SOME RESULTS ON FRAMES OF SUBSPACES AND G-FRAMES

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Abstract In this paper we discuss some aspects where frames of subspaces behave differently from frames. Several examples to make clearer the behaviour of frames of subspaces are given. We also improve some results on g-frames. Moreover, we extend the notion of redundancy to g-frames and show that most of the desirable properties of lower and upper redundancies on frames and frames of subspaces can carry over g-frames. We also study the relationship between redundancy of g-frames and their dual g-frames, redundancy for infinite g-frames and the excess of g-frames.

1 Introduction and Preliminaries

Frames were first introduced by Duffin and Schaeffer [11] in 1952 in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering (see e.g., [7],[8], [9], [10],[12],[14],[15]).

Besides traditional applications as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [12] and to design high-rate constellations with full diversity in multiple-antenna code design [14]. To handle these emerging applications of frames, new methods have to developed. One method is to first build frames locally and then piece them together to obtain frames for the whole space. With this idea, in [5] Casazza and Kutyniok introduced the definition of a frame of subspaces. It turns out that in many ways frames of subspaces behave as a generalization of frames. However, there are some aspects where frames of subspaces behave differently from frames. In this paper we discuss some of

[1]2010 *Mathematics Subject Classification*: 42C15; 46C05; 41A58.

[2]*Keywords*: Frame, Frame of subspaces, G-frame, Redundancy, Excess.

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these aspects. We also provide several examples to make clearer the behaviour of frames of subspaces.

In [19] Sun introduced the notion of g-frame which is a generalization of frame and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context. In this paper we deal with the question that when we get a g-frame after deleting an element from a given g-frame and present some improvements in results on g-frames in [13].

In [4] the authors introduced a quantitative notion of redundancy for finite frames which they called upper and lower redundancies, that match better with an intuitive understanding of redundancy for finite frames in a Hilbert space. In 2016, Rahimi et al., in [18] extended the concept of redundancy of frames to frames of subspaces. In this article we extend the notion of redundancy to g-frames and show that most of the desirable properties on frames and frames of subspaces can carry over g-frames. In addition, we also study the relationship between redundancy of g-frames and their dual g-frames, redundancy for infinite g-frames and the excess of g-frames.

First we briefly recall the definitions and some basic properties of frames for a Hilbert space. For more information we refer to the monograph of Daubechies [10] or the book of Christensen [7].

Throughout this paper, let H be a separable Hilbert space if not otherwise specified and I be a countable index set. A family $\{f_i\}_{i\in I}$ is a frame for H, if there exist $0 < A \le B < \infty$ such that for all $f \in H$,

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{1}$$

The numbers A, B are called lower and upper frame bounds, respectively (the largest A and the smallest B for which (1) holds are the optimal frame bounds). The family $\{f_i\}_{i\in I}$ is called a tight frame if in (1) the constants A and B can be chosen so that $A = B$, a Parseval frame if $A = B = 1$. We call a frame $\{f_i\}_{i\in I}$ uniform (or equal norm), if there exists a constant c such that $\|f_i\| = c$ for all $i \in I$. A frame is exact if it ceases to be a frame whenever any element is deleted from the sequence $\{f_i\}_{i\in I}$. We say that a frame $\{f_i\}_{i\in I}$ is a Riesz frame, if every subfamily of the sequence $\{f_i\}_{i\in I}$ is a frame for its closed linear span with uniform frame bounds A and B.

The frame operator $S_F(g) := \sum_{i\in I} \langle g, f_i\rangle f_i$ associated with a frame $F = \{f_i\}_{i\in I}$ is a bounded, invertible, and positive operator mapping H onto itself. This provides the reconstruction formula

$$g = \sum_{i\in I} \langle g, f_i\rangle \tilde{f}_i = \sum_{i\in I} \left\langle g, \tilde{f}_i \right\rangle f_i, \forall g \in H$$

where $\tilde{f}_i = S_F^{-1}(f_i)$. The family $\{\tilde{f}_i\}_{i\in I}$ is also a frame for H, called the canonical dual frame of $\{f_i\}_{i\in I}$. A frame $\{g_i\}_{i\in I}$ satisfying

$$g = \sum_{i\in I} \langle g, f_i\rangle\, g_i = \sum_{i\in I} \langle g, g_i\rangle\, f_i, \forall g \in H$$

is called an alternate dual frame of $\{f_i\}_{i\in I}$.

