A linearized stability theorem for nonlinear delay fractional differential equations

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***Abstract*— In this paper, we prove a theorem of linearized asymptotic stability for nonlinear fractional differential equations with a time delay. By using the method of linearization of a nonlinear equation along an orbit (Lyapunov's first method), we show that an equilibrium of a nonlinear Caputo fractional delay differential equation is asymptotically stable if its linearization at the equilibrium is asymptotically stable. Our approach is based on a technique which converts the linear part of the equation into a diagonal one. Then by using the properties of generalized Mittag-Leffler functions, the construction of an associated Lyapunov–Perron operator and the Banach contraction mapping theorem, we obtain the desired result.**

***Index Terms*— Asymptotic stability, delay differential equations with fractional derivatives, existence and uniqueness, fractional differential equations, growth and boundedness, stability.**

I. Introduction

Delay fractional differential equations (DFDEs) have received considerable research attention recently. They provide mathematical models of practical systems in which the fractional rate of change depends on the influence of their present and hereditary effects (see, [1]-[8] and the references therein). One of the simplest form of DFDEs is

$$\begin{cases} {}^C D^\alpha_{0+}x(t) = f(x(t), x(t-\tau)), \quad t \in [0,T], \\ x(t) = \phi(t), \quad \forall t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\alpha \in (0,1)$ is the order of the Caputo fractional derivative ${}^C D^\alpha_{0+}$, the initial condition ϕ is a continuous function on the interval $[-\tau, 0]$ with $\tau, T > 0$ are fixed real parameters. For this equation, the first basic, important problem is to show the existence and uniqueness of solutions under some reasonable conditions. It is well known that in the case of ordinary differential equations ($\alpha = 1$), under some Lipschitz conditions, a delay equation has a unique local solution (see [9, Section 2.2]). Furthermore, by using continuation property (see [9, Section 2.3]), one can derive global solutions as well. However, in the fractional case, the problem of existence and uniqueness of (local and global) solutions is more complex because of the *fractional order* feature of the equation which implies history dependence of the solutions. Hence, among others, the continuation property is not applicable. With regard to the existence of solutions to DFDEs, some results ([10], [11]) have been reported in the literature.

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Furthermore, whenever the solutions exist, it is of particular importance to know their asymptotic behavior. To the best of our knowledge, there have been only very few contributions to this problem until now. In [12] and [13], the authors considered the stability of some particular types of fractional differential equations with constant delays. In [6], the authors discussed stability and asymptotic properties of linear fractional-order differential systems involving both delayed and non-delayed terms. The stability and bifurcation analysis of a generalized scalar DFDE was discussed in [14]. The stability and performance analysis for positive fractional-order systems with time-varying delays was reported in [15]. However, the relationship between the stability of the trivial solution to a nonlinear delay fractional differential system and that of the linearized part is still an open problem.

This paper is devoted to the investigation of the asymptotic behavior for solutions near the equilibrium of (1) in the case the function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ has the form

$$f(x, y) = Ay + g(x, y).$$

Here, $A \in \mathbb{R}^{d\times d}$ and the function g satisfies the following conditions:

(H1) $g(0, 0) = 0$;

(H2) g is locally Lipschitz continuous in a neighborhood of the origin and

$$\lim_{\varrho \to 0} \ell_g(\varrho) = 0,$$

with

$$\ell_g(\varrho) := \sup_{\substack{x,y,\hat{x},\hat{y} \in B_{\mathbb{R}^d}(0,\varrho) \\ (x,y)\neq(\hat{x},\hat{y})}} \frac{\|g(x, y) - g(\hat{x}, \hat{y})\|}{\|x - \hat{x}\| + \|y - \hat{y}\|},$$

where $B_{\mathbb{R}^d}(0, \varrho) := \{x \in \mathbb{R}^d : \|x\| \le \varrho\}$.

Namely, we prove that the trivial solution to (1) is asymptotically stable if the trivial solution to the linearized equation

$$\begin{cases} {}^C D^\alpha_{0+}x(t) = Ax(t-\tau), \quad t \in (0,\infty), \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases}$$

where $\phi : [-\tau, 0] \to \mathbb{R}^d$ is a continuous function, is asymptotically stable. This means that the small nonlinear perturbation g does not affect the asymptotic stability of the trivial solution to the equation (1).

The rest of this paper is organized as follows. In Section II, we recall briefly a framework of delay fractional differential systems. Section III is devoted to the main result of this paper. In this section, we give a spectrum characterization of the asymptotic stability to nonlinear fractional differential systems.

II. Preliminaries

This section is devoted to recalling briefly a framework of DFDEs. We first introduce some notation which is used throughout this paper. Let $\mathbb{K}$ be the set of all real numbers or complex numbers and $\mathbb{K}^d$ be the d-dimensional Euclidean space endowed with a norm $\|\cdot\|$. Denote by I the real interval $[a,b]$ or $[a,\infty)$, let $C(I;\mathbb{K}^d)$ be the space of continuous functions $\xi : I \to \mathbb{K}^d$ with the sup norm $\|\cdot\|_\infty$, i.e.,

$$\|\xi\|_\infty := \sup_{t\in I}\|\xi(t)\|, \quad \forall \xi \in C(I;\mathbb{K}^d).$$

Finally, we denote by $B_{C([a,b];\mathbb{K}^d)}(0,\varrho)$ the closed ball centered at the origin with radius ϱ in the space $C([a,b];\mathbb{K}^d)$ and $B_{C_\infty}(0,\varrho)$ the closed ball with the center at the origin and radius ϱ in the space $C([a,\infty);\mathbb{K}^d)$.

