A NOTE ON NONDEGENERATE MATRIX POLYNOMIALS

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Abstract. In this paper, via Newton polyhedra, we define and study symmetric matrix polynomials, which are nondegenerate at infinity. From this we construct a class of (not necessarily compact) semialgebraic sets in $\mathbb{R}^n$ such that for each set K in the class we have the following two statements: (i) the space of symmetric matrix polynomials, whose eigenvalues are bounded on K, is described in terms of the Newton polyhedron corresponding to the generators of K (i.e., the matrix polynomials used to define K) and is generated by a finite set of matrix monomials; and (ii) a matrix version of Schmüdgen's Positivstellensätz holds: every matrix polynomial, whose eigenvalues are "strictly" positive and bounded on K, is contained in the preordering generated by the generators of K.

1. Introduction

The question of representing real polynomials by sums of squares of polynomials is a main topic in real algebraic geometry. Starting with Hilbert's question of whether every nonnegative real polynomial in several variables is a sum of squares of real rational functions, many questions have arisen in this field, and many interesting results are known. For more details we refer the reader to [2, 13, 24, 29] with the references therein.

Given a basic closed semialgebraic set K in $\mathbb{R}^n$ defined by finitely many polynomial inequalities $\{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$, where each g_i is a real polynomial, Positivstellensätze are results characterizing all polynomials, which are positive on K, in terms of sums of squares and the polynomials g_i used to describe K. Theorems about the existence of such representations have various applications, notably in problems of optimizing polynomial functions on semialgebraic sets (see, for example, [8, 11, 12, 13]).

In case K is compact, Schmüdgen [31] has proved that any polynomial, which is positive on K, is in the preordering generated by the g_i's, i.e., the set of finite sums of elements of the form $\sigma_e g_1^{e_1} \cdots g_m^{e_m}$, where $e_i \in \{0, 1\}$ and each σ_e is a sum of squares of polynomials.

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Putinar [25] has proved that, under a certain condition (which is slightly stronger than the compactness of K), the preordering can be replaced by the quadratic module generated by the g_i's, which is the set of sums $\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m$, where each σ_i is a sum of squares of polynomials.

If K is not compact, the above characterizations do not hold in general and can depend on the choice of generators. In fact, Scheiderer [26] has shown that Schmüdgen's Positivstellensätz does not hold if K is not compact and $\dim K \geq 3$, or $\dim K = 2$ and K contains a 2-dimensional cone. On the other hand, there exist non-compact semialgebraic sets K of any dimension for which Schmüdgen's Positivstellensätz (or even Putinar's Positivstellensätz) holds for polynomials, which are positive on K and satisfy certain extra conditions; see [7, 14, 19, 22, 23, 26, 27, 28, 32].

We also would like to note that both Schmüdgen's and Putinar's Positivstellensätz were extended from the usual real polynomials to the real symmetric matrix polynomials or operator polynomials; see [3, 5, 9, 30].

The aim of this paper is to extend the results obtained in [7] to matrix polynomials. More precisely, via Newton polyhedra, we define and study (symmetric) matrix polynomials, which are nondegenerate at infinity. From this we construct a class of (not necessarily compact) semialgebraic sets in $\mathbb{R}^n$ such that for each set K in the class we have the following two statements: (i) the space of symmetric matrix polynomials, whose eigenvalues are bounded on K, is described in terms of the Newton polyhedron corresponding to the matrix polynomials used to define K and is generated by a finite set of matrix monomials; and (ii) a matrix version of Schmüdgen's Positivstellensätz holds for matrix polynomials whose eigenvalues are "strictly" positive and bounded on K.

Notation. Throughout this paper, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{Z}_{\geq 0}$ the set of nonnegative integer numbers, and $\mathbb{R}^n$ denotes the Euclidean space of dimension n. The corresponding inner product (resp., norm) in $\mathbb{R}^n$ is defined by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^n$). We let $\mathbb{R}[x]$ denote the ring of real polynomials in n indeterminates.

In what follows, we fix a positive integer number d. We will denote by $\mathrm{Mat}_d(\mathbb{R}[x])$ the ring of all $d \times d$ matrices with entries from $\mathbb{R}[x]$ (elements in this ring will be called *matrix polynomials*) and by $\mathrm{Sym}_d(\mathbb{R}[x])$ the set of all symmetric matrix polynomials from $\mathrm{Mat}_d(\mathbb{R}[x])$. The unit of $\mathrm{Mat}_d(\mathbb{R}[x])$ is the identity matrix I_d.

Recall that a symmetric matrix $A \in \mathbb{R}^{d\times d}$ is called *positive semidefinite* if $\langle Av, v\rangle \geq 0$ for all vectors $v \in \mathbb{R}^d$. A is *positive definite* if it is positive semidefinite and invertible. For symmetric matrices A and B of the same size, we write $A \succeq B$ (resp., $A \succ B$) to express that $B - A$ is positive semidefinite (resp., positive definite). Geometrically, A is positive semidefinite if and only if all of its eigenvalues are nonnegative and A is positive definite if and only if all of its eigenvalues are positive.

Given a symmetric matrix polynomial $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ and a set $K \subset \mathbb{R}^n$, we write $F \succeq 0$ (resp., $F \succ 0$) on K if for all $x \in K$, the matrix $F(x)$ is positive semidefinite (resp., the matrix $F(x)$ is positive definite).

A subset $\mathcal{M}$ of $\mathrm{Sym}_d(\mathbb{R}[x])$ is said to be a *quadratic module* if $I_d \in \mathcal{M}, \mathcal{M} + \mathcal{M} \subset \mathcal{M}$ and $A^T\mathcal{M}A \subset \mathcal{M}$ for every $A \in \mathrm{Mat}_d(\mathbb{R}[x])$. The smallest quadratic module which contains a given subset $\mathcal{G}$ of $\mathrm{Sym}_d(\mathbb{R}[x])$ will be denoted by $\mathcal{M}_{\mathcal{G}}$. It consists of all finite sums of elements of the form A^TGA where $G \in \mathcal{G} \cup \{I_d\}$ and $A \in \mathrm{Mat}_d(\mathbb{R}[x])$. A subset $\mathcal{T}$ of the set $\mathrm{Sym}_d(\mathbb{R}[x])$ is said to be a *preordering* if $\mathcal{T}$ is a quadratic module and the set $\mathcal{T} \cap \mathbb{R}[x] \cdot I_d$ is closed under multiplication. The smallest preordering containing a given set $\mathcal{G} \subset \mathrm{Sym}_d(\mathbb{R}[x])$ will be denoted by $\mathcal{T}_{\mathcal{G}}$.

2. Nondegeneracy of matrix polynomials

Let $\mathcal{G} := \{G_1, \ldots, G_m\} \subset \mathrm{Sym}_d(\mathbb{R}[x])$. For each $i = 1, \ldots, m$, we can write

$$G_i(x) \quad = \quad \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} A_{i,\alpha} x^\alpha$$

for some symmetric matrices $A_{i,\alpha} \in \mathrm{Sym}_d(\mathbb{R})$. Then we define

$$\mathrm{supp}(\mathcal{G}) \quad := \quad \bigcup_{i=1}^m \{\alpha \in \mathbb{Z}^n_{\geq 0} \mid A_{i,\alpha} \neq 0\}.$$

The *Newton polyhedron (at infinity)* of $\mathcal{G}$, denoted by $\Gamma(\mathcal{G})$, is defined as the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(\mathcal{G})$. The system $\mathcal{G}$ is said to be *convenient* if $\Gamma(\mathcal{G})$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$.

