

Algorithms for finding global and local equilibrium points of Nash-Cournot equilibrium models involving concave cost

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Abstract

We consider Nash-Cournot oligopolistic equilibrium models involving separable concave cost functions. In contrast to the models with linear and convex cost functions, in these models a local equilibrium point may not be a global one. We propose algorithms for finding global and local equilibrium points for the models having separable concave cost functions. The proposed algorithms use the convex envelope of a separable concave cost function over boxes to approximate a concave cost model with an affine cost one. The latter is equivalent to a strongly convex quadratic program that can be solved efficiently. To obtain better approximate solutions the algorithms use an adaptive rectangular bisection which is performed only in the space of concave variables. Computational results on a lot number of randomly generated data show that the proposed algorithm for global equilibrium point are efficient for the models with moderate number of concave cost functions while the algorithm for local equilibrium point can solve efficiently the models with much larger size.

Keywords. Nash-Cournot oligopolistic model, concave cost, local, global equilibria, gap function, convex envelope, adaptive rectangular bisection.

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1 Introduction

The Nash- Cournot oligopolistic market model is one of fundamental models in economics that has been earned attention of many authors, see e.g. [1], [3], [4], [7], [9], [10], [11] and the references cited therein. In this model it is assumed that there are N -firms producing a common homogeneous commodity. Each firm i has a strategy set $D_i \subset R_+$ and a profit function f_i defined on the strategy set $D := D_1 \times \dots \times D_N$ of the model. Let $x_i \in D_i$ be a corresponding production level of firm i . Actually, each firm seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point (strategy) $x^* = (x_1^*, \dots, x_N^*) \in D$ is said to be a Nash equilibrium point of this Nash-Cournot oligopolistic market model if

$$f_i(x^*) \geq f_i(x^*[x_i]) \quad \forall x_i \in D_i, \quad \forall i,$$

where the vector $x^*[x_i]$ is obtained from x^* by replacing x_i^* with x_i .

In the linear Nash-Cournot model the profit function of firm i is given by

$$f_i(x) = (\alpha - \beta \sum_{j=1}^N x_j)x_i - h_i(x_i) \quad (i = 1, \dots, N), \quad (1.1)$$

where $\beta > 0$, $\alpha > 0$ and, for every i , the cost function h_i is affine that depends only on the quantity x_i of firm i . In this linear case, it has been shown that (see e.g. [9]) the model has a unique Nash equilibrium point which is the unique solution of a strongly convex quadratic program. In the case h_i is differentiable convex, the problem of finding a Nash equilibrium point can be formulated as a monotone variational inequality [3, 11] which can be solved by available methods for the monotone variational inequality.

In some practical applications, the cost for production of a unit commodity decreases as the quantity of the production gets larger. The cost function then is concave rather than convex. Nash-Cournot oligopolistic models with concave cost functions are considered in recent paper by Bigi and Passacantando in [2]. For these models, as it is shown [14] that the problem can be formulated as a mixed variational inequality of the form

$$\text{Find } x^* \in D : \langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in D.$$

In this problem F is not monotone and φ may not be convex, and therefore the existing methods for the monotone variational inequality cannot be applied. In

[14] an algorithm is proposed for finding a global equilibrium point of the model when some of the cost functions are piecewise linear concave. However the algorithm there is efficient only when the number of the piecewise linear concave cost functions is relatively small. In [17] a proximal point method was described for finding a stationary point of the model. However a stationary point may not be a global, even not a local equilibrium point.

In this paper we continue our work in [14] and [17] by considering Nash-Cournot models, where some of the cost functions are separable concave, the remaining costs are affine. Namely we approximate the model with concave cost functions by piecewise linear concave cost models that can be solved by an existing Search-and-Check algorithm in [14]. Thanks to the fact that the strategy set is a rectangle (box) and the cost functions are separable increasing, the model has particular features that can be employed to develop efficient algorithms for solving it. We propose two algorithms: the first one is a search-check-branch procedure that approximates the model with concave cost functions by the models with piecewise linear concave functions. Thanks to the affine property of the price function and separability of the concave cost function the latter models can be equivalently formulated as a strongly convex quadratic problem. In order to obtain better approximate solutions the algorithm use an adaptive rectangular bisection which is performed only in the space of the concave variables. The computational results on a lot number of randomly generated data show that this algorithm are efficient for models with a medium number (≤ 40) of the firms having concave cost functions, the number of total variables may be much larger. In order to solve the models with larger number of the firms having concave cost functions we use again the convex envelope of a concave function over a box to develop an algorithm for obtaining a local equilibrium point.

The remaining part of the paper is organized as follows. In the next section we define a gap function that can serve as a stopping criterion for the algorithms. The third section is devoted to description of the algorithms and analysis of their convergence. We close the paper with some computational results and experiences.

2 A Gap Function as a Stopping Criterion

In this section, we define a gap function for Nash-Cournot models involving concave cost functions. This gap function will serve as a stopping criterion for checking whether a point is equilibrium or not.

To be precise, we consider the Nash- Cournot oligopolistic market model presented above under the assumption that each profit function f_j is defined by (1.1) where h_j , $j = 1, \dots, n$ with $n \leq N$ is increasing concave while h_i with $i > n$ is

increasing affine. This assumption is motivated by the fact that for some firms the cost consists of both the production and transportation costs, while for the other ones, the production need not to transport. In practice the transportation cost function is concave (see the example in [2]).