For $[a,b] \subset \mathbb{R}$ and a measurable function $x : [a,b] \to \mathbb{R}$ such that $\int_a^b |x(\tau)|\, d\tau < \infty$, the Riemann–Liouville integral operator of order α is defined by

$$(I^\alpha_{a+}x)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\, ds, \quad t \in (a,b],$$

where Γ is the Gamma function. The *Caputo fractional derivative* ${}^C D^\alpha_{a+}x$ of a function $x \in AC([a,b];\mathbb{R})$ is defined by

$$({}^C D^\alpha_{a+}x)(t) := (I^{1-\alpha}_{a+}Dx)(t), \quad t \in (a,b],$$

where $AC([a,b];\mathbb{R})$ denotes the space of real functions x which is absolutely continuous, $D = \frac{d}{dt}$ is the usual derivative. The Caputo fractional derivative of a d-dimensional vector function $x(t) = (x_1(t),\cdots,x_d(t))^T$ is defined component-wise as

$$({}^C D^\alpha_{a+}x)(t) := ({}^C D^\alpha_{a+}x_1(t),\cdots,{}^C D^\alpha_{a+}x_d(t))^T.$$

Let τ be an arbitrary positive constant, and $\phi \in C([-r,0];\mathbb{R}^d)$ be a given continuous function. Consider the following delay Caputo fractional differential equations

$${}^C D^\alpha_{0+}x(t) = Ax(t-\tau) + g(x(t),x(t-\tau)), \quad t \in [0,\infty), \quad (2)$$

with the initial condition

$$x(t) = \phi(t), \quad \forall t \in [-\tau,0], \quad (3)$$

where $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous in a neighborhood of the origin.

For any $T > 0$, a function $\varphi(\cdot,\phi) \in C([-\tau,T];\mathbb{R}^d)$ is called a *solution* to the initial condition problem (2)–(3) over the interval $[-\tau,T]$ if ${}^C D^\alpha_{0+}\varphi(t,\phi) = A\varphi(t-\tau,\phi) + g(\varphi(t,\phi),\varphi(t-\tau,\phi))$ for all $t \in (0,T]$, and $\varphi(t,\phi) = \phi(t)$ for all $t \in [-\tau,0]$.

Since g is locally Lipschitz continuous in a neighborhood of the origin, [11, Theorem 3.1] implies the existence and uniqueness of solutions to the initial value problem (2)–(3) for any $\phi \in C([-\tau,0];\mathbb{R}^d)$. Let $I := [-\tau, t_{\max}(\phi))$, where $0 < t_{\max}(\phi) \le \infty$, be the maximal interval of existence to the solution $\varphi(\cdot,\phi)$. We now recall the definitions of stability and asymptotic stability of the trivial solution to the equation (2).

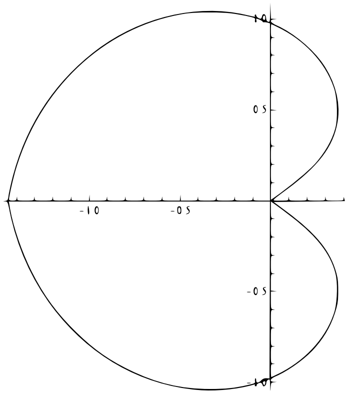

Fig. 1. Stability region $\mathcal{S}_{\alpha,\tau}$ (bounded by the grey curve) for $\alpha = 0.4$ and $\tau = 1$.

Definition 1: (i) The trivial solution of (2) is called stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $\|\phi\|_\infty \le \delta$, we have $t_{\max}(\phi) = \infty$ and

$$\|\varphi(t,\phi)\| \le \varepsilon, \quad \forall t \ge 0.$$

(ii) The trivial solution is called asymptotically stable if it is stable and there exists $\hat{\delta} > 0$ such that $\lim_{t\to\infty}\varphi(t,\phi) = 0$ whenever $\|\phi\|_\infty \le \hat{\delta}$.

In the case $g = 0$, the equation (2) is reduced to a linear delay fractional equation

$${}^C D^\alpha_{0+}x(t) = Ax(t-\tau), \quad \forall t \ge 0. \quad (4)$$

Denote by $\mathcal{S}_{\alpha,\tau}$ the set containing the complex number $\lambda \in \mathbb{C}\setminus\{0\}$ such that $|\lambda| < \left(\frac{|\arg(\lambda)| - \alpha\pi/2}{\tau}\right)^\alpha$ and $\frac{\alpha\pi}{2} < |\arg(\lambda)| \le \pi$. Due to [6, Theorem 2], we see that the trivial solution to (4) is asymptotically stable if and only if all eigenvalues of the matrix A are located in the domain $\mathcal{S}_{\alpha,\tau}$. Fig. 1 shows the stability region $\mathcal{S}_{\alpha,\tau}$ for $\alpha = 0.4$ and $\tau = 1$. For the nonlinear equation (2), we first focus on the case where the matrix A is diagonal and the function g is globally Lispchitz continuous. We define the generalized Mittag-Leffler function $E^{\lambda,\tau}_{\alpha,\beta}(t) : [0,\infty) \to \mathbb{C}$ by

$$E^{\lambda,\tau}_{\alpha,\beta}(t) := \begin{cases} \sum_{k=0}^\infty \frac{\lambda^k (t-k\tau)^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} H(t-k\tau), & \text{if } t > 0, \\ 1, \quad \text{if } t \le 0, \end{cases}$$

where $\beta > 0$, $\lambda \in \mathbb{C}$ and H is the Heaviside function defined by

$$H(t) = \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Note that for $\beta = 1$, and $0 < t \ll 1$, we have $E^{\lambda,\tau}_{\alpha,1} = t^0 = 1$. Hence, this function is continuous at $t = 0$.

The following result is a connection between the solutions to the equation (2) and its linear part.

Lemma 1: Consider the initial problem (2)–(3). Assume that g is globally Lipschitz continuous and

$$A = \mathrm{diag}(\lambda_1,\ldots,\lambda_d),$$

where $\lambda_i \in \mathbb{C}$, for $i = 1,\ldots,d$. Then, for any initial condition $\phi \in C([-\tau,0];\mathbb{C}^d)$, this problem has a unique solution on

$[-\tau,\infty)$. Denote this solution by $\varphi(\cdot,\phi)$. We have a representation of $\varphi(\cdot,\phi)$ as $\varphi(\cdot,\phi) := (\varphi^1(\cdot,\phi),\ldots,\varphi^d(\cdot,\phi))^{\mathrm{T}}$, in which, for $i\in\{1,\ldots,d\}$, the i-th component $\varphi^i(\cdot,\phi)$ for $\forall t\geq 0$ is as follows

$$E^{\lambda_i,\tau}_{\alpha,1}(t)\phi^i(0)+\lambda_i\int_{-\tau}^{0}E^{\lambda_i,\tau}_{\alpha,\alpha}(t-\tau-s)H(t-\tau-s)\phi^i(s)ds + \int_0^t E^{\lambda_i,\tau}_{\alpha,\alpha}(t-s)g^i(\varphi(s,\phi),\varphi(s-\tau,\phi))ds, \quad (5)$$

and $\varphi(t,\phi)=\phi(t)$ for all $t\in[-\tau,0]$.