Given a nonzero vector $q \in \mathbb{R}^n$, we define

$$\begin{aligned} \ell(q, \Gamma(\mathcal{G})) \quad &:= \quad \min\{\langle q, \alpha\rangle \ : \ \alpha \in \Gamma(\mathcal{G})\}, \\ \Delta(q, \Gamma(\mathcal{G})) \quad &:= \quad \{\alpha \in \Gamma(\mathcal{G}) \ : \ \langle q, \alpha\rangle = \ell(q, \Gamma(\mathcal{G}))\}. \end{aligned}$$

We say that a subset Δ of $\Gamma(\mathcal{G})$ is a *face* of $\Gamma(\mathcal{G})$ if there exists a nonzero vector $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma(\mathcal{G}))$. The dimension of a face Δ is defined as the minimum of the dimensions of the affine subspaces containing Δ. The faces of $\Gamma(\mathcal{G})$ of dimension 0 are called the *vertices* of $\Gamma(\mathcal{G})$. The *Newton boundary (at infinity)* of the system $\mathcal{G}$, denoted by $\Gamma_\infty(\mathcal{G})$, is defined as the union of all faces $\Delta(q, \Gamma(\mathcal{G}))$ for some $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$. For $i = 1, \ldots, m$ and $\Delta \in \Gamma_\infty(\mathcal{G})$ we denote by $G_{i,\Delta}$ the matrix polynomial $\sum_{\alpha \in \Delta} A_{i,\alpha} x^\alpha$.

Let

$$\rho(x) := \sum_{\alpha \in V(\mathcal{G})} |x^\alpha|,$$

where $V(\mathcal{G})$ is the set of all vertices of $\Gamma(\mathcal{G})$.

Remark 2.1. By the Tarski–Seidenberg theorem (see, for example, [1, 2]), it is easy to check that ρ is a semialgebraic function on $\mathbb{R}^n$. Furthermore, we can find a constant $c > 0$ such that (see also [20, Remark 3.1])

$$c\left(\sum_{\alpha\in\Gamma(\mathcal{G})\cap\mathbb{Z}^n_{\geq 0}} |x^\alpha|\right) \ \leq \ \rho(x) \ \leq \sum_{\alpha\in\Gamma(\mathcal{G})\cap\mathbb{Z}^n_{\geq 0}} |x^\alpha| \quad \text{for all} \quad x \in \mathbb{R}^n.$$

From now on for each $A \in \mathrm{Sym}_d(\mathbb{R})$, we denote by $\lambda(A)$ the largest eigenvalue of A. It is well-known that

$$\lambda(A) = \max\{\langle Av, v\rangle \mid v \in \mathbb{R}^n, \|v\| = 1\}.$$

The following definition is inspired from the works of Gindikin [6] and Mikhalov [16].

Definition 2.1. We say that the system $\mathcal{G} := \{G_1, \ldots, G_m\}$ is *nondegenerate (at infinity)* if and only if

$$\max_{i=1,\ldots,m} \lambda(G_{i,\Delta}(x)) > 0 \quad \text{for all} \quad x \in (\mathbb{R}\backslash\{0\})^n$$

for any face $\Delta \in \Gamma_\infty(\mathcal{G})$.

Example 2.1. Let $\alpha^1, \ldots, \alpha^m$ be nonzero vectors in $\mathbb{Z}^n_{\geq 0}$. Then the system

$$\{x^{2\alpha^1} \cdot I_d, \ldots, x^{2\alpha^m} \cdot I_d\} \subset \mathrm{Sym}_d(\mathbb{R})$$

is nondegenerate.

Lemma 2.1. *There exists a constant $c > 0$ such that*

$$\max_{i=1,\ldots,m} \lambda(G_i(x)) \ \leq \ c\rho(x) \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Proof. Take any $x \in \mathbb{R}^n$ and any $i \in \{1, \ldots, m\}$. Let λ be an eigenvalue of $G_i(x)$. By definition, there exists a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that $\lambda = \langle G_i(x)v, v\rangle$. Write $G_i(x) = \sum_\alpha A_{i,\alpha}x^\alpha$. Then

$$\begin{aligned} |\lambda| \ &= \ |\langle G_i(x)v, v\rangle| \ \leq \ \sum_\alpha |\langle A_{i,\alpha}v, v\rangle| \cdot |x^\alpha| \\ &\leq \ \sum_\alpha \|A_{i,\alpha}\| \cdot |x^\alpha| \ \leq \ \left(\max_\alpha \|A_{i,\alpha}\|\right) \cdot \sum_\alpha |x^\alpha|. \end{aligned}$$

Since λ is an arbitrary eigenvalue of $G_i(x)$, we get

$$\max_{i=1,\ldots,m} \lambda(G_i(x)) \ \leq \ \max_{i=1,\ldots,m} \left[\left(\max_\alpha \|A_{i,\alpha}\|\right) \cdot \sum_\alpha |x^\alpha|\right].$$

This, together with Remark 2.1, completes the proof. $\square$

We come now to the main result of this section.

Theorem 2.1. *The following two statements are equivalent.*

(i) *The system $\mathcal{G} := \{G_1, \ldots, G_m\}$ is nondegenerate.*

(ii) *There exist constants $c > 0$ and $R > 0$ such that*

$$c\rho(x) \quad \leq \quad \max_{i=1,\ldots,m} \lambda(G_i(x)) \qquad \text{for all} \qquad \|x\| \geq R.$$

Proof. (i) ⇒ (ii) By contradiction and using the Curve Selection Lemma at infinity ([17, 18]), we can find an analytic curve $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))$ for $s \in (0, \epsilon)$, such that

(a) $\|\varphi(s)\| \to +\infty$ as $s \to 0^+$; and

(b) $\rho(\varphi(s)) \gg \max_{i=1,\ldots,m} \lambda(G_i(\varphi(s)))$ as $s \to 0^+$.

Let $J := \{j \,|\, \varphi_j \not\equiv 0\} \subset \{1, \ldots, n\}$. By Condition (a), $J \neq \emptyset$. For $j \in J$, we can expand the coordinate φ_j in terms of the parameter: say

$$\varphi_j(s) = x_j^0 s^{q_j} + \text{ higher order terms in } s,$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. From Condition (a), we get $\min_{j \in J} q_j < 0$.

Let $\mathbb{R}^J := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \,|\, \alpha_j = 0 \text{ for } j \notin J\}$. We first suppose that $\Gamma(\mathcal{G}) \cap \mathbb{R}^J = \emptyset$. Then for each $\alpha := (\alpha_1, \ldots, \alpha_n) \in \Gamma(\mathcal{G})$, there exists an index $j \notin J$ such that $\alpha_j > 0$, and so $(\varphi_j(s))^{\alpha_j} \equiv 0$. Hence,

$$\begin{aligned}
\rho(\varphi(s)) \quad &= \quad \sum_{\alpha \in V(\mathcal{G})} |\varphi(s)|^\alpha = \sum_\alpha \left(\prod_{j \in J} |\varphi_j(s)|^{\alpha_j} \prod_{j \notin J} |\varphi_j(s)|^{\alpha_j} \right) \equiv 0, \\
G_i(\varphi(s)) \quad &= \quad \sum_\alpha A_{i,\alpha} \varphi(s)^\alpha = \sum_\alpha A_{i,\alpha} \left(\prod_{j \in J} \varphi_j(s)^{\alpha_j} \prod_{j \notin J} \varphi_j(s)^{\alpha_j} \right) \equiv 0,
\end{aligned}$$

which contradicts Condition (b).

Therefore, $\Gamma(\mathcal{G}) \cap \mathbb{R}^J \neq \emptyset$. Let ℓ be the minimal value of the linear function $\sum_{j \in J} q_j \alpha_j$ on $\Gamma(\mathcal{G}) \cap \mathbb{R}^J$, and let Δ be the (unique) maximal face[1] of $\Gamma(\mathcal{G}) \cap \mathbb{R}^J$ where this function takes its minimal value. Then $\Delta \in \Gamma_\infty(\mathcal{G})$ because $\min_{j \in J} q_j < 0$. Moreover, we can write

$$\rho(\varphi(s)) = \rho_\Delta(x^0) s^\ell + \text{ higher order terms in } s,$$

where $x^0 := (x_1^0, \ldots, x_n^0)$ and $x_j^0 := 1$ for $j \notin J$. Note that $\rho_\Delta(x^0) > 0$. Hence

$$\rho(\varphi(s)) \simeq s^\ell \qquad \text{as} \qquad s \to 0^+. \tag{1}$$

Let $i^* \in \{1, \ldots, m\}$ be an index such that

$$\lambda(G_{i^*,\Delta}(x^0)) \quad = \quad \max_{i=1,\ldots,m} \lambda(G_{i,\Delta}(x^0)) \ > \ 0. \tag{2}$$

By definition, there exists a vector $v \in \mathbb{R}^n$ with $\|v\| = 1$ such that

$$\lambda(G_{i^*,\Delta}(x^0)) \ = \ \langle G_{i^*,\Delta}(x^0) v, v \rangle.$$

[1]"maximal face" means with respect to the inclusion of faces.