We say that a sequence $\{f_i\}_{i\in I}$ in H is complete if the span of $\{f_i\}_{i\in I}$ is dense in H. We can check that if $\{f_i\}_{i\in I}$ is a frame for H then it must be complete.

Theorem 1. [7] *The removal of a vector from a frame leaves either a frame or an incomplete set.*

In [4], Bodmann, Casazza, Kutyniok gave the definition of redundancy function, the upper redundancy and the lower redundancy for a frame in a finite-dimensional Hilbert space as follows.

We denote the unit sphere in a Hilbert space H by $\mathbb{S} = \{x \in H : \|x\| = 1\}$. Let $< y >$ denote the span of some $y \in H$ and $\pi_{<y>}$ the orthogonal projection onto $< y >$.

Definition 1. [4] Let $F = \{f_i\}_{i=1}^k$ be a frame for a finite-dimensional real or complex Hilbert space H. For each $f \in \mathbb{S}$, the redundancy function $\mathcal{R}_F : \mathbb{S} \to \mathbb{R}^+$ is defined by

$$\mathcal{R}_F(f) = \sum_{i=1}^{k} \|\pi_{<f_i>}(f)\|^2 = \sum_{\{i:f_i\neq 0\}} \|f_i\|^{-2} |\langle f, f_i\rangle|^2.$$

The redundancy function measures the concentration of frame vectors around each point.

Since the redundancy function $\mathcal{R}_F$ is continuous on the unit sphere $\mathbb{S}$ which is compact in the finite-dimensional space H, the function assumes its maximum and its minimum on $\mathbb{S}$.

Definition 2. [4] Let $F = \{f_i\}_{i=1}^k$ be a frame for a finite-dimensional real or complex Hilbert space H. The upper redundancy of F is defined by

$$\mathcal{R}_F^+ = \max_{f\in\mathbb{S}} \mathcal{R}_F(f)$$

and the lower redundancy of F by

$$\mathcal{R}_F^- = \min_{f\in\mathbb{S}} \mathcal{R}_F(f).$$

Moreover, F has a uniform redundancy if $\mathcal{R}_F^- = \mathcal{R}_F^+$.

The upper and lower redundancies obtained from the redundancy function satisfy all of desirable properties (see [4]).

Now we briefly recall the definition of frames of subspaces (another name of frame of subspaces also used in literature is fusion frame). For more information we refer to the papers by Asgari and Khosravi [2], Casazza and Kutyniok [5], Casazza, Kutyniok, and Li [6].

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of H and $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{W_i\}_{i\in I}$ is called a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 \leq B\|f\|^2. \tag{2}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. The numbers A, B are called lower and upper frame bounds for the frame of subspaces, respectively (the largest A and the smallest B for which (2) holds are the optimal frame bounds). The family $\{W_i\}_{i\in I}$ is called a A-tight frame of subspaces with respect to $\{v_i\}_{i\in I}$ if in (2) the constants A and B can be chosen so that $A = B$, a Parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ provided that $A = B = 1$. We call a frame of subspaces with respect to $\{v_i\}_{i\in I}$ v-uniform, if $v = v_i = v_j$ for all $i, j \in I$. Moreover, we say that a frame of subspaces $\{W_i\}_{i\in I}$ is a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$, if there exist constants $A, B > 0$ so that every subfamily $\{W_i\}_{i\in J}$ with $J \subset I$ is a frame of subspaces with respect to $\{v_i\}_{i\in J}$ for its closed linear span with uniform frame bounds A and B.

A family of subspaces $\{W_i\}_{i\in I}$ for H is called complete if $\overline{\operatorname{span}}\,\{W_i\}_{i\in I} = H$.

To check completeness of a frame of subspaces, we have the following useful characterization.

Lemma 1. *[5] Let $\{W_i\}_{i\in I}$ be a family of subspaces with respect to $\{v_i\}_{i\in I}$ for H and for each $i \in I$ let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Then the following conditions are equivalent.*

(1) $\{W_i\}_{i\in I}$ is complete.

(2) $\{e_{ij}\}_{i\in I, j\in J_i}$ is complete in H.