Proof: From [11, Corollary 3.2], we see that the initial problem (2)–(3) has a unique solution with any initial condition $\phi\in C([-\tau,0];\mathbb{C}^d)$. On the other hand, due to [11, Theorem 4.1], all solutions to this problem are exponentially bounded. Using Laplace transform and similar arguments as in [6, Section 4], we obtain the variation of constants formula (5). ■

For the remainder of this section, we give some estimates involving the scalar generalized Mittag-Leffler functions $E^{\lambda,\tau}_{\alpha,\beta}$ with $\lambda\in\mathcal{S}_{\alpha,\tau}$ and $\beta=1$ or $\beta=\alpha$.

Lemma 2: Assume that $\lambda\in\mathcal{S}_{\alpha,\tau}$. Then, there exists a positive constant $C_{\alpha,\lambda}$ such that the following statements hold:

(i) $|E^{\lambda,\tau}_{\alpha,\alpha}(t)|\leq \frac{C_{\alpha,\lambda}}{t^{\alpha+1}},\quad \forall t\geq 1$;

(ii) $|E^{\lambda,\tau}_{\alpha,1}(t)|\leq \frac{C_{\alpha,\lambda}}{t^{\alpha}},\quad \forall t\geq 1$;

(iii) $\sup_{t\geq 0}\int_0^t |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\,ds\leq C_{\alpha,\lambda}$.

The proof of this lemma is given in the Appendix at the end of the paper.

III. Main result

Our aim in this section is to prove the following theorem. This is a generalization of [7, Theorem 3.1] for the delay fractional differential equations.

Theorem 1 (Linearized stability theorem): Consider the initial problem (2)–(3). Assume that the spectrum $\sigma(A)$ of the matrix A satisfies

$$\sigma(A)\subset\mathcal{S}_{\alpha,\tau}$$

and the function g satisfies the conditions (H_1) and (H_2). Then, the trivial solution to this problem is asymptotically stable.

To prove this theorem, we first transform the linear part of (2) to a diagonal matrix; then we construct a Lyapunov–Perron operator which is a contraction, and its fixed point is the solution to the initial problem (2)–(3). We then exploit the properties of the scalar generalized Mittag-Leffler function to obtain the conclusion of the theorem.

Transformation of the linear part: Using [16, Theorem 6.37, pp. 146], there exists a nonsingular matrix $T\in\mathbb{C}^{d\times d}$ transforming the matrix A in the equation (2) into the Jordan normal form, i.e.,

$$T^{-1}AT=\mathrm{diag}(A_1,\ldots,A_n),$$

for $i=1,\ldots,n$, the block A_i is of the following form

$$A_i=\lambda_i\,\mathrm{id}_{d_i\times d_i}+\eta_i\,N_{d_i\times d_i},$$

where $\mathrm{id}_{d_i\times d_i}$ is the identity matrix having the size $d_i\times d_i$, $\eta_i\in\{0,1\}$ and the nilpotent matrix $N_{d_i\times d_i}$ is given by

$$N_{d_i\times d_i}:=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\ddots&\ddots&\vdots\\0&0&\cdots&0&1\\0&0&\cdots&0&0\end{pmatrix}_{d_i\times d_i}.$$

Let γ be an arbitrary but fixed positive number. Using the transformation $P_i:=\mathrm{diag}(1,\gamma,\ldots,\gamma^{d_i-1})$, we obtain that

$$P_i^{-1}A_iP_i=\lambda_i\,\mathrm{id}_{d_i\times d_i}+\gamma_i\,N_{d_i\times d_i},$$

$\gamma_i\in\{0,\gamma\}$. Hence, under the transformation $y:=(TP)^{-1}x$, the equation (2) becomes

$$^{C}D^{\alpha}_{0+}y(t)=\mathrm{diag}(J_1,\ldots,J_n)y(t-\tau)+h(y(t),y(t-\tau)),\quad (6)$$

where $J_i:=\lambda_i\mathrm{id}_{d_i\times d_i}$ for $i=1,\ldots,n$ and the function h is given by

$$h(y(t),y(t-\tau)) :=\mathrm{diag}(\gamma_1N_{d_1\times d_1},\ldots,\gamma_nN_{d_n\times d_n})y(t-\tau) + (TP)^{-1}g(TPy(t),TPy(t-\tau)).\quad (7)$$

Remark 1: The function h in the equation (6) is locally Lipschitz continuous in a neighborhood of the origin and

$$h(0,0)=0,\quad\text{and}\quad \lim_{\varrho\to 0}\ell_h(\varrho)=\begin{cases}\gamma & \text{if there exists } \gamma_i=\gamma,\\ 0 & \text{otherwise.}\end{cases}$$

Remark 2: If the trivial solution to equations (6) is stable (or asymptotically stable), then the trivial solution to (2) is the same, i.e., it is also stable (or asymptotically stable).

Remark 3: By using the change of variable $y=(TP)^{-1}x$, the original equation (2) is transformed into the equation (6). In (6), the linear part is a diagonal matrix and the perturbation h is a locally Lipschitz continuous function with the Lipschitz coefficient γ which can be chosen small arbitrarily. This means that to study the asymptotic stability of the trivial solution to (2) in arbitrary finite dimensional spaces, we only need to consider this problem for scalar fractional differential equations which is simpler than the original one.

