of the lemma. $\square$

Note that, for fixed i, the vector field on U which coincides with v_i^β on each U_β is not necessarily a smooth vector field. In what follows, we will try to deform these fields to produce a rugose vector field in the sense of [16], which satisfies the assumption of Lemma 3.1. The process is carried out by induction on dimension.

For $2m \leqslant d \leqslant 2\dim_{\mathbb{C}} X$, let $I'_d := \{\beta \in I' : 2m \leqslant \dim X_\beta \leqslant d\}$, $B_d := \bigcup_{i \in I'_d} X_\beta$ and $U_d = B_d \cap U$. By induction on d, we construct a rugose vector field on a neighborhood of $U_{2\dim_{\mathbb{C}} X}$ in X with the property of Lemma 3.1. For $d = 2m$, let v_i^{2m} be the restriction to an open neighborhood of U_{2m} in X of the smooth vector field on B_{2m} which coincides with each v_i^β on X_β for $\beta \in I'_{2m}$. Then v_i^{2m} is clearly rugose, $df(v_i^{2m}) = \partial_i$ and by Lemma 3.2, $\|v_i^{2m}(x)\| \leqslant \frac{\|x\|+1}{c}$ for any $x \in U_{2m}$.

For each i, assume that we have constructed a rugose vector field, denoted by v_i^d, on a neighborhood $\widetilde{U}_d$ of U_d in B_d such that $d_x f\big(v_i^d(x)\big) = \partial_i$ and $\|v_i^d(x)\| \leqslant \frac{\|x\|+1}{c_d}$ for every $x \in \widetilde{U}_d$, where c_d is a positive constant. We need to extend each v_i^d to a rugose vector field v_i^{d+2} on a neighborhood of U_{d+2} in B_{d+2} such that $\|v_i^{d+2}(x)\| \leqslant \frac{\|x\|+1}{c_{d+2}}$ for every $x \in U_{d+2}$, where c_{d+2} is also a positive constant (recall that the strata of $\mathcal{S}$ have even dimension). Note that to construct v_i^{d+2}, it is enough to construct v_i^{d+2} separately on each stratum X_α with $\alpha \in I'_{d+2} \setminus I'_d$. Without loss of generality, suppose that $I'_{d+2} \setminus I'_d = \{\alpha\}$.

By similar arguments as in [16], for each $i = 1, \ldots, 2m$, there is a rugose vector field on a neighborhood $\widetilde{U}_{d+2}$ of U_{d+2} in $B_{d+2} = B_d \cup X_\alpha$, denoted by w_i^{d+2}, which extends v_i^d, such that:

(i) The restriction $w_i^{d+2}|_{U_\alpha}$ is a smooth vector field.
(ii) For $x \in U_\alpha$, we have $d_x f\big(w_i^{d+2}(x)\big) = \partial_i$.

For each $x \in U_d$, let X_β be the stratum containing x, and let W_x be a neighborhood of x given by Lemma 3.3. Since w_i^{d+2} is continuous, by shrinking W_x if necessary, we may assume that

$$\|w_i^{d+2}(y)\| < 2\|v_i^d(x)\|, \tag{3}$$

for any $y \in W_x$. Let $V_d := \bigcup_{x \in U_d} W_x$, then V_d is an open neighborhood of radius not bigger than 1 of U_d. By a smooth version of Urysohn's lemma, there is a smooth function $\varphi : \mathbb{R}^{2n} \to [0,1]$ such that $\varphi^{-1}(0) = \mathbb{R}^{2n} \setminus V_d$ and $\varphi^{-1}(1) = U_d$. For $x \in \widetilde{U}_{d+2}$, set

$$v_i^{d+2}(x) := \begin{cases} w_i^{d+2}(x) = v_i^d(x) & \text{if } x \in \widetilde{U}_{d+2} \cap \widetilde{U}_d \\ \big(1 - \varphi(x)\big) v_i^\alpha(x) + \varphi(x) w_i^{d+2}(x) & \text{if } x \in \widetilde{U}_{d+2} \setminus \widetilde{U}_d. \end{cases}$$