Hence, if we write $G_i(x) = \sum A_{i,\alpha} x^\alpha$, then we deduce successively

$$\begin{aligned}
\lambda(G_{i^*}(\varphi(s))) &\geq \langle G_{i^*}(\varphi(s))v, v\rangle = \sum_\alpha \langle A_{i^*,\alpha} v, v\rangle (\varphi(s))^\alpha \\
&= \left(\sum_{\alpha\in\Delta} \langle A_{i^*,\alpha} v, v\rangle\right) s^\ell + \text{ higher order terms in } s \\
&= \langle G_{i^*,\Delta}(x^0)v, v\rangle s^\ell + \text{ higher order terms in } s \\
&= \lambda(G_{i^*,\Delta}(x^0)) s^\ell + \text{ higher order terms in } s.
\end{aligned}$$

Combining this with (1), (2) and Condition (b), we get a contradiction.

(ii) $\Rightarrow$ (i) By contradiction, suppose that there exists a face $\Delta \in \Gamma_\infty(\mathcal{G})$ and a point $x^0 \in (\mathbb{R} \setminus \{0\})^n$ such that

$$\max_{i=1,\ldots,m} \lambda(G_{i,\Delta}(x^0)) \leq 0.$$

Let $q := (q_1, \ldots, q_n)$ be a nonzero vector in $\mathbb{R}^n$, with $\min_{j=1,\ldots,n} q_j < 0$, such that $\Delta = \Delta(q, \Gamma(\mathcal{G}))$. We define the monomial curve $\varphi\colon (0,1) \to \mathbb{R}^n$, $s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$, by setting

$$\varphi_j(s) := \begin{cases} x_j^0 s^{q_j} & \text{if } q_j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\varphi(s)\| \to +\infty$ as $s \to 0^+$. Moreover, we can write

$$\rho(\varphi(s)) = \rho_\Delta(x^0) s^\ell + \text{ higher order terms in } s,$$

where $\ell := \ell(q, \Gamma(\mathcal{G}))$.

On the other hand, for $i = 1, \ldots, m$, the functions

$$(0,1) \to \mathbb{R}^n, \quad s \mapsto \lambda(G_i(\varphi(s)),$$

are semialgebraic. By Monotonicity Lemma (see, for example, [1, 2]), these functions are C^1, and either constant or strictly monotone, for $0 < s \ll 1$. Therefore, there exists an index $i^* \in \{1, \ldots, m\}$ such that

$$\lambda(G_{i^*}(\varphi(s))) = \max_{i=1,\ldots,m} \lambda(G_i(\varphi(s))) \quad \text{for all} \quad 0 < s \ll 1.$$

Furthermore, if we write $G_i(x) = \sum_\alpha A_{i,\alpha} x^\alpha$, then we deduce successively

$$\begin{aligned}
\lambda(G_{i^*}(\varphi(s))) &= \max_{\|v\|=1} \langle G_{i^*}(\varphi(s))v, v\rangle \\
&= \max_{\|v\|=1} \left(\sum_\alpha \langle A_{i^*,\alpha} v, v\rangle (\varphi(s))^\alpha \right) \\
&= \max_{\|v\|=1} \left[\left(\sum_{\alpha\in\Delta} \langle A_{i^*,\alpha} v, v\rangle (x^0)^\alpha \right) s^\ell + \text{ higher order terms in } s \right] \\
&\leq \lambda(G_{i^*,\Delta}(x^0)) s^\ell + \text{ higher order terms in } s.
\end{aligned}$$

Therefore, by the assumption (ii), we get

$$0 \ < \ c\rho_\Delta(x^0) \ \leq \ \lambda(G_{i^*,\Delta}(x^0)) \ \leq \ \max_{i=1,\ldots,m} \lambda(G_{i,\Delta}(x^0)) \ \leq \ 0,$$

which is impossible. □

For any matrix $A \in \mathbb{R}^{d\times d}$, denote by $\mathfrak{r}(A)$ the *spectral radius* of A, that is,

$$\mathfrak{r}(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the set of all eigenvalues of A. The following is an immediate application of Theorem 2.1.

Corollary 2.1. *Let $\mathcal{G} := \{G_1, \ldots, G_m\} \subset \mathrm{Sym}_d(\mathbb{R}[x])$ be nondegenerate. Then there exist positive numbers c_1, c_2 and R such that*

$$c_1\rho(x) \ \leq \ \max_{i=1,\ldots,m} \mathfrak{r}(G_i(x)) \ \leq \ c_2\rho(x) \quad \textit{for all} \quad \|x\| \geq R.$$

Proof. By Lemma 2.1 and Theorem 2.1, there exist positive numbers c_1, R such that

$$c_1\rho(x) \ \leq \ \max_{i=1,\ldots,m} \lambda(G_i(x)) \quad \text{for all} \quad \|x\| \geq R.$$

Since $\lambda(G_i(x)) \leq \mathfrak{r}(G_i(x))$ for every $i = 1, 2, \ldots, m$, we get the first required inequality in the statement.

On the other hand, similarly to the proof of Lemma 2.1, we can find a constant $c_2 > 0$ such that

$$\max_{i=1,\ldots,m} \mathfrak{r}(G_i(x)) \ \leq \ c_2\rho(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

The proof is complete. □

3. Matrix polynomials having all eigenvalues bounded on a nondegenerate semialgebraic set

For a set $K \subset \mathbb{R}^n$ we will denote by $\mathcal{A}(K)$ the set of matrix polynomials $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ such that all eigenvalues of F are bounded on K, i.e.,

$$\mathcal{A}(K) \quad = \quad \{F \in \mathrm{Sym}_d(\mathbb{R}[x]) \mid \exists M > 0,\ M \cdot I_d \pm F(x) \succeq 0\ \forall x \in K\}.$$

A natural question is: *how to check whether a symmetric matrix polynomial is contained in* $\mathcal{A}(K)$? This question has a trivial answer when K is compact. The case of noncompact sets remains mainly unsolved. In this section we present a solution for semialgebraic sets defined by matrix polynomials that are nondegenerate.

To start with, notice that $\mathcal{A}(K)$ is not closed under multiplication because products of symmetric matrices are not symmetric in general. Furthermore, the following observations are clear.

Property 3.1. *Let K, L be subsets of $\mathbb{R}^n$. The following statements hold:*

(i) *If $K \subset L$ then $\mathcal{A}(L) \subset \mathcal{A}(K)$.*
(ii) $\mathcal{A}(K) = \mathcal{A}(\overline{K})$.
(iii) $\mathcal{A}(K \cup L) = \mathcal{A}(K) \cap \mathcal{A}(L)$.
(iv) $\mathcal{A}(K) = \mathrm{Sym}_d(\mathbb{R}[x])$ *provided that K is bounded.*
(v) *Every root in $\mathrm{Sym}_d(\mathbb{R}[x])$ of a monic polynomial with coefficients in $\mathcal{A}(K)$ belongs to $\mathcal{A}(K)$.*

Here and in the following, $\overline{K}$ stands for the closure of K.

Proof. For (v), let $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ be a root of a monic polynomial with coefficients in $\mathcal{A}(K)$; that is,

$$F^N + \sum_{i=1}^{N} A_i F^{N-i} = 0$$

for some matrices $A_i \in \mathcal{A}(K)$. Let $c_i := \sup_{x \in K} \|A_i(x)\| < +\infty$. For any $x \in K$, we have

$$\begin{aligned} \|F^N(x)\| \quad &= \quad \left\| \sum_{i=1}^{N} A_i(x) F^{N-i}(x) \right\| \ \leq \ \sum_{i=1}^{N} \|A_i(x)\| \|F(x)\|^{N-i} \\ &\leq \ \sum_{i=1}^{N} c_i (\max\{\|F(x)\|, 1\})^{N-i} \ \leq \ \sum_{i=1}^{N} c_i (\max\{\|F(x)\|, 1\})^{N-1}. \end{aligned}$$

Hence,

$$\|F(x)\| \ \leq \ \sum_{i=1}^{N} c_i + 1 \ < \ \infty,$$

which implies that $F \in \mathcal{A}(K)$. $\square$

The next property states that a polynomial automorphism induces an isomorphism of appropriate spaces.