Construction of an appropriate Lyapunov–Perron operator: We are now introducing a Lyapunov–Perron operator associated with (6). Before doing this, we discuss some conventions which are used in the remaining part of this section: The space $\mathbb{C}^d$ can be written as $\mathbb{C}^d=\mathbb{C}^{d_1}\times\cdots\times\mathbb{C}^{d_n}$. A vector $x\in\mathbb{C}^d$ can be written component-wise as $x=(x^1,\ldots,x^n)$.

For any $\phi\in C([-\tau,0];\mathbb{C}^d)$, the operator $\mathcal{T}_{\phi,r}$ from $C([-\tau,\infty);\mathbb{C}^d)$ to $C([-\tau,\infty);\mathbb{C}^d)$ is defined by

$$(\mathcal{T}_{\phi,\tau}\xi)(t)=((\mathcal{T}_{\phi,\tau}\xi)^1(t),\ldots,(\mathcal{T}_{\phi,\tau}\xi)^n(t))^{\mathrm{T}},$$

where for $i=1,\ldots,n$ and $t\geq 0$ the component $(\mathcal{T}_{\phi,\tau}\xi)^i(t)$ as

$$E^{\lambda_i,\tau}_{\alpha,1}(t)\phi^i(0)+\lambda_i\int_{-\tau}^{0}E^{\lambda_i,\tau}_{\alpha,\alpha}(t-\tau-s)H(t-\tau-s)\phi^i(s)\,ds + \int_0^t E^{\lambda_i,\tau}_{\alpha,\alpha}(t-s)h^i(\xi(s),\xi(s-\tau))\,ds,$$

and $(\mathcal{T}_{\phi,\tau}\xi)(t) = \phi(t)$ for any $t \in [-\tau, 0]$, is called the *Lyapunov-Perron operator associated with* (6). Next, we provide some estimates on the operator $\mathcal{T}_{\phi,\tau}$.

Proposition 1: Consider system (6) and suppose that

$$\sigma(A) \subset \mathcal{S}_{\alpha,\tau}.$$

Let ε_1 be a small positive parameter such that the function h is Lipschitz continuous on $B_{C_\infty}(0,\varepsilon_1) \times B_{C_\infty}(0,\varepsilon_1)$. Then, for any $\xi, \hat{\xi} \in B_{C_\infty}(0,\varepsilon_1)$, we have

$$\|\mathcal{T}_{\phi,\tau}\xi - \mathcal{T}_{\hat{\phi},\tau}\hat{\xi}\|_\infty \le \max\Big\{ \max_{1\le i\le n} \sup_{t\ge 0} \Big\{ |E^{\lambda_i,\tau}_{\alpha,1}(t)| + |\lambda_i| \int_{t-\tau}^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds \Big\} \times \|\phi - \hat{\phi}\|_\infty + \max_{1\le i\le n} \int_0^\infty |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds \times \ell_h(\hat{\varepsilon}_1) \times \|\xi - \hat{\xi}\|_\infty, \|\phi - \hat{\phi}\|_\infty \Big\}$$

for all $\phi, \hat{\phi} \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\varepsilon_1)$.

Proof: For $i = 1,\dots,n$ and $t \ge 0$, we get

$$\|(\mathcal{T}_{\phi,\tau}\xi)^i(t) - (\mathcal{T}_{\hat{\phi},\tau}\hat{\xi})^i(t)\| \le \|\phi - \hat{\phi}\|_\infty \Big(|E^{\lambda_i,\tau}_{\alpha,1}(t)| + |\lambda_i| \times \int_{t-\tau}^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds \Big) + \ell_h(\max\{\|\xi\|_\infty, \|\widehat{\xi}\|_\infty\}) \times \|\xi - \widehat{\xi}\|_\infty \times \int_0^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds.$$

Hence, for any $\xi, \hat{\xi} \in B_{C_\infty}(0,\varepsilon_1)$, we have

$$\|\mathcal{T}_{\phi,\tau}\xi - \mathcal{T}_{\hat{\phi},\tau}\hat{\xi}\|_\infty \le \max\Big\{ \max_{1\le i\le n} \sup_{t\ge 0} \Big\{ |E^{\lambda_i,\tau}_{\alpha,1}(t)| + |\lambda_i| \times \int_{t-\tau}^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds \Big\} \times \|\phi - \hat{\phi}\|_\infty + \max_{1\le i\le n} \int_0^\infty |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds \times \ell_h(\varepsilon_1) \times \|\xi - \hat{\xi}\|_\infty, \|\phi - \hat{\phi}\|_\infty \Big\}$$

for all $\phi, \hat{\phi} \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\varepsilon_1)$. The proof is complete. ■

From the proposition above, by letting $C(\lambda,\alpha) := \max_{1\le i\le n} \int_0^\infty |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds$, for any $\xi, \hat{\xi} \in B_{C_\infty}(0,\varepsilon_1)$, we have

$$\|\mathcal{T}_{\phi,\tau}\xi - \mathcal{T}_{\phi,\tau}\hat{\xi}\|_\infty \le C(\lambda,\alpha) \times \ell_h(\varepsilon_1) \times \|\xi - \hat{\xi}\|_\infty,$$

for all $\phi \in C([-\tau,0];\mathbb{C}^d)$. Note that the Lipschitz constant $C(\alpha,\lambda)$ is independent of the constant ε_1 which is hidden in the coefficients of system (6). From now on, we choose and fix the constant γ as $\gamma = \min\{\varepsilon_1, \frac{1}{2C(\alpha,\lambda)}\}$. The remaining question is now to choose a ball with small radius in $C_\infty(\mathbb{R}_{\ge 0}, \mathbb{C}^d)$ such that the restriction of the Lyapunov–Perron operator to this ball is strictly contractive.