First, we define the bifunction ϕ by taking

$$\phi(x, y) := \langle \tilde{B}_1 x - a, y - x \rangle + y^T B_1 y - x^T B_1 x + h(y) - h(x) \quad (2.1)$$

where

$$a := (\alpha, \alpha, \dots, \alpha)^T$$

$$B_1 := \begin{pmatrix} \beta & 0 & 0 & \dots & 0 \\ 0 & \beta & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \quad \tilde{B}_1 := \begin{pmatrix} 0 & \beta & \beta & \dots & \beta \\ \beta & 0 & \beta & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots \\ \beta & \beta & \beta & \dots & 0 \end{pmatrix}$$

and we suppose that

$$h(x) := \sum_{i=1}^N h_i(x_i).$$

Then the problem of finding an equilibrium point for the model can be formulated as a mixed variational inequality problem $MV(D)$ of the form (see e.g.[14])

$$\begin{cases} \text{find a point } x \in D \text{ such that} \\ \Phi(x, y) := \langle \tilde{B}_1 x - a, y - x \rangle + \varphi(y) - \varphi(x) \geq 0 \quad \forall y \in D, \end{cases} \quad MV(D)$$

where $\varphi(y) := y^T B_1 y + h(y)$, $\varphi(x) := x^T B_1 x + h(x)$. Clearly, φ is a DC separable function if each h_i is concave, in particular case, if each h_i is affine, then φ is a separable strongly quadratic convex function. In the latter case every local equilibrium point is global one and we have the following lemma.

Lemma 2.1 [9, 14] *Suppose that the cost function h is affine(classical model) given as $h(x) := \mu^T x + \xi$. Then variational inequality $MV(D)$ can be equivalently formulated as the convex quadratic programming problem*

$$\min\{x^T(2B_1 + \tilde{B}_1)x + (\mu - a)^T : x \in D\}.$$

Gap functions are commonly used to determine stopping rules in optimization, variational inequality and equilibrium problems as well as to reformulate them as a mathematical programming problem. Following this idea, we now define a

gap function for the Nash-Cournot equilibrium models with separable concave cost functions. Namely, for Problem MV(D) we define a gap function by taking, for each $x \in D$,

$$g(x) := -\min\{\Phi(x, y) : y \in D\}. \quad (2.2)$$

Lemma 2.2 *Suppose that cost function h_i is continuous on D_i for all $i = 1, 2, \dots, N$. Then*

- (i) *The function $g(x)$ is well defined, continuous and $g(x) \geq 0 \forall x \in D$;*
- (ii) *A point $x^* \in D$ is equilibrium for the model if and only if $g(x^*) = 0$.*

Proof. This lemma can be derived from Theorem 2.1 in [7]. Here we give a direct proof for MV(D).

(i) Since D is compact and, for each $x \in D$, $\Phi(x, \cdot)$ is continuous on D , $\Phi(x, \cdot)$ attains its minimum on D . Further, from property $\Phi(x, x) = 0$, it follows that $g(x) \geq 0$ for every $x \in D$.

(ii) Suppose that $x^* \in D$ is an equilibrium point, then

$$\Phi(x^*, y) \geq 0 \text{ for all } y \in D,$$

which implies $g(x^*) \leq 0$. Hence $g(x^*) = 0$. Conversely, if $g(x^*) = 0$, then from the definition of $g(x^*)$ one has $\Phi(x^*, y) \geq 0$ for all $y \in D$, that means that x^* is a equilibrium point of the model. \square

Motivated by this lemma, we call a point x_ϵ an ϵ -equilibrium point if $g(x_\epsilon) \leq \epsilon$.

We rewrite the bifunction Φ as

$$\Phi(x, y) = \langle \tilde{B}_1 x - a, y - x \rangle + \beta \sum_{i=1}^N y_i^2 + \sum_{i=1}^N h_i(y_i) - \beta \sum_{i=1}^N x_i^2 - \sum_{i=1}^N h_i(x_i),$$

the gap function g then can be rewritten as

$$g(x) = -\min_{y \in D} \left\{ \langle \tilde{B}_1 x - \alpha, y - x \rangle + \beta \sum_{i=1}^N y_i^2 + \sum_{i=1}^N h_i(y_i) \right\} + \beta \sum_{i=1}^N x_i^2 + \sum_{i=1}^N h_i(x_i) \quad (2.3)$$

Since D is the box of the form

$$D := \{x^T = (x_1, \dots, x_N) : 0 \leq l_i \leq x_i \leq u_i, i = 1, \dots, N\}$$

we can further write $g(x)$ as

$$\begin{aligned} g(x) = & -\sum_{i=1}^N \min_{l_i \leq y_i \leq u_i} \left\{ (\tilde{B}_1 x - \alpha)_i (y_i - x_i) + \beta y_i^2 + h_i(y_i) \right\} \\ & + \beta \left(\sum_{i=1}^N x_i \right)^2 + \sum_{i=1}^N h_i(x_i). \end{aligned} \quad (2.4)$$

A simple arrangement using (2.4) yields

$$g(x) = - \sum_{i=1}^N \min_{l_i \leq y_i \leq u_i} \left\{ \beta y_i^2 + \left(\beta \sigma^{(-i)}(x) - \alpha \right) y_i + h_i(y_i) \right\} \\ + \beta \left(\sum_{i=1}^N x_i \right)^2 - \alpha^T x + \sum_{i=1}^N h_i(x_i), \quad (2.5)$$

where $\sigma^{(-i)}(x) := \sum_{j \neq i}^N x_j$. From (2.5) it follows that evaluating $g(x)$, for each $x \in D$, one needs to solve N -optimal problems each of them is one-variable minimization problem of the form

$$\min_{l_i \leq y_i \leq u_i} \left\{ \beta y_i^2 + \left(\beta \sigma^{(-i)}(x) - \alpha \right) y_i + h_i(y_i) \right\}, \quad i = 1, 2, \dots, N. \quad (2.6)$$