Property 3.2. *If Φ is a polynomial automorphism of $\mathbb{R}^n$ then for any set $K \subset \mathbb{R}^n$, we have*

$$\mathcal{A}(K) = \Phi_* \mathcal{A}(\Phi(K)),$$

where Φ_* is the isomorphism $F \mapsto F \circ \Phi$ of $\mathrm{Sym}_d(\mathbb{R}[x])$.

Proof. It suffices to note that a matrix polynomial $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ belongs to $\mathcal{A}(\Phi(K))$ if and only if $F \circ \Phi$ belongs to $\mathcal{A}(K)$. □

From now on, we let $\mathcal{G} := \{G_1, \ldots, G_m\} \subset \mathrm{Sym}_d(\mathbb{R}[x])$ and consider the set

$$K \ := \ \{x \in \mathbb{R}^n \mid \Lambda_i - G_i(x) \succeq 0 \quad \text{for} \quad i = 1, \ldots, m\},$$

where $\Lambda_i \in \mathrm{Sym}_d(\mathbb{R}), i = 1, \ldots, m$. Since $\Lambda_i - G_i(x) \succeq 0$ can be presented as a system of polynomial inequalities, K is a basic closed semialgebraic set. The crucial role in our considerations is played by the following corollary, which follows from Theorem 2.1.

Corollary 3.1. *If the system $\mathcal{G} := \{G_1, \ldots, G_m\}$ is nondegenerate, then there exists a constant $r > 0$ such that*

$$K \ \subset \ \{x \in \mathbb{R}^n \mid |x^\alpha| \le r \ \ \textit{for} \ \ \alpha \in V(\mathcal{G})\}.$$

Proof. By Theorem 2.1, there exist constants $c > 0$ and $R > 0$ such that

$$c\rho(x) \le \max_{i=1,\ldots,m} \lambda(G_i(x)) \quad \text{for all} \quad \|x\| \ge R.$$

Hence, we have for all $x \in K$ with $\|x\| \ge R$,

$$\rho(x) \le \frac{1}{c} \max_{i=1,\ldots,m} \lambda(G_i(x)) \le \frac{1}{c} \max_{i=1,\ldots,m} \lambda(\Lambda_i).$$

Let

$$r := \max\{\max_{\|x\| \le R} \rho(x), \frac{1}{c} \max_{i=1,\ldots,m} \lambda(\Lambda_i)\} > 0.$$

Then it is clear that

$$K \subset \{x \in \mathbb{R}^n \mid \rho(x) \le r\} \subset \{x \in \mathbb{R}^n \mid |x^\alpha| \le r \ \ \text{for} \ \ \alpha \in V(\mathcal{G})\},$$

which completes the proof of the corollary. □

In the rest of the paper we will make the following assumptions: for all $i = 1, \ldots, m$, it holds that

(H1) $G_i(0) = 0$ (this can be achieved by replacing G_i by $G_i - G_i(0)$); and
(H2) the matrices $\Lambda_i \in \mathrm{Sym}_d(\mathbb{R})$ are positive definite.

Let $\mathcal{C}(\mathcal{G})$ be the cone with vertex at the origin generated by the Newton polyhedron $\Gamma(\mathcal{G})$ of the system $\mathcal{G}$, i.e.,

$$\mathcal{C}(\mathcal{G}) := \left\{ \sum_{\alpha \in V(\mathcal{G})} t_\alpha \alpha \mid t_\alpha \geq 0 \right\}.$$

The next result says that the space of symmetric matrix polynomials, whose eigenvalues are bounded on K, can be described in terms of the Newton polyhedron corresponding to the matrix polynomials used to define K.

Theorem 3.1. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate. Let $F \in \mathrm{Sym}_d(\mathbb{R}[x])$, then the following statements are equivalent.*

(i) $\mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})$.

(ii) $F \in \mathcal{A}(K)$, *i.e., there exists a constant $M > 0$ such that*

$$M \cdot I_d \pm F(x) \ \succeq \ 0 \quad \textit{for all} \quad x \in K.$$

Proof. We first remark from Lemma 2.1 and Theorem 2.1 that there exist positive constants c_1, c_2, and R such that

$$c_1 \rho(x) \leq \max_{i=1,\ldots,m} \lambda(G_i(x)) \leq c_2 \rho(x) \quad \text{for all} \quad \|x\| \geq R. \tag{3}$$

(i) $\Rightarrow$ (ii) By definition, we have for all $x \in K$ and all $i \in \{1, \ldots, m\}$,

$$\lambda(G_i(x)) \leq \lambda_i,$$

where $\lambda_i := \lambda(\Lambda_i)$ is the largest eigenvalue of the matrix Λ_i. It follows from (3) that

$$\sum_{\alpha \in V(\mathcal{G})} |x^\alpha| = \rho(x) \leq \max_{i=1,\ldots,m} \lambda_i / c_1 \quad \text{for all} \quad x \in K, \|x\| \geq R.$$

Since $\mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})$, for each $\beta \in \mathrm{supp}(F)$ we can find constants $t_\alpha \geq 0$ such that $\beta = \sum_{\alpha \in V(\mathcal{G})} t_\alpha \alpha$. Then

$$|x^\beta| \ = \ |x^{\sum_{\alpha \in V(\mathcal{G})} t_\alpha \alpha}| = \prod_{\alpha \in V(\mathcal{G})} |x^\alpha|^{t_\alpha} \leq \prod_{\alpha \in V(\mathcal{G})} \left(\max_{i=1,\ldots,m} \lambda_i / c_1 \right)^{t_\alpha} \quad \text{for all} \quad x \in K, \|x\| \geq R.$$

It follows that

$$\sup_{\beta \in \mathrm{supp}(F)} |x^\beta| \ \leq \ c_3 \quad \text{for all} \quad x \in K, \|x\| \geq R$$

for some $c_3 > 0$.

Now let us write $F(x) = \sum_\beta F_\beta x^\beta$, for some symmetric matrices $F_\beta \in \mathrm{Sym}_d(\mathbb{R})$. Take any vector $v \in \mathbb{R}^n$ with $\|v\| = 1$. We have for all $x \in K$ with $\|x\| \geq R$,

$$\begin{aligned} |\langle F(x)v, v\rangle| &= \left|\sum_\beta \langle F_\beta v, v\rangle x^\beta\right| \\ &\leq \sum_\beta |\langle F_\beta v, v\rangle||x^\beta| \\ &\leq \left(\sum_\beta |\langle F_\beta v, v\rangle|\right) c_3. \end{aligned}$$

Let

$$M := \max_{\|v\|=1} \left(\sum_\beta |\langle F_\beta v, v\rangle|\right) c_3.$$

Then we have

$$|\langle F(x)v, v\rangle| \leq M \quad \text{for all} \quad x \in K, \|x\| \geq R \ \text{ and } \ v \in \mathbb{R}^n, \|v\| = 1.$$

This, together with the compactness of the ball $\{x \in \mathbb{R}^n \mid \|x\| \leq R\}$, proves (ii).

(ii) $\Rightarrow$ (i) By contrary, suppose that there exists $\beta \in \mathrm{supp}(F) \setminus \mathcal{C}(\mathcal{G})$. By the separation theorem, there exists a nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$ such that

$$\langle q, \alpha\rangle \geq 0 > \langle q, \beta\rangle \quad \text{for all} \quad \alpha \in \mathcal{C}(\mathcal{G}). \tag{4}$$

For simplicity, we let

$$\begin{aligned} \Delta &:= \Delta(q, \Gamma(F)), & \ell &:= \ell(q, \Gamma(F)), \\ \Delta' &:= \Delta(q, \Gamma(\mathcal{G})), & \ell' &:= \ell(q, \Gamma(\mathcal{G})). \end{aligned}$$

Then, by (4), we have

$$\ell < 0 \leq \ell' \quad \text{and} \quad \min_{j=1,\ldots,n} q_j < 0.$$

In particular, by definition, $\Delta' \in \Gamma_\infty(\mathcal{G})$.