Clearly, the restriction of v_i^{d+2} on each stratum is a smooth vector field. Moreover for $x \in \widetilde{U}_{d+2} \setminus \widetilde{U}_d$, we have

$$\begin{aligned} d_x f\big(v_i^{d+2}(x)\big) &= d_x f\Big(\big(1 - \varphi(x)\big) v_i^\alpha(x) + \varphi(x) w_i^{d+2}(x)\Big) \\ &= \big(1 - \varphi(x)\big) d_x f\big(v_i^\alpha(x)\big) + \varphi(x) d_x f\big(w_i^{d+2}(x)\big) \\ &= \big(1 - \varphi(x)\big)\partial_i + \varphi(x)\partial_i = \partial_i. \end{aligned}$$

Let us prove that v_i^{d+2} is a rugose vector field. For any $x \in \widetilde{U}_{d+2} \cap \widetilde{U}_d$, let X_β be the stratum containing x. For $x' \in W_x \cap X_\beta$ and $y \in W_x \cap X_\alpha$ with $\beta \in I'_d$, we have

$$\begin{aligned} \|v_i^{d+2}(y) - v_i^{d+2}(x')\| &= \big\|\big(1 - \varphi(y)\big) v_i^\alpha(y) + \varphi(y) w_i^{d+2}(y) - v_i^d(x')\big\| \\ &= \big\|\big(1 - \varphi(y)\big) v_i^\alpha(y) - \big(1 - \varphi(y)\big) w_i^{d+2}(y) + w_i^{d+2}(y) - v_i^d(x')\big\| \\ &\leqslant \big(1 - \varphi(y)\big)\|v_i^\alpha(y) - w_i^{d+2}(y)\| + \|w_i^{d+2}(y) - v_i^d(x')\| \\ &\leqslant \big(1 - \varphi(y)\big)(\|v_i^\alpha(y)\| + \|w_i^{d+2}(y)\|) + \|w_i^{d+2}(y) - v_i^d(x')\|. \end{aligned}$$

We note the following facts:

- Since $1 - \varphi(y)$ is a smooth function, it is locally Lipschitz; with no loss of generality, assume that $1 - \varphi(y)$ is Lipschitz on W_x with constant c_1. Then

$$1 - \varphi(y) = \big(1 - \varphi(y)\big) - \big(1 - \varphi(x')\big) \leqslant c_1\|y - x'\|.$$

- By Lemma 3.3 and by the continuity of w_i^{d+2}, there is a positive constant c_2 depending only on x such that $\|v_i^\alpha(y)\| + \|w_i^{d+2}(y)\| \leqslant c_2$
(we can take $c_2 := \max\big\{2\|v_i^\beta(x)\|, \sup_{y \in \overline{W}_x \cap X_\alpha}\|w_i^{d+2}(y)\|\big\}$).
- Since w_i^{d+2} is rugose, it follows that there is a positive constant c_3 depending only on x such that $\|w_i^{d+2}(y) - v_i^d(x')\| \leqslant c_3\|y - x'\|$.

Hence

$$\|v_i^{d+2}(y) - v_i^{d+2}(x')\| \leqslant (c_1c_2 + c_3)\|y - x'\|,$$

i.e., v_i^{d+2} is rugose. Now it remains to show that there is a positive constant c_{d+2} such that $\|v_i^{d+2}(y)\| \leqslant \frac{\|y\|+1}{c_{d+2}}$ for every $y \in \widetilde{U}_{d+2}$. Obviously, the statement holds for $y \in (\widetilde{U}_{d+2} \cap \widetilde{U}_d)$ by the induction assumption and for $y \in (\widetilde{U}_{d+2} \setminus V)$ by Lemma 3.2, so we can suppose that $y \in (V \cap \widetilde{U}_{d+2}) \setminus \widetilde{U}_d$, which clearly implies that $y \in X_\alpha$. By construction and by Lemma 3.3, there are a point $x \in U_d$ and an open neighborhood W_x of radius not greater than 1 of x containing y such that $\|v_i^\alpha(y)\| < 2\|v_i^\beta(x)\|$, where β is the index of the stratum X_β containing x. By Lemma 3.2, it follows that