In order to compare the Cournot model presented above with existing models let us consider the Bertrand model. In a Bertrand model the firms producing a common homogenous commodity. In contrast to the Cournot model, here each firm sets prices rather than the production quantity. So, in such a model, the demand is a function of price and the customers buy from firms with lowest price. However, often this assumption is not realistic, since usually the products of the firms are not entirely interchangeable, and thus some consumers may prefer one product to the other even it costs somewhat more.

Suppose that the quantity level x_i produced by firm i depends on the price and given by

$$x_i(p) = \gamma_i - \sigma_i p_i + \sum_{j \neq i}^n \lambda_{ij} p_j, \quad i = 1, \dots, n$$

where $\gamma_i, \sigma_i > 0, \lambda_{ij} \geq 0$ if $(j \neq i)$. The condition $\sigma_i > 0$ means that the demand for firm i decreases as its price increases, while $\lambda_{ij} \geq 0, (i \neq j)$ means that the demand for firm i increases when other firms increase their price.

The profit function of firm i then is given as

$$f_i(p) := p_i x_i - h_i(x_i),$$

where, following [2], we assume that the cost $h_i(\cdot)$ is a concave function of the production level and is given by

$$h_i(x_i) = \nu_i x_i - d_i x_i^2 \quad \text{with } d_i \geq 0.$$

Then an elementary computation shows that the cost is a function of the price as

$$h_i(p) = -d_i \sigma_i^2 p_i^2 + \sigma_i \left[2d_i \left(\gamma_i + \sum_{j \neq i}^n \lambda_{ij} p_j \right) - \nu_i \right] p_i + \nu_i \left(\gamma_i + \sum_{j \neq i}^n \lambda_{ij} p_j \right) \\ - d_i \left(\gamma_i + \sum_{j \neq i}^n \lambda_{ij} p_j \right)^2$$

The profit function then takes the form

$$f_i(p) = \sigma_i(d_i\sigma_i - 1)p_i^2 + [\sigma_i\nu_i + (\gamma_i + \sum_{j \neq i}^n \lambda_{ij}p_j)(1 - 2d_i\sigma_i)]p_i + d_i(\gamma_i + \sum_{j \neq i}^n \lambda_{ij}p_j)^2 - \nu_i(\gamma_i + \sum_{j \neq i}^n \lambda_{ij}p_j)$$

Each firm i attempts to maximize its profit by choosing a corresponding price level on its strategy set $[0, T_i]$ by solving the optimization problem

$$f_i(p) = \max_{y_i \in [0, T_i]} f_i(p[y_i]), \quad \forall i = 1, \dots, n,$$

where $p[y_i]$ is the vector obtained from p by replacing p_i with y_i .

By the same technique as in the Nash-Cournot model the problem of finding a Nash equilibrium point of this Bertrand model can be formulated as a mixed variational inequality of the form

$$\text{Find } p \in T := T_1 \times \dots \times T_n : \Phi(p, y) := \langle Gp - y, p - y \rangle + \psi(y) - \psi(p) \geq 0 \quad \forall y \in T$$

where

$$G = \begin{pmatrix} 0 & \lambda_{12}(1 - 2d_1\sigma_1) & \dots & \lambda_{13}(1 - 2d_1\sigma_1) & \lambda_{1n}(1 - 2d_1\sigma_1) \\ \lambda_{21}(1 - 2d_2\sigma_2) & 0 & \dots & \lambda_{23}(1 - 2d_2\sigma_2) & \lambda_{2n}(1 - 2d_2\sigma_2) \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n1}(1 - 2d_n\sigma_n) & \dots & \dots & \lambda_{n,n-1}(1 - 2d_n\sigma_n) & 0 \end{pmatrix}$$

with

$$r_i = \gamma_i(1 - 2d_i\sigma_i), \quad i = 1, \dots, n,$$

$$\psi(y) = \sum_{i=1}^n \sigma_i(d_i\sigma_i - 1)y_i^2,$$

So as the Nash-cournot model, the Bertrand model can be formulated as a mixed variational inequality $MV(D)$. Note that since $\sigma_i(d_i\sigma_i - 1)$, $i = 1, \dots, n$ may be negative, the function $\psi(\cdot)$ may not be convex.

3 An Algorithm for Global Equilibria

In this section we describe an algorithm for approximating a global equilibrium point of the model. The idea of the proposed algorithm is quite natural, it uses the convex envelope of the concave cost function to approximate the original model with the one having piecewise linear concave costs. The latter can be solved by an algorithm developed in [14] to obtain an approximate equilibrium point x . Then

by evaluating the gap function we can check whether or not the obtained point x is an ϵ -equilibrium point. If not, we use an adaptive rectangular bisection to get a better approximate point. Thanks to the rectangular structure of the strategy set and separability of the cost function, the proposed algorithm can be implemented easily.