On the other hand, the assumption that the matrices $\Lambda_i, i = 1, \ldots, m$, are positive definite gives us a real number $\lambda_* > 0$ such that

$$\Lambda_i - \lambda_* \cdot I_d \succ 0 \quad \text{for all} \quad i = 1, \ldots, m.$$

Furthermore, since $G_i(0) = 0, i = 1, \ldots, m$, we have that $0 \notin \Gamma(\mathcal{G})$ and so $\rho_{\Delta'}(0) = 0$. Therefore, we can find a point $x^0 \in (\mathbb{R} \setminus \{0\})^n$ satisfying the following conditions

$$\rho_{\Delta'}(x^0) \ll \frac{1}{c_2}\lambda_* \quad \text{and} \quad F_\Delta(x^0) \not\equiv 0. \tag{5}$$

Consider the monomial curve

$$\phi\colon (0,1) \to \mathbb{R}^n, \quad s \mapsto (x_1^0 s^{q_1}, \ldots, x_n^0 s^{q_n}).$$

We have $\|\varphi(s)\| \to +\infty$ as $s \to 0^+$ because $\min_{j=1,\ldots,n} q_j < 0$. Furthermore, a simple calculation shows that

$$\rho(\phi(s)) = \rho_{\Delta'}(x^0)s^{\ell'} + \text{ higher order terms in } s.$$

Since $\ell' \geq 0$, it follows from the first inequality of (5) that

$$\rho(\phi(s)) < \frac{1}{c_2}\lambda_* \quad \text{for all} \quad |s| \ll 1.$$

Thanks to (3), hence

$$\max_{i=1,\ldots,m} \lambda(G_i(\phi(s))) < \lambda_* \quad \text{for all} \quad |s| \ll 1.$$

Therefore $\phi(s) \in K$ for $|s| \ll 1$.

On the other hand, the second condition of (5) gives us a vector $v \in \mathbb{R}^n$, $\|v\| = 1$, such that $\langle F_\Delta(x^0)v, v\rangle \neq 0$. Then a simple calculation shows that

$$\langle F(\phi(s))v, v\rangle = \langle F_\Delta(x^0)v, v\rangle s^{\ell} + \text{ higher order terms in } s.$$

It follows from the facts $\ell < 0$ and $\langle F_\Delta(x^0)v, v\rangle \neq 0$ that

$$\lim_{s\to 0^+} \langle F(\phi(s))v, v\rangle = \infty,$$

which contradicts the assumption that $F \in \mathcal{A}(K)$. □

Corollary 3.2. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate. Then the system $\mathcal{G}$ is convenient if and only if K is a compact set.*

Proof. Indeed, assume that the system $\mathcal{G}$ is convenient, i.e., the Newton polyhedron $\Gamma(\mathcal{G})$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$. By definition, this is equivalent to the fact that $\mathcal{C}(\mathcal{G}) = \mathbb{R}^n_{\geq 0}$. Let $F(x) := (\sum_{j=1}^n x_j^2) \cdot I_d \in \mathrm{Sym}_d(\mathbb{R}[x])$. We have $\mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})$. By Theorem 3.1, $F \in \mathcal{A}(K)$. Consequently, K is a compact set.

Conversely, assume that the set K is compact but the system $\mathcal{G}$ is not convenient. By definition, there exists a vector $\beta \in \mathbb{R}^n_{\geq 0} \setminus \mathcal{C}(\mathcal{G})$. Then, by a similar argument as in the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 2.1, we can construct a curve $\phi\colon (0,1) \to \mathbb{R}^n$ such that $\|\varphi(s)\| \to +\infty$ as $s \to 0^+$ and $\phi(s) \in K$ for all s sufficiently small, which contradicts the compactness of K. □

As a direct consequence of Theorem 3.1, we get the following stability of $\mathcal{A}(K)$; see also [7, Lemma 3.1] and [10, Theorem 3.1].

Corollary 3.3. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate. Then the space of matrix polynomials which are bounded on the semialgebraic set:*

$$\{x \in \mathbb{R}^n \mid \Lambda_i - G_i(x) \succeq 0 \quad \textit{for} \quad i = 1, \ldots, m\}$$

is independent on the positive define matrices Λ_i for $i = 1, \ldots, m$.

We say that the set $\mathcal{A}(K) \subset \mathrm{Sym}_d(\mathbb{R}[x])$ is *finitely generated* if one can find a finite set $\{\zeta_1, \ldots, \zeta_k\} \subset \mathcal{A}(K)$ such that for any matrix polynomial $F \in \mathcal{A}(K)$ there exists a matrix polynomial $G \in \mathrm{Sym}_d(\mathbb{R}[x])$ such that $F = G(\zeta_1, \ldots, \zeta_k)$.

Now we can state a theorem on the structure of sets of matrix polynomials, whose eigenvalues are bounded on nondegenerate semialgebraic sets in $\mathbb{R}^n$. For related results, see [10, Theorem 2.5], [15, Therorem 2.1], and [21].

Theorem 3.2. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate. Then the space of matrix polynomials which are bounded on the semialgebraic set:*

$$\{x \in \mathbb{R}^n \mid \Lambda_i - G_i(x) \succeq 0 \quad \textit{for} \quad i = 1, \ldots, m\}$$

is generated by the matrix monomials $x^\alpha \cdot I_d$ for $\alpha \in \mathrm{co}(\{0\} \cup \mathrm{supp}(\mathcal{G}))$, and hence is finitely generated.

Proof. By Theorem 3.1, it suffices to show that there exists a finite set $L \subset \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n$ satisfying the condition: for any $\beta \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n$, there exist some constants $t_\alpha \in \mathbb{Z}_{\geq 0}$ for $\alpha \in L$ such that $\beta = \sum_{\alpha \in L} t_\alpha \alpha$.

In fact, let $k := \dim \mathcal{C}(\mathcal{G}) \leq n$. Then there exist cones $\mathcal{C}_1, \ldots, \mathcal{C}_p$ and vectors $\omega_{ij} \in \mathbb{Z}^n_{\geq 0}$ for $i = 1, \ldots, p, j = 1, \ldots, k$, such that

- $\mathcal{C}(\mathcal{G}) = \cup_{i=1}^p \mathcal{C}_i$, $\dim \mathcal{C}_i = k$ for $i = 1, \ldots, p$;
- for each index i, the vectors $\omega_{i1}, \ldots, \omega_{ik}$ are linearly independent and the greatest common divisor of the coordinates of these vectors is equal to 1; and
- each cone $\mathcal{C}_i$ is generated by the vectors $\omega_{i1}, \ldots, \omega_{ik}$.

Next we define the k-dimensional parallelepiped P_i to be the set of all α in $\mathbb{R}^n$ such that

$$\alpha = c_1\omega_{i1} + \cdots + c_k\omega_{ik}$$

for some scalars c_j with $0 \leq c_j \leq 1$. Then it is easy to see, by a linear isomorphism, that $\mathcal{C}_i$ and P_i, respectively, are identified to the cone $\mathbb{R}^k_{\geq 0}$ and the cube $[0, 1]^k$. Note that

$$\mathbb{R}^k_{\geq 0} = \{\alpha + (t_1, \ldots, t_k) \mid \alpha \in [0, 1]^k \text{ and } t_j \in \mathbb{Z}_{\geq 0}\}.$$

Consequently, we have

$$\mathcal{C}_i = \{\alpha + t_1\omega_{i1} + \cdots + t_k\omega_{ik} \mid \alpha \in P_i \text{ and } t_j \in \mathbb{Z}_{\geq 0}\}.$$

Hence, if we put $L_i := P_i \cap \mathbb{Z}^n$, then

$$\mathcal{C}_i \cap \mathbb{Z}^n = \{\alpha + t_1\omega_{i1} + \cdots + t_k\omega_{ik} \mid \alpha \in L_i \text{ and } t_j \in \mathbb{Z}_{\geq 0}\}.$$

Clearly, the set $L := \cup_{i=1}^p L_i$ has the desired properties. □

Example 3.1. Let $n := 2$ and $\mathcal{G} := \{\pm x \cdot I_d, \pm xy \cdot I_d\} \subset \mathrm{Sym}_d(\mathbb{R}[x, y])$ and consider the corresponding semialgebraic set

$$K := \{(x, y) \in \mathbb{R}^2 \mid I_d \pm x \cdot I_d \succeq 0 \quad \text{and} \quad I_d \pm xy \cdot I_d \succeq 0\}.$$

By Theorem 3.2, it is easy to see that $\mathcal{A}(K)$ is generated by the matrix monomials $x \cdot I_d$ and $xy \cdot I_d$, i.e., $\mathcal{A}(K) = \mathrm{Sym}_d(\mathbb{R}[x, xy])$. This example is inspired by [21, Example 3.10].