$$\|v_i^\alpha(y)\| < 2\frac{\|x\|+1}{c} \leqslant 2\frac{\|y\|+\|x-y\|+1}{c} \leqslant 2\frac{\|y\|+2}{c} \leqslant 4\frac{\|y\|+1}{c}, \tag{4}$$

where c is the constant in the same lemma. Similarly, in view of (3) and the induction assumption, we have

$$\|w_i^{d+2}(y)\| < 2\|v_i^d(x)\| \leqslant 2\frac{\|x\|+1}{c_d} \leqslant 2\frac{\|y\|+\|x-y\|+1}{c_d} \leqslant 2\frac{\|y\|+2}{c_d} \leqslant 4\frac{\|y\|+1}{c_d}. \tag{5}$$

Thus (4) and (5) yield

$$\begin{array}{rcl} \|v_i^{d+2}(y)\| & = & \big|\big(1-\varphi(y)\big)v_i^\alpha(y) + \varphi(y)w_i^{d+2}(y)\big\| \\ & \leqslant & \big(1-\varphi(y)\big)\|v_i^\alpha(y)\| + \varphi(y)\|w_i^{d+2}(y)\| \\ & \leqslant & \big(1-\varphi(y)\big)4\frac{\|y\|+1}{c} + \varphi(y)4\frac{\|y\|+1}{c_d} \\ & < & \left(\frac{4}{c}+\frac{4}{c_d}\right)(\|y\|+1). \end{array}$$

Set $c_{d+2} = \min\left\{\frac{1}{\frac{4}{c}+\frac{4}{c_d}}, c, c_d\right\}$, then $\|v_i^{d+2}(y)\| \leqslant \frac{\|y\|+1}{c_{d+2}}$ for every $y \in \widetilde{U}_{d+2}$. By induction, there exists a rugose vector field on a neighborhood of $U_{2\dim_{\mathbb{C}} X}$ in X with the property of Lemma 3.1. Then the theorem follows by applying Lemma 3.1. $\square$

The following corollary follows immediately from Theorem 3.1.

Corollary 3.1. *Let $X \subset \mathbb{C}^n$ be an affine variety with an affine Whitney stratification $\mathcal{S}$ and let $f : X \to \mathbb{C}^m$ be a polynomial dominant map. Assume that for any stratum $X_\beta \in \mathcal{S}$, the restriction $f|_{X_\beta}$ is a submersion and $K_\infty(f|_{X_\beta}) = \emptyset$. Then f is a locally trivial fibration.*

4. Computation of the sets of stratified generalized critical values

In this section, we will compute the set $K(f)$ of stratified generalized critical values of f, for which we need to construct an affine Whitney stratification of X and then apply [8] for each stratum of this stratification. The process is a bit different from the construction in Subsection 2.3 since we only need to construct such an affine Whitney stratification "partially", by remarking the following facts:

- As the construction of Whitney stratifications is by induction on dimension, we only need to proceed until the dimension shrinks below m since the restriction of f to any stratum of dimension $< m$ is always singular.

- For any algebraic set $Z \subseteq X$, let

$$r_Z := \max_{x \in Z \setminus V(p_{Z,\emptyset})} \mathrm{rank} Jac_x(f|_Z) \quad \text{and} \quad H(Z) := \overline{\{x \in Z \setminus V(p_{Z,\emptyset}) : \mathrm{rank} Jac_x(f|_Z) < r_Z\}}^{\mathcal{Z}}.$$

  Then at any step of the induction process, the construction in Subsection 2.3 can be omitted if $r_Y < m$.

Let us now construct such a stratification. With the same notations as in Lemma 2.3, let

$$\Gamma_3 := \bigcup_{k=1}^{t} \left\{ \begin{array}{c} (x, x, w, v, \gamma, \lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t : \\ h_1(x, x, w, v) = \cdots = h_q(x, x, w, v) = 0 \\ \gamma \sum_{j=1}^{n} v_j w_j = 1 \\ \lambda p_{Y,\emptyset}(x) = 1 \\ \mu_k M_k^{(m,p)}(x) = 1 \end{array} \right\},$$