In 2016, Rahimi, Zandi, and Daraby [18] introduced the notion of redundancy of frames of subspaces. Many results on redundancy of frames remain true in the context of frames of subspaces.

Definition 3. [18] Let $W = \{W_i\}_{i=1}^k$ be a frame of subspaces with respect to $\{v_i\}_{i=1}^k$ for a finite-dimensional Hilbert space H . For each $f \in \mathbb{S}$, the redundancy function $\mathcal{R}_W : \mathbb{S} \to \mathbb{R}^+$ is defined by $\mathcal{R}_W(f) = \sum\limits_{i=1}^{k} \|P_{W_i}(f)\|^2$.

This notion is reminiscent of the definition of redundancy function for finite frames if dim $W_i = 1$ for all i.

Definition 4. [18] For the frame of subspaces $W = \{W_i\}_{i=1}^k$ with respect to $\{v_i\}_{i=1}^k$ for a finite-dimensional Hilbert space H, the upper redundancy is defined by

$$\mathcal{R}_W^+ = \max_{f \in \mathbb{S}} \mathcal{R}_W(f)$$

and the lower redundancy of W by

$$\mathcal{R}_W^+ = \min_{f \in \mathbb{S}} \mathcal{R}_W(f).$$

We say W has a uniform redundancy if $\mathcal{R}_W^- = \mathcal{R}_W^+$.

Definition 5. [5] We call a frame of subspaces $\{W_i\}_{i \in I}$ with respect to $\{v_i\}_{i \in I}$ for H a Riesz decomposition of H, if every $f \in H$ has a unique representation $f = \sum\limits_{i \in I} f_i, f_i \in W_i$.

Now we give a short review of g-frames. For more information we refer to the papers [1, 13, 16, 17, 19]. Let H and K be two separable Hilbert spaces and $\{W_i\}_{i \in I}$ be a sequence of closed subspaces of K and $L(H, W_i)$ be the collection of all bounded linear operators from H into W_i.

We call $\Lambda = \{\Lambda_i\}_{i \in I}$, where $\Lambda_i \in L(H, W_i)$, a generalized frame, or simply a g-frame, for H with respect to $\{W_i\}_{i \in I}$ if there are two positive constants A and B such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i(f)\|^2 \leq B\|f\|^2. \tag{3}$$

We call $\{\Lambda_i\}_{i \in I}$ a tight g-frame if in (3) the constants A and B can be chosen so that $A = B$, a Parseval g-frame provided that $A = B = 1$. We call $\{\Lambda_i\}_{i \in I}$ an exact g-frame if it ceases to be a g-frame whenever any element is deleted from the sequence. We call this family a g-frame for H with respect to W whenever $W_i = W$ for all $i \in I$. A family $\{\Lambda_i\}_{i \in I}, \Lambda_i \in L(H, W_i)$ is called g-complete, if $\{f \in H : \Lambda_i(f) = 0, \forall i \in I\} = \{0\}$. If $\{\Lambda_i\}_{i \in I}$ is g-complete and there are positive constants A and B such that for any finite subset $I_1 \subset I$ and $g_i \in W_i, i \in I_1$,

$$A \sum_{i \in I_1} \|g_i\|^2 \leq \sum_{i \in I_1} \|\Lambda_i^* g_i\|^2 \leq B \sum_{i \in I_1} \|g_i\|^2,$$

then we say $\{\Lambda_i\}_{i \in I}$ is a g-Riesz basis for H with respect to $\{W_i\}_{i \in I}$.

We say that two g-frames $\Lambda = \{\Lambda_i\}_{i \in I}$, $\Lambda_i \in L(H, W_i)$ and $\Phi = \{\Phi_i\}_{i \in I}$, $\Phi_i \in L(K, W_i)$ are similar if there is a bounded invertible operator $U : H \to K$ so that $\Lambda_i = \Phi_i U$ for all $i \in I$.

In this paper, we call g-frame $\{\Lambda_i\}_{i \in I}$ equal norm if there is a constant $c > 0$ such that $\|\Lambda_i\| = c$ for all $i \in I$ and unit norm if $c = 1$.