Example 3.2. Let $d := 1$ and $n := 2$. Take the Motzkin polynomial $\mathfrak{m}(x, y) := 1+x^2y^2(x^2+y^2-3) \in \mathbb{R}[x, y]$ and consider the semialgebraic set

$$K_c := \{(x, y) \in \mathbb{R}^2 \mid c - \mathfrak{m}(x, y) \geq 0\},$$

where c is a real parameter. It is easy to check that the system $\mathcal{G} := \{\mathfrak{m}-1\}$ is nondegenerate and $\mathfrak{m}(0, 0) - 1 = 0$. If $c < 1$, then K_c is a compact set, and hence $\mathcal{A}(K_c) = \mathbb{R}[x, y]$. If $c = 1$, then the algebra $\mathcal{A}(K_c)$ does not admit a finite set of generators (see [10, Example 3.5] and [15, Example 3.2]). On the other hand, if $c > 1$ then we deduce from Theorem 3.2 that $\mathcal{A}(K_c) = \mathbb{R}[xy, x^2y, xy^2]$ is finitely generated.

Example 3.3. Let $d = n = 2$ and consider $K_c := \{(x, y) \mid \Lambda_c - G(x, y) \succeq 0\}$, where c is a real parameter,

$$G(x, y) \;=\; \begin{pmatrix} x^2y^2 & 0 \\ 0 & x^2 \end{pmatrix} \quad \text{and} \quad \Lambda_c \;:=\; \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

By definition, the cone $\mathcal{C}(\mathcal{G})$ (with $\mathcal{G} := \{G\}$) is the convex cone generated by the vectors $(1, 0)$ and $(1, 1)$. Hence,

$$\{F \in \mathrm{Sym}_2(\mathbb{R}[x, y]) \mid \mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})\} \;=\; \mathrm{Sym}_2(\mathbb{R}[x, xy]).$$

The matrix Λ_c has eigenvalues 1 and c. If $c < 0, K_c = \emptyset$ and so $\mathcal{A}(K_c) = \mathrm{Sym}_2(\mathbb{R}[x, y])$. If $c > 0$, we can use Theorem 4.1 to get $\mathcal{A}(K_c) = \mathrm{Sym}_2(\mathbb{R}[x, xy])$. If $c = 0$, the set K_c is the y-axis, hence $\mathcal{A}(K_c) = \mathrm{Sym}_2(\mathbb{R}[x, y]x + \mathbb{R})$.

Finally, we can see that $\mathcal{A}(K)$ is "absorbing" in the following sense.

Corollary 3.4. *Suppose that $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate. Denoted by $\mathcal{V}$ the linear subspace of $\mathbb{R}^n$ spanned by the cone $\mathcal{C}(\mathcal{G})$. Then for any matrix polynomial $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ with $\mathrm{supp}(F) \subset \mathcal{V}$, there exists a vector $\beta \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$ such that*

$$x^\beta \cdot F \in \mathcal{A}(K).$$

Proof. Let $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ be a matrix polynomial with $\mathrm{supp}(F) \subset \mathcal{V}$. Assume that we have proved the claim that: for each $\alpha \in \mathcal{V} \cap \mathbb{Z}^n_{\geq 0}$, there exists $\beta(\alpha) \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$ such that

$$\alpha + \beta(\alpha) \in \mathcal{C}(\mathcal{G}).$$

This, of course, implies that if we let $\beta := \sum_{\alpha\in\mathrm{supp}(F)} \beta(\alpha) \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$, then $\alpha + \beta \in \mathcal{C}(\mathcal{G})$ for all $\alpha \in \mathrm{supp}(F)$. By Theorem 3.1, $x^\beta \cdot F \in \mathcal{A}(K)$.

So we are left with proving the claim. To this end, let α be an arbitrary vector in $\mathcal{V} \cap \mathbb{Z}^n_{\geq 0}$. Then there exist numbers $\mu_i \in \mathbb{R}$ and vectors $\alpha^i \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$ for $i = 1, \ldots, k$, such that

$$\alpha = \mu_1\alpha^1 + \cdots + \mu_k\alpha^k.$$

Take integers $\nu_i > 0$ for $i = 1, \ldots, k$, and let

$$\gamma := \nu_1\alpha^1 + \cdots + \nu_k\alpha^k \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}.$$

Choose $t \in \mathbb{Z}_{\geq 0}$ large enough such that $\mu_i + t\nu_i \geq 0$ for all $i = 1, \ldots k$, and let $\beta(\alpha) := t\gamma$. Then the vectors $\beta(\alpha)$ and $\alpha + \beta(\alpha)$ belong to $\mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$. □

4. Positivstellensätze for matrix polynomials on nondegenerate semialgebraic sets

In this section, we establish a matrix version of Schmüdgen's Positivstellensätz for matrix polynomials whose eigenvalues are "strictly" positive and bounded on K. To do this, recall that

$$K := \{x \in \mathbb{R}^n \mid \Lambda_i - G_i(x) \succeq 0 \quad \text{for} \quad i = 1, \ldots, m\}.$$

Definition 4.1. We say that the cone $\mathcal{C}(\mathcal{G})$ is *unimodular* if there exist n vectors $\alpha^1, \ldots, \alpha^n \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$ such that the following two conditions hold:

(a) $\det A = 1$, where $A := [\alpha^1, \ldots, \alpha^n]$; and
(b) $\mathcal{C}(\mathcal{G})$ is generated by $\alpha^1, \ldots, \alpha^n$, i.e.,

$$\mathcal{C}(\mathcal{G}) = \left\{ \sum_{j=1}^n t_j\alpha^j \mid t_j \geq 0 \right\}.$$

Suppose that the cone $\mathcal{C}(\mathcal{G})$ is unimodular. Then it is straightforward to show that for any $\beta \in \mathcal{C}(\mathcal{G}) \cap (\mathbb{Z}_{\geq 0})^n$, there are nonnegative integers $t_1, \ldots, t_n$ such that $\beta = \sum_{j=1}^n t_j\alpha^j$. As a consequence,

$$\{F \in \mathrm{Sym}_d(\mathbb{R}[x]) \mid \mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})\} \quad = \quad \mathrm{Sym}_d(\mathbb{R}[x^{\alpha^1}, \ldots, x^{\alpha^n}]).$$

Let us make a change of variables $u := x^A$, that is $u_j = x^{\alpha^j}, j = 1, \ldots, n$, and $u = (u_1, \ldots, u_n)$. Then for any $\beta \in \mathcal{C}(\mathcal{G}) \cap \mathbb{Z}^n_{\geq 0}$, there is a representation

$$\beta = \sum_{j=1}^n t_j\alpha^j,$$

for some nonnegative integers $t_j, j = 1, \ldots, n$; and so

$$x^\beta = \prod_{j=1}^{n} x^{t_j \alpha^j} = \prod_{j=1}^{n} (x^{\alpha^j})^{t_j} = \prod_{j=1}^{n} (u_j)^{t_j}.$$

Consequently, for each $i = 1, \ldots, m$, we can define a matrix polynomial $\widetilde{G}_i \in \mathrm{Sym}_d(\mathbb{R}[u])$ with the property that $\widetilde{G}_i(x^A) = G_i(x)$ for all $x \in \mathbb{R}^n$. Let

$$\widetilde{K} \quad := \quad \{u \in \mathbb{R}^n \mid \Lambda_i - \widetilde{G}_i(u) \succeq 0, \ i = 1, \ldots, m\}.$$

Lemma 4.1. *Define the monomial mapping* $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$ *by* $\Phi(x) = x^A$. *We have*

(i) $\overline{\Phi(K)} \subset \widetilde{K}$;

(ii) *The restriction* $\Phi \colon K \cap (\mathbb{R} \setminus \{0\})^n \longrightarrow \widetilde{K} \cap (\mathbb{R} \setminus \{0\})^n$ *is one-to-one and onto. In particular,*

$$\Phi(K \cap (\mathbb{R} \setminus \{0\})^n) = \Phi(K) \cap (\mathbb{R} \setminus \{0\})^n = \widetilde{K} \cap (\mathbb{R} \setminus \{0\})^n.$$

Proof. Suppose that $x \in K$. Then $\widetilde{G}_i(\Phi(x)) = G_i(x)$, which implies (i).