3.1 A Search-Check-Branch Algorithm

First we recall [8] that the convex envelope of a function φ on a convex set C is the convex function on C , denoted by $co_C\varphi$ such that $co_C\varphi(x) \leq \varphi(x)$ for every $x \in C$, and if ξ is any convex function on C satisfying $\xi(x) \leq \varphi(x)$ for every $x \in C$, then $\xi(x) \leq co_C\varphi(x)$ for every $x \in C$. It is well known [8] that the convex envelope of a concave function is affine, and that if $C = C_1 \times \dots \times C_N$ is compact and φ is separable, i.e., $\varphi(x_1, \dots, x_N) = \sum_{j=1}^N \varphi_j(x_j)$ then $co\varphi(x) = \sum_{j=1}^N co\varphi_j(x_j)$ where $co\varphi_j$ is the convex envelope of φ_j over C_j . Clearly, since $h_i, i > n$ is affine, $h_i \equiv coh_i$ on every convex set.

The algorithm we are going to describe is a search-check-branch procedure. For a given tolerance $\epsilon \geq 0$, at each iteration, the algorithm consists of three steps. The search-step requires solving convex quadratic programs for the approximate model with piecewise linear concave cost functions to obtain an approximate equilibrium point. The check-step uses the gap function presented in the preceding section to check whether the obtained solution is an ϵ -equilibrium point or not yet. The branch-step employs an adaptive rectangular bisection performed in the space of concave variables to obtain a better approximation for the model.

To be precise, suppose that the strategy set $D := D_1 \times \dots \times D_N$. Let

$$I^0 := D_1 \times \dots \times D_n, \quad J^0 := D_{n+1} \times \dots \times D_N.$$

For a n -dimensional subbox $I \subseteq I^0$, define

$$D_I := \{x^T := (x_1, \dots, x_N) : (x_1, \dots, x_n) \in I, (x_{n+1}, \dots, x_N) \in J^0\} \quad (3.1)$$

and consider the convex mixed variational inequality $CMV(D_I)$ defined as

$$\begin{aligned} &\text{Find } x^{D_I} \in D_I \text{ such that:} \\ &\langle \tilde{B}_1 x^{D_I} - \alpha, y - x^{D_I} \rangle + y^T B_1 y + co_I h(y) + \sum_{j=n+1}^N h_j(y) \\ &- (x^T B_1 x^{D_I} + co_I h(x^{D_I}) + \sum_{j=n+1}^N h_j(x^{D_I})) \geq 0 \quad \forall y \in D_I. \end{aligned}$$

In what follows we write $x^{D_I} = (x^I, x^J)$ with $x^I \in I, x^J \in J^0$.

Since $co_I h(\cdot)$ is affine, by Lemma 2.1, this problem is reduced to the strongly convex quadratic program

$$\min_{x \in D_I} \{x^T Q x + (c^J)^T x\}, \quad (QD_I)$$

where $Q := \frac{1}{2}\tilde{B}_1 + B_1$, $c^I = (c^{I_1}, \dots, c^{I_N})^T$ with $c^{I_j} := (a^{I_j} - \alpha)(j = 1, 2, \dots, N)$.

Suppose that each strategy set D_j ($j = 1, \dots, n$) has been divided into interval $D_{j,1}, \dots, D_{j,k_j}$ on each of them the cost function is affine. Let Δ be the set of n -dimensional subboxes defined as

$$\Delta := \{B := I_1 \times \dots \times I_n : I_j \in \{D_{j,1}, \dots, D_{j,k_j}\}, j = 1, \dots, n\}.$$

Define Σ as the family of N -dimensional subboxes by taking

$$\Sigma := \{I = B \times J^0 : B \in \Delta\}.$$

Let us define the gap function for the model with piecewise concave cost function, that is

$$\bar{g}(x) := -\min_{y \in D} \bar{\phi}(x, y) \quad (3.2)$$

where

$$\bar{\phi}(x, y) := \langle \tilde{B}_1 x - a, y - x \rangle + y^T B_1 y - x^T B_1 x + \bar{h}(y) - \bar{h}(x), \quad (3.3)$$

where \bar{h} is the piecewise linear concave function obtained by taking the convex envelope of h on each element of Σ .

Note that, since h_i is affine on D_i for every $i = n+1, \dots, N$, the convex envelope of h_i on any subbox coincides with h_i for every $i = n+1, \dots, N$. In particular, $co_D h$ is affine and

$$co_D h(x) = \sum_{j=1}^n co_{I^0} h_j(x) + \sum_{i=n+1}^N h_i(x).$$

First we briefly describe the algorithm in [14] as follows.

Algorithm 1 (Search-and-Check). Choose a tolerance $\epsilon \geq 0$.

Step 1: Select a subbox $I \in \Sigma$.

Step 2: Solve the strongly convex quadratic problem (QD_I) to obtain its unique solution x^{D_I} .

Step 3: a) If $\bar{g}(x^{D_I}) \leq \epsilon$, terminate: x^{D_I} is an ϵ -equilibrium point for piecewise concave cost model.

(b) If $\bar{g}(x^{D_I}) > \epsilon$ and $\Sigma = \emptyset$, then terminate: the model has no equilibrium point. Otherwise, replace Σ by $\Sigma \setminus \{I\}$ and return to Step 1.

It is obvious that in the worst case, the algorithm searches all subboxes in Σ , however the computational results reported in [14] show that by using the gap

function, in general, the algorithm finds an ϵ -equilibrium point without searching all elements of Σ .