Lemma 3: The following statements hold:

(i) There exists $\varepsilon > 0$ such that

$$q := C(\alpha,\lambda) \times \ell_h(\varepsilon) < 1. \tag{8}$$

(ii) Choose and fix $\varepsilon > 0$ satisfying (8). Define δ by the maximum value according to index i of the set below

$$\frac{\varepsilon(1-q)}{\sup_{t\ge 0}\Big\{ |E^{\lambda_i,\tau}_{\alpha,1}(t)| + |\lambda_i| \int_{t-\tau}^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)| ds + 1 \Big\}}.$$

Then, for any $\phi \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\delta)$, we have $\mathcal{T}_{\phi,\tau}(B_{C_\infty}(0,\varepsilon)) \subset B_{C_\infty}(0,\varepsilon)$ and $\|\mathcal{T}_{\phi,\tau}\xi - \mathcal{T}_{\phi,\tau}\hat{\xi}\|_\infty \le q\|\xi - \hat{\xi}\|_\infty$ for all $\xi, \hat{\xi} \in B_{C_\infty}(0,\varepsilon)$.

Proof: (i) By Remark 1, $\lim_{\varrho\to 0} \ell_h(\varrho) \le \gamma$. Hence, we can choose a positive constant ε such that

$$q := C(\alpha,\lambda) \times \ell_h(\varepsilon) < 1,$$

and the assertion (i) is proved.

(ii) According to Proposition 1, for any $\phi \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\delta)$ and any $\xi \in B_{C_\infty}(0,\varepsilon)$, we obtain that

$$\begin{aligned}\|\mathcal{T}_{\phi,\tau}\xi\|_\infty &\le \max_{1\le i\le n} \sup_{t\ge 0} \Big\{ |E^{\lambda_i,\tau}_{\alpha,1}(t)| + |\lambda_i| \int_{t-\tau}^t |E^{\lambda_i,\tau}_{\alpha,\alpha}(s)|\, ds + 1 \Big\} \times \|\phi\|_\infty + C(\alpha,\lambda) \times \ell_h(\varepsilon) \times \|\xi\|_\infty \\ &\le (1-q)\varepsilon + q\varepsilon,\end{aligned}$$

which proves that $\mathcal{T}_{\phi,\tau}(B_{C_\infty}(0,\varepsilon)) \subset B_{C_\infty}(0,\varepsilon)$. Furthermore, we also have

$$\begin{aligned}\|\mathcal{T}_{\phi,\tau}\xi - \mathcal{T}_{\phi,\tau}\widehat{\xi}\|_\infty &\le C(\alpha,\lambda) \times \ell_h(\varepsilon) \times \|\xi - \widehat{\xi}\|_\infty \\ &\le q\|\xi - \widehat{\xi}\|_\infty,\end{aligned}$$

which concludes the proof. ■

We are now in a position to give the proof of Theorem 1.

Proof: Due to Remark 2, it is sufficient to prove the asymptotic stability for the trivial solution of system (6). For this purpose, let δ be defined as in (9) and $\phi \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\delta)$ be arbitrary. Using Lemma 3 and the Contraction Mapping Principle, there exists a unique fixed point $\xi \in B_{C_\infty}(0,\varepsilon)$ of $\mathcal{T}_{\phi,\tau}$. According to Lemma 1, this point is also a solution to (6) with the initial condition $\xi(t) = \phi(t)$ for all $t \in [-\tau, 0]$. Since the equation (6) has unique global solution in $B_{C_\infty}(0,\varepsilon)$ for each initial condition $\phi \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\delta)$, the trivial solution is stable. To complete the proof of the theorem, we have to show that the trivial solution is attractive. Suppose that $\xi(t) = (\xi^1(t),\dots,\xi^n(t))$ is the solution to (6) which satisfies $\xi(t) = \phi(t)$ for every $t \in [-\tau,0]$, where $\phi \in B_{C([-\tau,0];\mathbb{C}^d)}(0,\delta)$. From Lemma 3, we see that $\|\xi\|_\infty \le \varepsilon$. Put $a := \limsup_{t\to\infty} \|\xi(t)\|$, then $a \in [0,\varepsilon]$. Let $\hat{\varepsilon}$ be a positive number small enough. Then, there exists $T(\hat{\varepsilon}) > 0$ such that

$$\|\xi(t)\| \le (a + \hat{\varepsilon}) \qquad \text{for any } t \ge T(\hat{\varepsilon}).$$

For each $i = 1,\dots,n$, we will estimate $\limsup_{t\to\infty} \|\xi^i(t)\|$. According to Lemma 2(i) and 2(ii), we obtain

(i) $\lim_{t\to\infty} E^{\lambda_i,\tau}_{\alpha,1}(t) = 0$;

(ii) $\lim_{t\to\infty} \int_{-\tau}^0 E^{\lambda_i,\tau}_{\alpha,\alpha}(t-\tau-s) H(t-\tau-s)\phi^i(s)\, ds = 0$;

(iii)

$$\begin{aligned}&\limsup_{t\to\infty} \left\| \int_0^{T(\hat{\varepsilon})} E^{\lambda_i,\tau}_{\alpha,\alpha}(t-s) h^i(\xi(s))\, ds \right\| \\ &\le \max_{t\in[0,T(\varepsilon)]} \|h^i(\xi(t))\| \limsup_{t\to\infty} \int_0^{T(\hat{\varepsilon})} \frac{C_{\alpha,\lambda_i}}{(t-s)^{\alpha+1}} ds \\ &= 0.\end{aligned}$$

Therefore, from the fact that $\xi^i(t) = (\mathcal{T}_x\xi)^i(t)$, we have

$$\begin{aligned}\limsup_{t\to\infty}\|\xi^i(t)\| &= \limsup_{t\to\infty}\left\|\int_{T(\hat\varepsilon)}^t E^{\lambda_i,\tau}_{\alpha,\alpha}(t-s)h^i(\xi(s))ds\right\|\\ &\le \ell_h(\varepsilon)\times C_{\alpha,\lambda_i}\times(a+\hat\varepsilon),\end{aligned}$$

where we use the estimate

$$\begin{aligned}\left|\int_{T(\hat\varepsilon)}^t E^{\lambda_i,\tau}_{\alpha,\alpha}(t-s)\,ds\right| &= \left|\int_0^{t-T(\hat\varepsilon)} E^{\lambda_i,\tau}_{\alpha,\alpha}(u)\,du\right|\\ &\le C_{\alpha,\lambda_i},\end{aligned}$$

see Lemma 2(iii), to obtain the inequality above. Thus,

$$\begin{aligned}a &\le \max\left\{\limsup_{t\to\infty}\|\xi^1(t)\|,\dots,\limsup_{t\to\infty}\|\xi^n(t)\|\right\}\\ &\le \ell_h(\tau)\times C(\alpha,\lambda)\times(a+\hat\varepsilon).\end{aligned}$$