If $u \in (\mathbb{R} \setminus \{0\})^n$ then $x := u^{A^{-1}}$ is again an element in $(\mathbb{R} \setminus \{0\})^n$ and $\Phi(x) = u$. Then (ii) follows easily. □

Recall that the *preordering* generated by the matrix polynomials $\Lambda_1 - G_1, \ldots, \Lambda_m - G_m$, denoted by $\mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}$, is defined to be the smallest quadratic module in $\mathrm{Sym}_d(\mathbb{R}[x])$ which contains $\Lambda_1 - G_1, \ldots, \Lambda_m - G_m$ and whose intersection with the set $\mathbb{R}[x] \cdot I_d$ is closed under multiplication. By definition, every matrix polynomial in $\mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}$ is positive semidefinite on K. The converse does not hold in general. On the other hand we have the following statement.

Theorem 4.1. *Assume that the system* $\mathcal{G} := \{G_1, \ldots, G_m\}$ *is nondegenerate, the cone* $\mathcal{C}(\mathcal{G})$ *is unimodular and that* $\overline{\Phi(K)} = \widetilde{K}$. *Let* $F \in \mathcal{A}(K)$ *be such that*

$$\inf_{x \in K} \lambda_{\min}(F(x)) > 0,$$

where $\lambda_{\min}(F(x))$ *is the smallest eigenvalue of* $F(x)$. *Then*

$$F \in \mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}.$$

Proof. Since $\mathcal{G}$ is nondegenerate, by Corollary 3.1, there exists a constant $r > 0$ such that

$$K \quad \subset \quad \{x \in \mathbb{R}^n \mid |x^\alpha| \le r \ \text{ for } \ \alpha \in V(\mathcal{G})\}.$$

This, together with the assumptions that $\mathcal{C}(\mathcal{G})$ is unimodular and $\overline{\Phi(K)} = \widetilde{K}$, implies easily that

$$\widetilde{K} := \{u \in \mathbb{R}^n \mid \Lambda_i - \widetilde{G}_i(u) \succeq 0, \ i = 1, \ldots, m\} \subset \{u \in \mathbb{R}^n \mid |u_j| \le \widetilde{r}, \ j = 1, \ldots, n\}$$

for some $\widetilde{r} > 0$. Consequently, the set $\widetilde{K}$ is compact.

On the other hand, by Theorem 3.1, we have $\mathrm{supp}(F) \subset \mathcal{C}(\mathcal{G})$ because of $F \in \mathcal{A}(K)$. Since $\mathcal{C}(\mathcal{G})$ is unimodular, we can define a matrix polynomial $\widetilde{F} \in \mathrm{Sym}_d(\mathbb{R}[u])$ with the property that $\widetilde{F} \circ \Phi(x) = F(x)$ for all $x \in \mathbb{R}^n$. We will show that $\widetilde{F} \succ 0$ on $\widetilde{K}$. In fact, it is clear that $\widetilde{F} \succeq 0$ on $\widetilde{K}$ because we have that $F \succ 0$ on K and $\overline{\Phi(K)} = \widetilde{K}$. Assume that there exists a point $\widetilde{a} \in \widetilde{K}$ such that the smallest eigenvalue of $\widetilde{F}(\widetilde{a})$ is equal to zero. There are two cases to be considered.

Case 1: There exists a point $a \in K$ such that $\widetilde{a} = a^A$. By definition, then

$$\lambda_{\min}(F(a)) = \lambda_{\min}(\widetilde{F}(\widetilde{a})) = 0,$$

which contradicts the assumption.

Case 2: There is no point $a \in K$ such that $\widetilde{a} = a^A$. Then by the hypothesis and Lemma 4.1, there exists a sequence $\{a^k\}_{k \geq 1} \subset K \cap (\mathbb{R} \setminus \{0\})^n$ such that $\widetilde{a}^k := \Phi(a^k) = (a^k)^A \in \widetilde{K}$ for all $k \geq 1$ and $\lim_{k \to \infty} \widetilde{a}^k = \widetilde{a}$. Hence,

$$\lim_{k \to \infty} \lambda_{\min}(F(a^k)) = \lim_{k \to \infty} \lambda_{\min}(\widetilde{F}((a^k)^A)) = \lim_{k \to \infty} \lambda_{\min}(\widetilde{F}(\widetilde{a}^k)) = \lambda_{\min}(\widetilde{F}(\widetilde{a})) = 0,$$

which contradicts the assumption again.

Therefore, $\widetilde{F} \succ 0$ on $\widetilde{K}$. By [5, Theorem 6], we get

$$\widetilde{F} \ \in \ \mathcal{T}_{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m}.$$

Note from [4, Lemma 2] that

$$\mathcal{T}_{\{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m\}} \ = \ \mathcal{M}_{\{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m\} \cup (\prod(\{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m\})' \cdot I_d)},$$

where $\prod(\{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m\})'$ is the set of all finite products of elements from

$$(\{\Lambda_1 - \widetilde{G}_1, \ldots, \Lambda_m - \widetilde{G}_m\})' \ = \ \{\widetilde{p}^T(\Lambda_i - \widetilde{G_i})\widetilde{p} \mid i = 1, \ldots, m, \text{and } \widetilde{p} \in (\mathbb{R}[u])^d\}.$$

By definition, therefore $F \in \mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}$. □

Define the function $\theta \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\theta(u) := \min_{i=1,\ldots,m} \lambda_{\min}(\Lambda_i - \widetilde{G}_i(u)),$$

where $\lambda_{\min}(\Lambda_i - \widetilde{G}_i(u))$ is the smallest eigenvalue of the matrix $\Lambda_i - \widetilde{G}_i(u)$. Then it is easy to see that the function θ is continuous and satisfies

$$\widetilde{K} = \{u \in \mathbb{R}^n \mid \theta(u) \geq 0\}.$$

Corollary 4.1. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate, the cone $\mathcal{C}(\mathcal{G})$ is unimodular and 0 is not a local maximum value of θ. Let $F \in \mathcal{A}(K)$ be such that*

$$\inf_{x \in K} \lambda_{\min}(F(x)) > 0,$$

where $\lambda_{\min}(F(x))$ *is the smallest eigenvalue of* $F(x)$. *Then*

$$F \in \mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}.$$

Proof. By Theorem 4.1, it suffices to show that $\widetilde{K}$ is equal to the closure of $\widetilde{K} \cap (\mathbb{R} \setminus \{0\})^n$. To this end, let u be an element in $\widetilde{K}$ which does not belong to $(\mathbb{R} \setminus \{0\})^n$. Then $\theta(u) \geq 0$.

We first assume that $\theta(u) > 0$. By continuity, there exists a real number $\eta > 0$ such that

$$\theta(v) > 0 \quad \text{for all} \quad v \in \mathbb{B}(u, \eta),$$

where $\mathbb{B}(u, \eta)$ denotes the open ball centered at u with radius η. In particular, $\mathbb{B}(u, \eta) \subset \widetilde{K}$. Note that $\mathbb{B}(u, \epsilon) \cap (\mathbb{R} \setminus \{0\})^n \neq \emptyset$ for all $\epsilon \in (0, \eta)$. Therefore, $u \in \overline{\widetilde{K} \cap (\mathbb{R} \setminus \{0\})^n}$.