Proposition 1. [17] *A family* $\{\Lambda_i\}_{i\in I}, \Lambda_i \in L(H, W_i)$ *is a g-complete if and only if*

$$\overline{\operatorname{span}}\{\Lambda_i^*(H_i)\}_{i\in I} = H.$$

Definition 6. [19] We say that $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for H with respect to $\{W_i\}_{i\in I}$ if it satisfies the following conditions:
1) $\left\langle \Lambda_{i1}^* g_{i_1}, \Lambda_{i_2}^* g_{i_2} \right\rangle = \delta_{i_1,i_2} \left\langle g_{i_1}, g_{i_2} \right\rangle, \forall i_1, i_2 \in I, g_{i_1} \in W_{i_1}, g_{i_2} \in W_{i_2}$,
where Λ_i^* is the adjoint operator of Λ_i.
2) $\sum\limits_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2$.

Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{W_i\}_{i\in I}$. Sun [19] defined the g-frame operator S_Λ associated with Λ as follows:

$$S_\Lambda(f) = \sum_{i\in I} \Lambda_i^* \Lambda_i(f).$$

Similarly to the frame operator, the g-frame operator is a bounded, invertible, and positive operator mapping H onto itself. In particular, when $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame, the g-frame operator is the identity operator of H.

Let $\tilde{\Lambda}_i = \Lambda_i S^{-1}$. Then we also have the reconstruction formula

$$f = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in I} \tilde{\Lambda}_i^* \Lambda_i f.$$

and $\{\tilde{\Lambda}_i\}_{i\in I}$ is also a g-frame for H with respect to $\{W_i\}_{i\in I}$. $\{\tilde{\Lambda}_i\}_{i\in I}$ is called the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Similarly to frames, a g-frame which is not a g-Riesz basis has an alternate dual g-frame which is different from its canonical dual (see [1]).

Definition 7. [1] Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Phi = \{\Phi_i\}_{i\in I}$ be two g-frames of H with respect to $\{W_i\}_{i\in I}$ such that $f = \sum_{i\in I} \Lambda_i^* \Phi_i(f) = \sum_{i\in I} \Phi_i^* \Lambda_i(f)$ for all $f \in H$. Then Φ is called an alternate dual of Λ.

For each sequence $\{W_i\}_{i\in I}$, we define the space $\left(\sum\limits_{i\in I} \oplus W_i\right)_{l_2}$ by

$$\left(\sum_{i\in I} \oplus W_i\right)_{l_2} = \left\{\{f_i\}_{i\in I} : f_i \in W_i \text{ and } \sum_{i\in I} \|f_i\|^2 < +\infty\right\},$$

with the inner product defined by $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \rangle = \sum\limits_{i\in I} \langle f_i, g_i \rangle$. Then $\left(\sum\limits_{i\in I} \oplus W_i\right)_{l_2}$ is a Hilbert space with pointwise operators.

The analysis operator for $\Lambda = \{\Lambda_i\}_{i\in I}$ is the operator

$$T_\Lambda : H \to \left(\sum_{i\in I} \oplus W_i\right)_{l_2}$$

defined by $T_\Lambda(f) = \{\Lambda_i(f)\}_{i\in I}$. The following Propositions show similar results in frame theory can be extended to g-frames.

Proposition 2. [1] *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{W_i\}_{i\in I}$. Then $\tilde{\Lambda}$ is the canonical dual of Λ if and only if $\|T_{\tilde{\Lambda}}(f)\| \le \|T_\Phi(f)\|$ for all $f \in H$ and for each alternate dual g-frame Φ of Λ.*

Proposition 3. [1]*Let $\Lambda = \{\Lambda_i\}_{i\in I}$, $\Phi = \{\Phi_i\}_{i\in I}$ be g-frames for H with respect to $\{W_i\}_{i\in I}$. Λ and Φ are similar if and only if their analysis operators have the same ranges.*

In [19], Sun gave a characterization of g-frames and g-orthonormal bases as follows.

Theorem 2. [19] *Let $\Lambda_i \in L(H, W_i)$, $i \in I$ and $\{e_{i,j}\}_{j\in J_i}$ is an orthonormal basis for W_i where J_i is a subset of $\mathbb{N}$, $i \in I$. Set $u_{i,j} = \Lambda_i^* e_{i,j}, i \in I, j \in J_i$. Then $\{\Lambda_i\}_{i\in I}$ is a g-frame (respectively g-orthonormal basis) for H with respect to $\{W_i\}_{i\in I}$ if and only if $\{u_{i,j}\}_{i\in I, j\in J_i}$ is a frame (respectively orthonormal basis) for H.*

2 Frames of subspaces

We start with the question whether Theorem 1 is still valid for frames of subspaces? The answer is No. We consider the following examples.