Letting $\hat\varepsilon\to 0$, we have

$$a \le \ell_h(\varepsilon)\times C(\alpha,\lambda)\times a.$$

Due to the fact $\ell_h(\varepsilon)\times C(\alpha,\lambda) < 1$, we get that $a = 0$ and the proof is complete. ■

Remark 4: Consider the equation (2). If the function g also depends on the time t, i.e. $g = g(t,x,y)$, then by using the same arguments as above, Theorem 1 is still true provided that the following conditions are satisfied:

(H1)' $g(t,0,0) = 0$ for all $t \ge 0$;

(H2)' for any $t\ge 0$, $g(t,\cdot,\cdot)$ is locally Lipschitz continuous in a neighborhood of the origin and

$$\lim_{\varrho\to 0}\ell_{g(t,\cdot,\cdot)}(\varrho) = 0,\quad \forall t\ge 0,$$

where for any $t \ge 0$,

$$\ell_{g(t,\cdot,\cdot)}(\varrho) := \sup_{\substack{x,y,\hat x,\hat y\in B_{\mathbb{R}^d}(0,\varrho)\\ (x,y)\neq(\hat x,\hat y)}}\frac{\|g(t,x,y)-g(t,\hat x,\hat y)\|}{\|x-\hat x\|+\|y-\hat y\|}.$$

IV. Illustrative Example

Let us consider the following nonlinear delay fractional differential equation

$$^C D^{0.5}_{0+}x(t) = -x(t-1) + x^2(t) + x^2(t-1),\quad t>0, \quad (9)$$

with the initial condition $x(t) = 0.5$ for all $t\in[-1,0]$.

In this case, we see that $-1\in\mathcal{S}_{0.5,1}$ and the perturbation $x^2(t)+x^2(t-1)$ satisfies the conditions (H1) and (H2) in Theorem 1. Hence, the trivial solution to (9) is asymptotically stable. In particular, the solution $\varphi(\cdot,0.5)$ to (9) tends to zero as $t\to\infty$. Denote by $\hat\varphi(\cdot,0.5)$ the solution to the linearized part

$$^C D^{0.5}_{0+}x(t) = -x(t-1),\quad \forall t>0,$$

with the initial condition $x(t) = 0.5$ for all $t\in[-1,0]$. The trajectories of $\hat\varphi(\cdot,0.5)$ and $\varphi(\cdot,0.5)$ are depicted in Fig. 2.

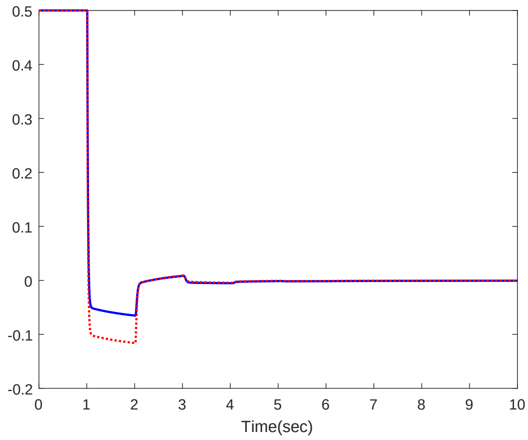

Fig. 2. Trajectories of the solutions $\hat\varphi(\cdot,0.5)$ (the dotted line) and $\varphi(\cdot,0.5)$ (the solid line).

V. Conclusion

This paper has studied the asymptotic behavior of solutions to nonlinear fractional differential equations with a time delay. We have shown that an equilibrium of a nonlinear Caputo fractional differential equation with a time delay is asymptotically stable if its linearization at the equilibrium is asymptotically stable. That is, we have given a sufficient condition of the asymptotic stability based on the characteristic spectrum of the linear part to the original equation. This is a new contribution in the qualitative theory of nonlinear fractional differential equations with delays. In the future, we hope to obtain a characteristic spectrum for the stability of fractional differential equations with multi-delays in high dimensional spaces.

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Appendix

In this section, we will prove Lemma 2.

Proof: For $\mu>0$ and $\theta\in(0,\pi)$, we denoted by $\gamma(\mu,\theta)$ the oriented contour formed by three segments:

- $\arg(z) = -\theta,\quad |z|\ge\mu$;
- $-\theta\le\arg(z)\le\theta,\quad |z| = \mu$;
- $\arg(z) = \theta,\quad |z|\ge\mu$.

From [6, Proposition 1 (ii)], we can choose a positive constant δ such that all zeros z_i of the function $s^\alpha - \lambda\exp(-\tau s)$ satisfy $|\arg(z_i)|\neq\frac{\pi}{2}+\delta$, and there are only finitely many of them satisfying $|\arg(z_i)|\le\frac{\pi}{2}+\delta$. Hence, there exist $R>\varepsilon>0$ such that all z_i lie to the left of $\gamma(R,\frac{\pi}{2}+\delta)$ and those satisfying $|\arg(z_i)|\le\frac{\pi}{2}+\delta$ are located to the right of $\gamma(\varepsilon,\frac{\pi}{2}+\delta)$, see [6, p. 116]. Let $\beta\in\{1,\alpha\}$. For $t\ge 1$, from [6, p. 116], we have

$$\begin{aligned}E^{\lambda,\tau}_{\alpha,\beta}(t) &= \frac{1}{2\pi i}\int_{\gamma(R,\frac{\pi}{2}+\delta)}\frac{s^{\alpha-\beta}\exp(ts)}{s^\alpha-\lambda\exp(-\tau s)}\,ds\\ &= I^1(t)+I^2(t),\end{aligned}$$

where

$$I^1(t) = \frac{1}{2\pi i}\int_{\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)} \frac{s^{\alpha-\beta}\exp(ts)}{s^\alpha - \lambda\exp(-\tau s)}\,ds,$$

and

$$I^2(t) = \frac{1}{2\pi i}\int_{\gamma(R,\frac{\pi}{2}+\delta)-\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)} \frac{s^{\alpha-\beta}\exp(ts)}{s^\alpha - \lambda\exp(-\tau s)}\,ds.$$