Using Algorithm 1 described above we can develop an algorithm for approximating an equilibrium point of the model where some of the cost functions are concave. The idea is quite natural. In fact, at each iteration we use the convex envelope of the concave cost function to obtain a model with piecewise linear concave cost function to which we can apply the search-and-check algorithm 1 to obtain an approximate equilibrium point. If the obtained point is not yet an ϵ -equilibrium point, we use an adaptive rectangular bisection (Rule 1 below) to reduce the difference between the concave function and its convex envelope to obtain a better approximate equilibrium point for the original model, and so on.

An adaptive rectangular bisection (Rule 1). Let I be a given n -dimensional subbox of $D_1 \times \dots \times D_n$. For $x^I \in I$, define

$$j_{max} := \operatorname{argmax}_{1 \leq j \leq n} \{h_j(x_j^I) - \operatorname{coh}_j(x_j^I)\}.$$

Then we bisect I into two boxes via the middle point of edge $I_{j_{max}}$. We call this middle point the *bisection point* and j_{max} the *bisection index*.

For this bisection we have the following lemma whose proof can be found, e.g., in [12], [13].

Lemma 3.1 *Let $\{I^k\}$ be an infinite sequence of boxes generated by the adaptive rectangular bisection Rule 1 such that $I^{k+1} \subset I^k$ for every k . Let b^k be the bisection point and j_k be the bisection index for I^k . Then $\lim_{k \rightarrow \infty} (h_{j_k}(b^k) - \operatorname{co}_{I^k} h_{j_k}(b^k)) = 0$. Consequently, $\{I_{j_k}\}$ tends to a singleton, provided h_{j_k} is (concave) not affine on I_{j_k} for every j_k .*

For each subbox I having n -edges I_j ($j = 1, \dots, n$) we define

$$\rho(I_j) := \max_{t \in I_j} \{h_j(t) - \operatorname{coh}_j(t)\}$$

and

$$\rho(I) := \max\{\rho(I_j) : j = 1, \dots, n\}. \quad (3.4)$$

The algorithm now can be described as follows:

Algorithm 2 (Search-Check-Branch for global equilibria).

Initial step. Choose a tolerance $\epsilon \geq 0$. Take the initial box $I^0 := D_1 \times \dots \times D_n$. Solve the convex mixed variational inequality $\operatorname{CMV}(D)$ defined as

$$\text{Find } x \in D : \bar{\Phi}_0(x, y) := \langle \tilde{B}_1 x - a, y - x \rangle + y^T B_1 y - x^T B_1 x + \operatorname{co}_D h(y) - \operatorname{co}_D h(x) \geq 0 \forall y \in D,$$

which is equivalent to the strongly convex quadratic program (QD_{I^0}) to obtain its unique solution u^0 .

Let $\Sigma_0 := \{I^0\}$. $x^0 := u^0$.

Iteration k ($k = 0, 1, \dots$)

At the beginning of each iteration k we have:

- Σ_k : a finite family of n -dimensional subboxes of I^0 ;
- $u^k = (u^{k_1}, u^{k_2})$ with $u^{k_1} \in I^0$, $u^{k_2} \in J^0$, the equilibrium point of the model with piecewise linear concave function;
- $x^k \in D$: the currently best feasible point, i.e., $g(x^k)$ is smallest among the obtained feasible points so far.

Step 1. a) If $g(x^k) \leq \epsilon$, terminate: x^k is an ϵ -equilibrium point of the original model.

b) If $g(x^k) > \epsilon$, choose $I^k \in \Sigma_k$ such that

$$\rho(I^k) = \max\{\rho(I); I \in \Sigma_k\}.$$

Step 2. Use the bisection Rule 1 described above to bisect I^k into two boxes I^{k+} and I^{k-} . Let j_k be the bisection index for I^k .

Step 3. Solve the strongly convex quadratic program (QD_I) with $I = I^{k-}$ and $I = I^{k+}$ to obtain x^{k+} and x^{k-} respectively.

Step 4. If either $g(x^{k+}) \leq \epsilon$ or $g(x^{k-}) \leq \epsilon$, terminate.

Otherwise, update x^k , Σ_k and the linear piecewise concave cost function by taking respectively

$$x^{k+1} \in \{x^k, x^{k+}, x^{k-}\} \text{ such that } g(x^{k+1}) = \min\{g(x^k), g(x^{k-}), g(x^{k+})\},$$

$$\Sigma_{k+1} = (\Sigma_k \setminus \{I^k\}) \cup \{I^{k-}, I^{k+}\}.$$

Step 5. Compute the convex envelope of function h_{j_k} on the edge j_k of the subboxes I^{k-} , I^{k+} , thereby to obtain the new approximation bifunction

$$\bar{\Phi}_{k+1}(x, y) := \langle \tilde{B}_1 x - a, y - x \rangle + y^T B_1 y - x^T B_1 x + co_{k+1} h(y) - co_{k+1} h(x),$$

where $co_{k+1} h$ is the convex envelope of h obtained by replacing the convex envelope of h_{j_k} on the edge j_k of I^k by the convex envelope of h_{j_k} on the edge j_k of I^{k-} and I^{k+} . Then use Algorithm 1 with the just obtained piecewise linear concave cost function to solve the newly approximated piecewise linear concave model to obtain u^{k+1} .

Increase k by one and go to Step 1 of iteration k .

Suppose that every model with piecewise linear concave cost function has an ϵ -equilibrium point for any $\epsilon > 0$. Then we have the following convergence result.