Example 1. Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^\infty$. We define

$W_0 := \overline{\text{span}}\{e_{2i}\}_{i=1}^\infty, v_0 = 1.$

For even number i , $W_i := \overline{\text{span}}\{e_i\}, v_i = \dfrac{1}{i}.$

For odd number i , $W_i := \overline{\text{span}}\{e_i\}, v_i = 1.$

Then $\{W_i\}_{i=0}^\infty$ is a frame of subspaces with respect to the above sequence of weights. Indeed, for all $f \in H$, we compute

$$\begin{aligned}
\sum_{i=0}^{\infty} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 &= \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 + \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 \frac{1}{(2i)^2} + \sum_{i=0}^{\infty} |\langle f, e_{2i+1}\rangle|^2 \\
&= \|f\|^2 + \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 \frac{1}{(2i)^2} \\
&\le \|f\|^2 + \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 \\
&\le 2\|f\|^2 .
\end{aligned}$$

Therefore, $\|f\|^2 \le \sum\limits_{i=0}^{\infty} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 \le 2\|f\|^2$ and $\{W_i\}_{i=0}^{\infty}$ is a frame of subspaces with respect to the above sequence of weights.

However, $\{W_i\}_{i=1}^{\infty}$ is complete but it is not a frame of subspaces with respect to $\{v_i\}_{i=1}^{\infty}$. Indeed, for all $f \in H$, we have

$$\sum_{i=1}^{\infty} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 = \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 \frac{1}{(2i)^2} + \sum_{i=0}^{\infty} |\langle f, e_{2i+1}\rangle|^2 .$$

Choose $f = e_{2l}$. Then $\sum\limits_{i=1}^{\infty} v_i^2 \left\|\pi_{W_i}(f)\right\|^2 = \frac{1}{(2l)^2}$ which converges to 0 as $l \to \infty$.

In the above example, there is a subsequence ($v_i = \frac{1}{i}$ for even number i) of the sequence of the weights converging to 0 as $i \to \infty$ and it seems very important to make the lower condition of the definition of frame of subspaces fail. One question is that whether there is another example in which no subsequence of the sequence of the weights converges to 0?

Example 2. Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. We define

$W_i^1 = \text{span}\{e_{2i} + \frac{1}{i}e_{2i+1}\}, i = 1, 2, 3, ...$

$W_i^2 = \text{span}\{e_{2i}\}, i = 1, 2, 3, ...$

$W^3 = \text{span}\{e_1\}$

$W^4 = \overline{\text{span}}\{e_{2i+1}\}_{i=1}^{\infty}$.

Then the family $\{W_i^1\}_{i=1}^{\infty}, \{W_i^2\}_{i=1}^{\infty}, W^3, W^4$ forms a frame of subspaces for H with

respect to $v_i = 1$ for all i. Indeed,

$\sum\limits_{i=1}^{\infty} \left\|\pi_{W_i^1}(f)\right\|^2 + \sum\limits_{i=1}^{\infty} \left\|\pi_{W_i^2}(f)\right\|^2 + \left\|\pi_{W^3}(f)\right\|^2 + \left\|\pi_{W^4}(f)\right\|^2$

$$= \sum_{i=1}^{\infty} \frac{\left|\left\langle f, e_{2i} + \frac{1}{i}e_{2i+1}\right\rangle\right|^2}{1+\frac{1}{i^2}} + \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 + \sum_{i=0}^{\infty} |\langle f, e_{2i+1}\rangle|^2$$

$$= \sum_{i=1}^{\infty} \frac{\left|\left\langle f, e_{2i} + \frac{1}{i}e_{2i+1}\right\rangle\right|^2}{1+\frac{1}{i^2}} + \|f\|^2$$

$$\leq \sum_{i=1}^{\infty} \left|\langle f, e_{2i}\rangle + \frac{1}{i}\langle f, e_{2i+1}\rangle\right|^2 + \|f\|^2$$

$$\leq 2(\sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 + \frac{1}{i^2}|\langle f, e_{2i+1}\rangle|^2) + \|f\|^2$$

$$\leq 2(\sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 + |\langle f, e_{2i+1}\rangle|^2) + \|f\|^2$$

$$\leq 3\|f\|^2.$$

On the other hand,

$$\sum_{i=1}^{\infty} \frac{\left|\left\langle f, e_{2i} + \frac{1}{i}e_{2i+1}\right\rangle\right|^2}{1+\frac{1}{i^2}} + \|f\|^2 \geq \|f\|^2.$$

Therefore, the above family is a frame of subspaces for H.