We now assume that $\theta(u) = 0$. Since 0 is not a local maximum value of θ, we can find a sequence $\{u^k\} \subset \mathbb{R}^n$ tending to u as $k \to \infty$ such that $\theta(u^k) > 0$. By the previous argument, we know that u^k belongs to the closure of $\widetilde{K} \cap (\mathbb{R} \setminus \{0\})^n$, and so does u. □

Example 4.1. We illustrate here some examples where we can or cannot apply Corollary 4.1. Let $d = n = 2$ and c be a positive number, and consider the set

$$K_c := \{(x, y) \in \mathbb{R}^2 \mid cI_2 - G(x, y) \succeq 0\},$$

where

$$G(x, y) = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} x^8 y^4 + \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} x^8 + \begin{pmatrix} 0 & -1 \\ -1 & -3 \end{pmatrix} x^4 y^4 + \begin{pmatrix} 2 & 5 \\ 5 & 11 \end{pmatrix} x^2 y^2.$$

Then the support of G is $\{(8, 4), (8, 0), (4, 4), (2, 2)\}$. The cone $C(\mathcal{G})$ is the convex cone generated by $(1, 0), (1, 1)$ and is equal to $\{(\alpha + \beta, \beta) \mid \alpha \geq 0, \beta \geq 0\}$, where $\mathcal{G}$ is a singleton $\{G\}$. Hence, $C(\mathcal{G})$ is unimodular. The set $V(\mathcal{G})$ of vertices of the Newton polygon of G consists four points $\{(8, 4), (8, 0), (4, 4), (2, 2)\}$. Making change of variables $u = x, v = xy$, (i.e., $\Phi(x, y) = (x, xy)$), we have

$$\widetilde{K}_c := \{(u, v) \in \mathbb{R}^2 \mid cI_2 - \widetilde{G}(u, v) \succeq 0\},$$

where

$$\widetilde{G}(u, v) = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} u^4 v^4 + \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} u^8 + \begin{pmatrix} 0 & -1 \\ -1 & -3 \end{pmatrix} v^4 + \begin{pmatrix} 2 & 5 \\ 5 & 11 \end{pmatrix} v^2.$$

A direct calculation shows that $\widetilde{K}_c$ is a subset of $\{(u, v) \in \mathbb{R}^2 \mid u^8 \leq c, v^2 \leq c\}$ and hence, is compact. Furthermore,

$$\widetilde{K}_c = \overline{\Phi(K_c)} \cup \{(0, v) \mid \max\{v^4 - v^2, -v^4 + 3v^2\} \leq c\}.$$

If $c \neq 2$, $\widetilde{K}_c = \overline{\Phi(K_c)}$ and in this case, we can apply Corollary 4.1. However, if $c = 2$, then the function

$$\mathbb{R} \to \mathbb{R}, \quad v \mapsto \max\{v^4 - v^2, -v^4 + 3v^2\},$$

attains its local minimum at the points $v = 0, -\sqrt{2}, \sqrt{2}$ and its local minimal values are $0, 2$. Hence, 0 is a local maximal value of the function θ in Corollary 4.1, where

$$\theta(u, v) = 2 - \max\{u^4v^4 + u^8 + v^4 - v^2, u^4v^4 + u^8 - v^4 + 3v^2\}.$$

In addition, $\widetilde{K}_2$ contains and does not equal $\overline{\Phi(K_2)}$. Indeed, $(0, \sqrt{2})$ belongs to $\widetilde{K}_2$ while it does not lie in $\overline{\Phi(K_2)}$. Hence, we can not apply Corollary 4.1 in this case.

Example 4.2. Let K be a *logarithmic polyhedron* determined by

$$K := \{x \in \mathbb{R}^n \mid (r_1^2 - x^{2\alpha^1}) \cdot I_d \succeq 0, \ldots, (r_m^2 - x^{2\alpha^m}) \cdot I_d \succeq 0\}$$

where $r_i > 0$ and $\alpha^i \in \mathbb{Z}^n_{\geq 0}$ for $i = 1, \ldots, m$.

It is easy to see that the system $\mathcal{G} := \{x^{2\alpha^1} \cdot I_d, \ldots, x^{2\alpha^m} \cdot I_d\}$ is nondegenerate and its Newton polyhedron has even vertices. Suppose that $\mathcal{C}(\mathcal{G})$ is unimodular, $F \in \mathrm{Sym}_d(\mathbb{R}[x])$ is bounded on K, and that

$$\inf_{x \in K} \lambda_{\min}(F(x)) > 0.$$

By Theorem 4.1, F belongs to the preordering generated by the matrix polynomials $(r_1^2 - x^{2\alpha^1}) \cdot I_d, \ldots, (r_m^2 - x^{2\alpha^m}) \cdot I_d$ in $\mathrm{Sym}_d(\mathbb{R}[x])$. Moreover, by a similar argument as in the proof of [7, Theorem 2.2]), we can show that F belongs to the quadratic module generated by $(r_1^2 - x^{2\alpha^1}) \cdot I_d, \ldots, (r_m^2 - x^{2\alpha^m}) \cdot I_d$ in $\mathrm{Sym}_d(\mathbb{R}[x])$.

In the rest of this paper, for simplicity, we write $\mathcal{T}$ instead of $\mathcal{T}_{\{\Lambda_1 - G_1, \ldots, \Lambda_m - G_m\}}$-the preordering generated by $\Lambda_1 - G_1, \ldots, \Lambda_m - G_m$ in $\mathrm{Sym}_d(\mathbb{R}[x])$. Set

$$\begin{aligned}
\mathcal{T}^\vee &= \{\mathscr{L} \colon \mathrm{Sym}_d(\mathbb{R}[x]) \to \mathbb{R} \mid \mathscr{L} \text{ is linear}, \mathscr{L}(I_d) = 1, \mathscr{L}(\mathcal{T}) \geq 0\},\\
\mathcal{T}^{\vee\vee} &= \{F \in \mathrm{Sym}_d(\mathbb{R}[x]) \mid \mathscr{L}(F) \geq 0, \ \forall \mathscr{L} \in \mathcal{T}^\vee\},\\
\mathcal{T}^{Sat} &= \{F \in \mathrm{Sym}_d(\mathbb{R}[x]) \mid F(x) \succeq 0, \ \forall x \in K\}.
\end{aligned}$$

Clearly, $\mathcal{T} \subset \mathcal{T}^{\vee\vee}$. Furthermore, we have the following statement.

Corollary 4.2. *Assume that the system $\mathcal{G} = \{G_1, \ldots, G_m\}$ is nondegenerate, the cone $\mathcal{C}(\mathcal{G})$ is unimodular, and that $\widetilde{K} = \overline{\Phi(K)}$. Then,*

$$\mathcal{A}(K) \cap \mathcal{T}^{Sat} \quad \subset \quad \mathcal{T}^{\vee\vee}.$$

Proof. Let $F \in \mathcal{A}(K) \cap \mathcal{T}^{Sat}$ and take any $\epsilon > 0$. We have $F + \epsilon \cdot I_d \in \mathcal{A}(K)$ and $\lambda_{\min}(F(x) + \epsilon \cdot I_d) \geq \epsilon$ for all $x \in K$. By Theorem 4.1, $F + \epsilon \cdot I_d$ belongs to $\mathcal{T}$. Hence for all $\mathscr{L} \in \mathcal{T}^\vee$,

$$\mathscr{L}(F) + \epsilon \ = \ \mathscr{L}(F + \epsilon \cdot I_d) \ \geq \ 0,$$

so by taking $\epsilon \to 0$, we get $\mathscr{L}(F) \geq 0$. Therefore, $F \in \mathcal{T}^{\vee\vee}$. □

We conclude the paper by the following remark.

Remark 4.1. Let $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial isomorphism. Let K_1, K_2 be two semialgebraic sets in $\mathbb{R}^n$ such that $K_1 = \Psi(K_2)$. If K_2 is determined by a nondegenerate system of matrix polynomials, then the results obtained in this paper hold also for K_1.

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