Convergence Theorem . (i) *If the algorithm terminates at iteration k then x^k is an ϵ -equilibrium point.*

(ii) *If the algorithm does not terminate, it generates an infinite sequence $\{x^k\}$ such that any its cluster point is an equilibrium point whenever the model has an equilibrium point. Furthermore $g(x^k) \searrow 0$ as $k \rightarrow \infty$.*

Proof. The statement (i) is obvious. To prove statement (ii) we suppose that the algorithm never terminates. Let x^* be any cluster point of $\{x^k\}$. Then there exists a subsequence of $\{x^{k_q}\}$ that tends to x^* . Thus the corresponding sequence of selected intervals has a nested sequence, which, by taking a subsequence if necessary, we denote also by I^{k_q} . Since I^{k_q} is the box to be bisected at iteration k_q , by Lemma 3.1, $\{I^{k_q}\}$ tends to a singleton, which implies that $h_{j_q}(x_{j_q}) - \text{coh}_{j_q}(x_{j_q}) \rightarrow 0$ as $q \rightarrow \infty$ (j_q is the bisection index at iteration k_q). By the rule for selecting the bisection index, we have $h_j(x_j) - \text{coh}_j(x_j) \rightarrow 0$ for every j . Since u^{k_q} is an equilibrium point of the model with piecewise linear concave cost function, we have $\bar{g}_{k_q}(u^{k_q}) = 0$ for every q , where \bar{g}_{k_q} is the gap function for the piecewise linear concave cost model at iteration k_q . By the definition of the gap function g for the original model and of \bar{g} for the approximate model, and the rule for selecting bisection index, we can write

$$\bar{g}(u^{k_q}) - 2\sigma_{k_q} \leq g(u^{k_q}) \leq \bar{g}(u^{k_q}) + 2\sigma_{k_q} \quad \forall q.$$

Letting $q \rightarrow \infty$, since $\sigma_{k_q} \rightarrow 0$, $u^{k_q} \rightarrow u^*$, by continuity of g , we obtain $g(u^*) = 0$.

On the other hand, since x^{k_q} is the currently best feasible point obtained at iteration k_q , we have $0 \leq g(x^{k_q}) \leq g(u^{k_q})$. Letting $q \rightarrow \infty$, by continuity of g , we obtain $0 \leq g(x^*) = g(u^*) = 0$, which means that x^* is an equilibrium of the model. Note that, since x^k is the currently best feasible point obtained at iteration k , by definition, the sequence $\{g(x^k)\}$ is nonincreasing. Since the whole sequence $\{x^k\}$ is bounded, it has a subsequence $\{x^{k_j}\}$ converging to some \bar{x} . Then, as we just have shown, \bar{x} is an equilibrium point which implies $g(\bar{x}) = 0$. Then the whole sequence $\{g(x^k)\}$ tends to 0 as well. \square

Remark 3.2 *In order to save the memory, we may use a criterion to delete every subbox that does not contain an equilibrium point in it.*

The following lemma gives a criterion that can be used to check whether a subbox contains an equilibrium point or not. In fact, for a subbox $D_I := \{x \in D :$

$l^I \leq x \leq u^I$ }, let us define the number

$$\tilde{g}(D_I) := - \min_{y \in D_I} \{ \langle \tilde{B}_1 u^I - a, y \rangle + y^T B_1 y + h(y) \} - (l^I)^T B_1 l^I + a^T u^I - h(l^I).$$

Then we have the following lemma:

Lemma 3.3 *Suppose x^{D_I} is an optimal solution of Problem (QD_I) .*

(i) *If $co_I h(x^{D_I}) = h(x^{D_I})$ then x^I is the equilibrium point the model restricted on D_I .*

(ii) *If $\tilde{g}(D_I) > 0$, the subbox D_I contains no equilibrium point of the model.*

Proof (i) Since x^{D_I} is the solution of (QD_I) , we have

$$\langle \tilde{B}_1 x^{D_I} - a, y - x^{D_I} \rangle + y^T B_1 y + co_I h(y) - (x^I)^T B_1 x^{D_I} - co_I h(x^{D_I}) \geq 0, \forall y \in D_I.$$

Note that $h(y) \geq co_I h(y)$, $\forall y \in D_I$, by the assumption, $co_I h(x^{D_I}) = h(x^{D_I})$, we obtain

$$\langle \tilde{B}_1 x^{D_I} - a, y - x^{D_I} \rangle + y^T B_1 y + h(y) - (x^I)^T B_1 x^{D_I} - h(x^{D_I}) \geq 0$$

for every $y \in D_I$, which means that x^{D_I} is the equilibrium point the model restricted on D_I .

(ii) We now prove that $g(x) > 0$ for all $x \in D_I$. Indeed, by definition

$$\Phi(x, y) = \langle \tilde{B}_1 x - a, y \rangle + y^T B_1 y + h(y) - x^T B_1 x + a^T x - h(x).$$

Since $y \geq 0$, \tilde{B}_1 and B_1 are non-negative matrices, $h_i(\cdot)$ ($i = 1, 2, \dots, n$) are increasing functions and $l^I \leq x \leq u^I$ for every $x \in D_I$, we can write, for every $y \in D$ and $x \in D_I$,

$$\begin{aligned} \Phi(x, y) &= \langle \tilde{B}_1 x - a, y \rangle + y^T B_1 y + h(y) - x^T B_1 x + a^T x - h(x) \\ &\leq \langle \tilde{B}_1 u^I - a, y \rangle + y^T B_1 y + h(y) - (l^I)^T B_1 l^I + a^T u^I - h(l^I). \end{aligned} \quad (3.5)$$