For $I^1(t)$, we use the representation

$$I^1(t) = I^1_1(t) + I^1_2(t) + I^1_3(t),$$

where

$$I^1_1(t) = -\frac{1}{\lambda 2\pi i}\int_{\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)} s^{\alpha-\beta}\exp((\tau+t)s)\,ds,$$

$$I^1_2(t) = -\frac{1}{\lambda^2 2\pi i}\int_{\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)} s^{2\alpha-\beta}\exp((2\tau+t)s)\,ds,$$

$$I^1_3(t) = \frac{1}{\lambda^2 2\pi i}\int_{\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)} \frac{s^{3\alpha-\beta}\exp((2\tau+t)s)}{s^\alpha - \lambda\exp(-\tau s)}\,ds.$$

Using the change of variable $s = \frac{u^{1/\alpha}}{t}$, we have

$$I^1_1(t) = -\frac{1}{\lambda}\frac{1}{2\alpha\pi i}\int_{\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} u^{\frac{1-\beta}{\alpha}}\exp((1+\frac{\tau}{t})u^{1/\alpha})\,du \times \frac{1}{t^{\alpha-\beta+1}},$$

which, by changing the variable $\nu = (1+\frac{\tau}{t})^\alpha u$, implies

$$\begin{aligned} I^1_1(t) =& -\frac{1}{\lambda}\frac{1}{2\alpha\pi i}\int_{\gamma((1+\frac{\tau}{t})^\alpha\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \frac{\nu^{\frac{1-\beta}{\alpha}}\exp(\nu^{1/\alpha})}{(1+\frac{\tau}{t})^{\alpha-\beta+1}}\,d\nu \\ &\times \frac{1}{t^{\alpha-\beta+1}} \\ =& -\frac{1}{\lambda}\frac{1}{2\alpha\pi i}\int_{\gamma((1+\frac{\tau}{t})^\alpha\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \nu^{\frac{1-\beta}{\alpha}}\exp(\nu^{1/\alpha})\,d\nu \\ &\times \frac{1}{(t+\tau)^{\alpha-\beta+1}} \\ =& -\frac{1}{\lambda}\times\Big(\frac{1}{\Gamma(z)}\Big)_{|z=\beta-\alpha}\times\frac{1}{(t+\tau)^{\alpha-\beta+1}},\quad \forall t\geq 1, \end{aligned} \tag{10}$$

where we used the integral representation

$$\Big(\frac{1}{\Gamma(z)}\Big)_{|z=\beta-\alpha} = \frac{\int_{\gamma((1+\frac{\tau}{t})^\alpha\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \frac{\nu^{\frac{1-\beta}{\alpha}}\exp(\nu^{1/\alpha})}{(1+\frac{\tau}{t})^{\alpha-\beta+1}}\,d\nu}{2\alpha\pi i},$$

see e.g. [17, Formula (1.52), p. 16].

For $I^1_2(t)$, by using the variable $s = \frac{u^{1/\alpha}}{t}$, we have

$$\begin{aligned} I^1_2(t) =& -\frac{1}{\lambda^2}\frac{1}{2\pi i}\int_{\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \frac{u^{\frac{2\alpha-\beta}{\alpha}}\exp((1+\frac{2\tau}{t})u^{1/\alpha})}{t^{2\alpha-\beta}} \\ &\times\frac{1}{\alpha t}u^{\frac{1}{\alpha}-1}\,du \\ =& -\frac{1}{\lambda^2}\frac{\int_{\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} u^{\frac{\alpha-\beta+1}{\alpha}}\exp((1+\frac{2\tau}{t})u^{1/\alpha})\,du}{2\alpha\pi i} \\ &\times\frac{1}{t^{2\alpha-\beta+1}}. \end{aligned}$$

Put $\nu = (1+\frac{2\tau}{t})^\alpha u$, we obtain

$$\begin{aligned} I^1_2(t) =& -\frac{1}{\lambda^2}\frac{1}{2\alpha\pi i}\int_{\gamma((1+\frac{2\tau}{t})^\alpha\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \nu^{\frac{\alpha-\beta+1}{\alpha}}\exp(\nu^{1/\alpha})\,d\nu \\ &\times\frac{1}{(t+2\tau)^{2\alpha-\beta+1}} \\ =& -\frac{1}{\lambda^2}\times\Big(\frac{1}{\Gamma(z)}\Big)_{|z=\beta-2\alpha}\times\frac{1}{(t+2\tau)^{2\alpha-\beta+1}} \end{aligned} \tag{11}$$

for all $t\geq 1$. For $I^1_3(t)$, we have

$$\begin{aligned} I^1_3(t) =& \frac{1}{\lambda^2}\times\frac{\int_{\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} \frac{u^{\frac{2\alpha-\beta+1}{\alpha}}\exp((1+\frac{3\tau}{t})u^{1/\alpha})}{u\exp(\frac{\tau u^{1/\alpha}}{t})-\lambda t^\alpha}\,du}{2\alpha\pi i} \\ &\times\frac{1}{t^{2\alpha-\beta+1}}. \end{aligned} \tag{12}$$

Note that there exists a positive constant C_1 such that

$$|s^\alpha - \lambda\exp(-\tau s)| \geq C_1,\quad \forall s\in\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta),\ \forall t\geq 1.$$

Thus, for any $u\in\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)$, we have

$$\Big|u\exp(\frac{\tau u^{1/\alpha}}{t}) - \lambda t^\alpha\Big| \geq C_1 t^\alpha|\exp(\frac{\tau u^{1/\alpha}}{t})|.$$