However, the family $\{W_i^1\}_{i=1}^{\infty}, \{W_i^2\}_{i=1}^{\infty}, W^3$ is complete but it is not a frame of subspaces for H with respect to $v_i = 1$ for all i. Indeed, by using Lemma 1, we can check that $\overline{\text{span}}\{\{W_i^1\}_{i=1}^{\infty}, \{W_i^2\}_{i=1}^{\infty}, W^3\} = H$. Towards a contradiction assume that there is a constant $A > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \left\|\pi_{W_i^1}(f)\right\|^2 + \sum_{i=1}^{\infty} \left\|\pi_{W_i^2}(f)\right\|^2 + \|\pi_{W^3}(f)\|^2, \forall f \in H.$$

Then $A\|f\|^2 \leq \sum_{i=1}^{\infty} \frac{\left|\left\langle f, e_{2i} + \frac{1}{i}e_{2i+1}\right\rangle\right|^2}{1+\frac{1}{i^2}} + \sum_{i=1}^{\infty} |\langle f, e_{2i}\rangle|^2 + |\langle f, e_1\rangle|^2$ for all $f \in H$.

Choose$f = e_{2k+1}, k \geq 1$. Then $A \leq \frac{1}{k^2+1} \to 0$ as $k \to \infty$. We have a contradiction.

Now we consider a problem relating to Riesz frames of subspaces. Suppose that $\{W_i\}_{i\in I}$ is a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H and

$\{f_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for W_i with Riesz frame bounds A and B for all $i \in I$. We may ask whether $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for H. The following example shows that this is not always the case.

Example 3. Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. We define $W_i = \text{span}\{e_i, e_{i+1}\}, i = 1, 2, 3, ...$ Then $\{W_i\}_{i=1}^{\infty}$ is a Riesz frame of subspaces with respect to $\{v_i\}$ with $v_i = 1$ for all i with Riesz frame bounds 1 and 2. For each i, $\{e_i + e_{i+1}, e_{i+1}\}$ is a Riesz frame for W_i and we can choose the same Riesz frame bounds A and B for all i. But $\{e_i + e_{i+1}, e_{i+1}\}_{i=1}^{\infty}$ is not a Riesz frame for H because $\{e_i + e_{i+1}\}_{i=1}^{\infty}$ is not a frame for its closed linear span (By Example 5.4.6 in [7], the sequence $\{e_i + e_{i+1}\}_{i=1}^{\infty}$ is a complete sequence of H but it is not a frame for H).

Remark 1. If $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H and for each i, $\widetilde{W_i}$ is a closed subspace of W_i then it is not necessary that $\{\widetilde{W_i}\}_{i\in I}$ is a frame of subspaces for its closed linear span with the same sequence of weights. For example, let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ and

$W_i^1 := \text{span}\{e_{2i} + \frac{1}{i} e_{2i+1}, e_{2i+1}\}, i = 1, 2, 3, ...$

$W_i^2 := \text{span}\{e_{2i}\}, i = 1, 2, 3, ...$

$W^3 := \text{span}\{e_1\}.$

$\widetilde{W_i^1} := \text{span}\{e_{2i} + \frac{1}{i} e_{2i+1}\}, i = 1, 2, 3, ...$ Then the family $\{W_i^1\}_{i=1}^{\infty}, \{W_i^2\}_{i=1}^{\infty}, W^3$ forms a frame of subspaces for H with respect to $v_i = 1$ for all i and $\overline{\text{span}}_{i\in\mathbb{N}}\{\widetilde{W_i^1}, W_i^2, W^3\} = H$. However, we can see as in Example 2, that $\{\widetilde{W_i^1}\}_{i=1}^{\infty}, \{W_i^2\}_{i=1}^{\infty}, W^3$ does not form a frame of subspaces for H with respect to the above sequence of weights.