By the definition of $\tilde{g}(D_I)$, it follows from (3.5) that

$$g(x) := - \min_{y \in D} \phi(x, y) \geq - \min_{y \in D_I} \phi(x, y) \geq \tilde{g}(D_I) > 0 \quad \forall x \in D_I,$$

which implies that D_I does not contain an equilibrium point. \square

4 An Algorithm for Local Equilibria

Using the fact that a point $x^* \in D$ is an equilibrium point of the model if and only if the gap function $g(x^*) = 0$, we say that a point \bar{x} is a *local equilibrium point* of the model if there exists an open set $B \subset D$ such that $\bar{x} \in B$, $g_B(\bar{x}) = 0$, where g_B stands for the gap function of the model restricted on B . Note that because of concavity of the cost function, in this equilibrium Nash-Cournot model, a local equilibrium point may not be a global one.

In this section we propose an algorithm for approximating a local equilibrium point of the model by using again the gap function.

Namely, for a subbox

$$I := \{x = (x_1, \dots, x_n)^T : l_i \leq x_i \leq u_i, i = 1, \dots, n\},$$

let, as before, D_I be the subbox of D consists of all points $x^T = (x_1, \dots, x_n, \dots, x_N)$ such that $(x_1, \dots, x_n) \in I$. That is

$$D_I = \{x = (x_1, \dots, x_N)^T, l_i \leq x_i \leq u_i, i = 1, \dots, N\}.$$

Then define the gap function g_{D_I} restricted on D_I by taking

$$\begin{aligned} g_{D_I}(x) = & - \sum_{i=1}^N \min_{l_i \leq y_i \leq u_i} \left\{ \beta y_i^2 + \left(\beta \sigma_{(-i)}(x) - \alpha \right) y_i + h_i(y_i) \right\} \\ & + \beta \sum_{i=1}^N x_i^2 - a^T x + \sum_{i=1}^N h_i(x_i), \end{aligned} \tag{4.1}$$

where $\sigma^{(-i)}(x) := \sum_{j \neq i}^N x_j$.

As before we use the convex envelope of the concave function h on each subbox D_I to obtain a convex mixed variational inequality whose solution can be obtained by solving a strongly convex quadratic over D_I . If it happens that at the obtained solution the values of the cost function and its convex envelope on D_I coincide, this solution is a local equilibrium point of the model. Otherwise we bisect I to reduce the difference between the cost function and its convex envelope on D_I . Note that if $g_{D_I}(x) = 0$ for some $x \in D_I$, then x is a local equilibrium point. Thus, if $x \in D_I$ and $g_I(x) \leq \epsilon$, then x is an ϵ -local equilibrium point.

Since x^{D_I} is the equilibrium point of the model with respect to D_I , from the definitions of the convex envelope of h and the gap function restricted on D_I , it follows that $h(x^{D_I}) - \text{co}_I h(x^{D_I}) = 0$ implies $g_{D_I}(x^{D_I}) = 0$.

The algorithm now can be described as follows.

Algorithm 3 (Search-Check-Branch for local equilibria).

Initial step. Choose tolerances $\epsilon > 0$ and solve Problem (QD) to obtain its optimal solution x^{I^0} .

Compute $\rho_0 := \rho(I^0)$ and $\epsilon_0 := g_D(x^{I^0})$. Set the initial box I^0 and let $\Gamma_0 := \{I^0\}$.

Iteration k ($k = 0, 1, \dots$). At the beginning of each iteration k we have

- Γ_k : a finite family of n -dimensional subboxes of I^0 ;
- $\epsilon_k = \min\{g_{D_I}(x^{D_I}) : I \in \Gamma_k\}$: where x^{D_I} is the optimal solution of the convex quadratic program (QD $_I$);

Step 1. (Stopping criteria) If $\epsilon_k \leq \epsilon$, terminate: x^{D_I} with $\epsilon_k = g_{D_I}(x^{D_I})$ is an ϵ -local equilibrium point.

Step 2. (Selection) Choose $I^k \in \Gamma_k$ such that

$$\rho_k := \rho(I^k) = \max\{\rho(I) : I \in \Gamma_k\}.$$

Step 3. (Bisection): Divide the subbox I^k into two subboxes I^{k+} and I^{k-} by the bisection Rule 1.

Step 4. Solve the strongly convex quadratic programs (QD $_I$) with $I := I^{k+}$ and $I := I^{k-}$ to obtain the optimal solutions x^{k+} and x^{k-} respectively. Compute $\rho(I^{k+})$ and $\rho(I^{k-})$.

Step 5. Let $\epsilon_{k+1} := \operatorname{argmin}\{\epsilon_k, g_{D_I}(x^{D_I})\}$ with $I = I^{k-}$ and $I = I^{k+}$.

Step 6. (Updating) If $g(x^{D_I}) > 0$ delete I from further consideration.

Let Γ_{k+1} be the remaining set. If $\Gamma_{k+1} = \emptyset$, terminate: the model has no ϵ -local equilibrium point. Otherwise, go to iteration k with $k := k + 1$.

Convergence. *The algorithm terminates after a finite iteration yielding an ϵ -local equilibrium point whenever it does exist.*

The proof of this convergence result is evident because of the fact that $\epsilon > 0$, that the sequence of selected boxes tends to a singleton and that the gap function is continuous.