On the other hand, there exists a constant positive C_2 satisfying

$$\int_{\gamma(\varepsilon^\alpha,\frac{\alpha\pi}{2}+\alpha\delta)} |u|^{\frac{2\alpha-\beta+1}{\alpha}}|\exp((1+\frac{2\tau}{t})u^{1/\alpha})||du| \leq C_2,\quad \forall t\geq 1.$$

This implies that

$$|I^1_3(t)| \leq \frac{C_2}{C_1|\lambda|^2 2\alpha\pi t^{3\alpha-\beta+1}},\quad \forall t\geq 1. \tag{13}$$

We now estimate the quantity $I^2(t)$. Because the domain bounded by $\gamma(R,\frac{\pi}{2}+\delta)-\gamma(\frac{\varepsilon}{t},\frac{\pi}{2}+\delta)$ is a compact set in the complex plane $\mathbb{C}$ and $s^\alpha-\lambda\exp(-\tau s)$ is analytic in this set, there is a finite number of zeros of $s^\alpha-\lambda\exp(-\tau s)$ in this domain. Let us denote these zeros by $z_1,\dots,z_k$. According to [6, Lemma 2], $z_1,\dots,z_k$ are single zeros of $s^\alpha-\lambda\exp(-\tau s)$. From [4, p. 101], we have

$$\begin{aligned} I^2(t) =& \sum_{i=1}^k \mathrm{Res}_{z_i}\Big\{\frac{\exp(ts)}{s^{\beta-\alpha}(s^\alpha-\lambda\exp(-\tau s))}\Big\} \\ =& \sum_{i=1}^k \frac{\exp(tz_i)}{\beta {z_i}^{\beta-1} - ((\beta-\alpha)\lambda z_i^{\beta-\alpha-1} - \tau\lambda z_i^{\beta-\alpha})\exp(-\tau z_i)}, \end{aligned} \tag{14}$$

where Res_{z_i} is the residue at z_i of $s^\alpha-\lambda\exp(-\tau s)$. Hence, there is a constant $C_3>0$ such that

$$|I^2(t)| \leq C_3\sum_{i=1}^k|\exp(z_i t)|,\quad \forall t\geq 1.$$

(i) Note that

$$\Big(\frac{1}{\Gamma(z)}\Big)_{|z=0} = 0.$$

For $\beta=\alpha$, from (10), (11), (13) and (14), we can find a constant $C_{\alpha,\lambda}>0$ such that

$$|E^{\lambda,\tau}_{\alpha,\alpha}(t) \leq \frac{C_{\alpha,\lambda}}{t^{\alpha+1}},\quad \forall t\geq 1.$$

(ii) Similarly, For $\beta = 1$, from (10), (11), (13) and (14), there exists a constant $C_{\alpha,\lambda} > 0$ such that

$$|E^{\lambda,\tau}_{\alpha,1}(t)| \leq \frac{C_{\alpha,\lambda}}{t^{\alpha}}, \quad \forall t \geq 1.$$

(iii) First, from the representation of the function $E^{\lambda,\tau}_{\alpha,\alpha}$, we see that

$$\int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds$$

is bounded. Indeed, Consider the following cases.

The case $\tau \geq 1$**.** We have

$$\begin{aligned}
&\int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds \\
&\qquad \leq \int_0^1 \sum_{0\leq k\tau\leq s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds \\
&\qquad = \int_0^1 \frac{s^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&\qquad = \frac{1}{\Gamma(\alpha+1)}.
\end{aligned}$$

The case $0 \leq \tau < 1$**.** Let $k_0 \in \mathbb{N}$ be the number satisfying $k_0\tau < 1$ and $(k_0+1)\tau \geq 1$. We can partition the interval $[0,1]$ into k_0+1 subintervals as $[0,\tau], \ldots, [k_0\tau, 1]$. Then,

$$\begin{aligned}
&\int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds \\
&\leq \int_0^1 \sum_{0\leq k\tau\leq s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds \\
&= \sum_{i=0}^{k_0-1} \int_{i\tau}^{(i+1)\tau} \sum_{0\leq k\tau\leq s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds \\
&\quad + \int_{k_0\tau}^1 \sum_{0\leq k\tau\leq s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds.
\end{aligned}$$

Furthermore, for $0 \leq i \leq k_0 - 1$, we see that

$$\begin{aligned}
&\int_{i\tau}^{(i+1)\tau} \sum_{0\leq k\tau\leq s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds \\
&= \sum_{k=0}^{i} \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha)} \int_{i\tau}^{(i+1)\tau} (s-k\tau)^{\alpha k+\alpha-1}\, ds \\
&= \sum_{k=0}^{i} \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha+1)} \left(((i+1-k)\tau)^{\alpha k+\alpha} - ((i-k)\tau)^{\alpha k+\alpha}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{k_0\tau}^1 \sum_{0\leq k\tau<s} \frac{|\lambda|^k (s-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} H(s-k\tau)\, ds \\
&= \sum_{k=0}^{k_0} \int_{k_0\tau}^1 \frac{|\lambda|^k (\tau-k\tau)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}\, ds \\
&= \sum_{k=0}^{k_0} \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha+1)} \left((1-k\tau)^{\alpha k+\alpha} - ((k_0-k)\tau)^{\alpha k+\alpha}\right),
\end{aligned}$$

which imply that

$$\int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds$$

is bounded. Now, for $t > 1$, we use the representation

$$\int_0^t |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds = \int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds + \int_1^t |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds.$$

From (i), there exists a positive constant $\hat{C}$ such that

$$\begin{aligned}
\int_1^t |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds &\leq \hat{C} \int_1^t \frac{1}{s^{\alpha+1}}\, ds \\
&\leq \frac{\hat{C}}{\alpha}.
\end{aligned}$$

Put $C_{\alpha,\lambda} := \int_0^1 |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds + \frac{\hat{C}}{\alpha}$. Then,

$$\sup_{t\geq 0} \int_0^t |E^{\lambda,\tau}_{\alpha,\alpha}(s)|\, ds \leq C_{\alpha,\lambda}.$$

The proof is complete. ■

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