where each $M_k^{(m,p)}(x)$ is a minor of the matrix

$$A(x) := \begin{bmatrix} d_x f_1 \\ \vdots \\ d_x f_m \\ d_x g_1 \\ \vdots \\ d_x g_r \\ d_x \widetilde{g}_{r+1} \\ \vdots \\ d_x \widetilde{g}_p \end{bmatrix},$$

obtained by deleting $n - m - p$ columns. So Γ_3 differs from Γ_2 in the last t equations since we are only interested in finding the points where the Whitney condition (b) is not satisfied, outside $P(Y, \emptyset)$. Let

$$\pi_3 : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t \to \mathbb{C}^n$$

be the projection on the first n coordinates. By Lemma 2.3, $\pi_3(\Gamma_3)$ is the set of points where the Whitney condition (b) fails. Obviously $\pi_3(\Gamma_3) \subset \mathrm{reg}(f|_{Y \setminus P(Y)})$ and $\dim \pi_3(\Gamma_3) < \dim Y$. Set $\widetilde{W} := \overline{\pi_3(\Gamma_3)}^{\mathcal{Z}}$. Then obviously $\dim \widetilde{W} < \dim Y$. Again, we can compute a Gröbner basis of the ideal $I(\widetilde{W})$.

Finally, set

- $X_0 := X$,
- $X_1 := X_0 \cap V(p_{X_0,\emptyset}),\ S_1 = \mathrm{sing}(f|_{X_0 \setminus X_1}), \ldots,$
- $X_i := X_{i-1} \cap V(p_{X_{i-1}, \bigcup_{j=0}^{i-2} \widetilde{W}(X_j, X_{i-1})}),\ S_i = \mathrm{sing}(f|_{X_{i-1} \setminus X_i}), \ldots$

By induction, we can construct a finite filtration of algebraic sets $X = X_0 \supset X_1 \supset \cdots \supset X_q \supset X_{q+1} \supseteq \emptyset$ with $\dim X_i > \dim X_{i+1}$ and $r_{X_{q+1}} < m$. It is clear that this filtration does not induce an affine Whitney stratification of X. However, it shows that there is an affine Whitney stratification $\mathcal{S}$ such that

$$\mathrm{sing}(f, \mathcal{S}) = \bigcup_{i=1}^{q} S_i \cup X_{q+1}.$$

Let $B_i := X_i \setminus X_{i+1}$. Then $\{B_i\}_{i=0,...,q}$ is an affine Whitney stratification of $X \setminus X_{q+1}$. Every variety B_i can be realized as a closed affine variety in $\mathbb{C}^{n+1}$, by the embedding $B_i \ni x \mapsto \big(x, 1/P_{X_i, \bigcup_{j=0}^{i-1} \widetilde{W}(X_j, X_i)}(x)\big) \in \mathbb{C}^{n+1}$ for $i > 0$ or the embedding $B_0 \ni x \mapsto \big(x, 1/P_{X_0,\emptyset}(x)\big) \in \mathbb{C}^{n+1}$. Let $K_\infty(f|_{B_i})$ be the set of asymptotic critical values of $f|_{B_i}$, which now can be computed by applying [8]. Then from the construction, it is clear that the set of stratified generalized critical values of f is given by

$$K(f) := \bigcup_{i=1}^{q} K_\infty(f|_{B_i}) \cup \mathrm{sing}(f, \mathcal{S}),$$

and $K(f)$ can be computed effectively. Note that Remark 2.1 and elementary properties of Gröbner basis implies:

Corollary 4.1. *Let $X \subset \mathbb{C}^n$ be an affine variety of pure dimension and let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial mapping. Let $\mathbb{F} \subset \mathbb{C}$ be a subfield generated by coefficients of generators of $I(X)$ and all coefficients of polynomials $f_i,\ i = 1, ..., m$. Then there is a nowhere dense affine variety $K(f) \subset \mathbb{C}^m$, which is described by polynomials from $\mathbb{F}[x_1, ..., x_m]$, such that all bifurcation values $B(f)$ of f are contained in $K(f)$. In particular, for $m = 1$, if X and f are described by polynomials from $\mathbb{Q}[x_1, ..., x_n]$, then all bifurcation values of f are algebraic numbers.*

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