Remark 4.1 *If for every i , the cost function h_i satisfies the condition*

$$\nabla^2 h_i(y_i) \geq -2\beta_i \quad \forall y_i \in D_i. \quad (4.2)$$

Then the model admits a solution.

Indeed, for each $x \in D$, let $H_i(x)$ be the solution set of the problem

$$\min_{y_i \in D_i} \left\{ \varphi_i(x_{-i}, y_i) := \beta y_i^2 + \left(\beta \sum_{j \neq i}^N x_j - \alpha_i \right) y_i + h_i(y_i) \right\}. \quad (QD_i(x)).$$

It is easy to check that condition (4.2) ensures that the object function of this problem is convex in y_i . Thus $H_i(x)$ is a closed convex of the interval D_i . Since the objective function of this problem is continuous and the feasible is compact, the solution set $H_i(x)$ is a upper semicontinuous mapping from D into itself, by well-known Kakutani fixed point, the mapping $H(x) := H_1(x) \times \dots \times H_N(x)$ has a fixed point x^* , which is also an equilibrium point of the model.

Note that both the cost functions

$$h_i(y_i) = \ell_i y_i - d_i y_i^2$$

with $\beta_i > d_i > 0$ for all $i = 1, \dots, n$. used in [2] and

$$h_i(y_i) = \mu_i y_i + \ln(1 + \gamma_i y_i),$$

with $\gamma_i > 0$ and $\gamma_i^2 \leq 2\beta_i$, $\forall i = 1, \dots, n$ satisfy condition (4.2).

5 Computational Results and Experiments

The proposed two algorithms were implemented in MATLAB. The programs were executed on a PC Core 2Duo 2*2.0 GHz, RAM 2GB. We tested the program on different groups of problems, each of them contains ten problems of different sizes N and n , but having randomly generated input data. Namely, for each problem, the numbers α , β , μ_i ($i = n + 1, \dots, N$) are randomly generated in the interval [20.30], [0.001, 0.005] and [10.20] respectively. We take the cost functions of the form

$$h_j(x_j) = a_j x_j + \ln(1 + \gamma_j x_j), (j = 1, \dots, n), h_i(x_j) := \mu_i x_i (i = n + 1, \dots, N). \quad (5.1)$$

where γ_j and a_j are randomly generated in [7, 15] and [2, 7] respectively. The strategy set of firm i is $D_i := [0, u_i]$ where each u_i is randomly generated in the interval [100.500].

The obtained results are reported in Table 4.1 below, where we use the following headings:

- N : number of the firms;
- n : number of the firms having concave (but not affine) cost;
- *Average time*: the average time (in second) needed to solve one problem;
- *Average iter*: the average numbers of iterations for one problem.

- *Glob-GSCB*: number of problems for which an equilibrium point was obtained by Search-Check-Branch Algorithm for global equilibria.
- *Glob-LCB*: number of problems for which a global optimal solution was obtained by Search-Check-Branch for local equilibria.

Size		GSCB-Alg.			LSCB-Alg.		
N	n	Average time	Average iter.	Glob-GSCB	Average time	Average iter	Glob-LSCB
5	5	0.00	1	10	0.03	1	10
50	5	8.98	133	10	0.06	1	10
100	5	17.89	171	10	0.18	2	8
200	5	1.78	7	10	0.29	2	8
10	10	9.65	308	10	0.05	1	8
50	10	82.35	1141	10	0.22	4	4
100	10	47.05	445	10	0.43	5	7
200	10	41.06	203	10	0.33	2	7
20	20	127.15	2478	10	1.29	24	1
50	20	98.10	1231	10	0.50	7	3
100	20	105.00	914	10	1.72	16	3
200	20	440.88	2216	10	1.94	11	5
30	30	286.57	3754	10	0.89	13	2
50	30	246.44	2901	10	1.23	17	1
100	30	872.27	7193	10	0.73	7	2
200	30	750.72	3514	10	2.70	15	4
40	40	515.10	5944	10	3.09	40	2
50	40	1332.10	14820	9	7.69	97	0
100	40	646.53	5213	10	2.85	26	0
200	40	898.09	4169	9	3.83	21	1
100	100	Skip	-	-	20.21	148	0
200	100	Skip	-	-	132.64	568	0
200	200	Skip	-	-	107.63	400	0
300	200	Skip	-	-	252.67	579	0

Table 4.1

From the obtained results reported in Table 4.1 we can conclude the followings for the tested concave cost functions given as (5.1).

- Algorithm 2 for global equilibrium point can solve models with a moderate number ($n \leq 40$) of concave cost functions, while Algorithm 3 can solve models where the number of concave cost functions much larger.

- For models where the number of the firms having concave cost is somewhat large ($n \geq 40$), the local equilibrium point obtained by the local algorithm is often not a global one.

6 Conclusion.

A Nash-Cournot oligopolistic equilibrium model involving concave cost functions may have local equilibrium points that are not global ones. We have approximated such a model with the one having piecewise linear concave function by using the convex envelope of a separable concave function over a box. Based upon this approximation we have proposed two algorithms for approximating a global as well as local equilibrium points that employ a gap function as a stopping criterion for the algorithms, and an update rectangular bisection to make the approximation better. Some computational results have been reported showing efficiency of the proposed algorithms for models where the number of the concave (but not affine) cost functions is not large ($n \leq 40$) for global algorithm, and ($n \leq 200$) for local one. An open question that would be interesting for further consideration is to find a differentiable gap function, for which a local optimization algorithms such as descent ones in [6] or DCA in [16] could be applied efficiently.

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