Let $W = \{W_i\}_{i=1}^{k}$ be a Riesz decomposition of H. We may ask if the redundancy of W is equal to 1. The answer is No. We consider the following example.

Example 4. Let $H = \mathbb{R}^2$, $e_1 = (0, 1)^T$, $e_2 = (1, 1)^T$, $W_1 = \text{span}\{e_1\}, W_2 = \text{span}\{e_2\}$, $v_1 = v_2 = 1$. Then $W = \{W_i\}_{i=1}^{2}$ is a Riesz decomposition of H and $\mathcal{R}_W(f) = \|P_{W_1}(f)\|^2 + \|P_{W_2}(f)\|^2 = |\langle f, e_1\rangle|^2 + \frac{|\langle f, e_2\rangle|^2}{2}$. It is not hard to see that $\mathcal{R}_W^- = \frac{1}{2}$ and $\mathcal{R}_W^+ = 1$.

3 G-frames

In [19] Sun introduced the notion of g-frames and proved that g-frames share many useful properties with frames. However not all the properties are similar.

Sun gave an example to show that Theorem 1 is not valid for g-frames and also gave the following theorem.

Theorem 3. [19] *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{W_i\}_{i\in I}$ and $\{\tilde{\Lambda}_i\}_{i\in I}$ be the canonical dual g-frame. Suppose that $i_0 \in I$.*

1) If there is some $g_0 \in W_{i_0}\backslash\{0\}$ such that $\tilde{\Lambda}_{i_0}\Lambda^*_{i_0}g_0 = g_0$ then $\{\Lambda_i\}_{i\in I\backslash i_0}$ is not g-complete in H.

2) If there is some $f_0 \in H\backslash\{0\}$ such that $\Lambda^*_{i_0}\tilde{\Lambda}_{i_0}f_0 = f_0$ then $\{\Lambda_i\}_{i\in I\backslash i_0}$ is not g-complete in H.

3) If $I_{W_{i_0}} - \Lambda_{i_0}\tilde{\Lambda}^*_{i_0}$ or $I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0}$ is bounded invertible on W_{i_0} then $\{\Lambda_i\}_{i\in I\backslash i_0}$ is a g-frame for H.

From this theorem we get the following corollary.

Corollary 1. *Suppose $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{W_i\}_{i\in I}$ and $\{\tilde{\Lambda}_i\}_{i\in I}$ is its canonical dual g-frame. If $\{\Lambda_i\}_{i\in I\backslash i_0}$ is a g-frame for H then* $\mathrm{Ker}(I_H - \tilde{\Lambda}^*_{i_0}\Lambda_{i_0}) = \{0\}$.

Proof. Suppose that there is $f_0 \in H\backslash\{0\}$ such that $\tilde{\Lambda}^*_{i_0}\Lambda_{i_0}(f_0) = f_0$. By Theorem 3 part 2, $\{\tilde{\Lambda}_i\}_{i\in I\backslash i_0}$ is not g-complete in H. By definition, there is $g_0 \in H\backslash\{0\}$ such that $\tilde{\Lambda}_i(g_0) = 0$ for all $i \in I\backslash i_0$. Since for each $i \in I$, $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, where S_Λ is the g-frame operator for $\{\Lambda_i\}_{i\in I}$, we have $\Lambda_i S_\Lambda^{-1}(g_0) = 0$ for all $i \in I\backslash i_0$. Since $\{\Lambda_i\}_{i\in I\backslash i_0}$ is a g-frame for H, it follows that $\{\Lambda_i\}_{i\in I\backslash i_0}$ is g-complete. Hence $S_\Lambda^{-1}(g_0) = 0$. So $g_0 = 0$ and we have a contradiction. □

In 2015, Guo [13] received the following Lemma, which is an improvement of Sun's result.

Lemma 2. [13] Suppose $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{W_i\}_{i\in I}$ and $\{\tilde{\Lambda}_i\}_{i\in I}$ is its canonical dual g-frame.

1) If $\mathrm{Ker}(I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0}) \neq \{0\}$ or $\overline{\mathrm{Ran}}(I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0}) \neq W_{i_0}$ then $\{\Lambda_i\}_{i\in I\backslash i_0}$ is not g-complete in H.

2) If $I_{W_{i_0}} - \Lambda_{i_0}\tilde{\Lambda}^*_{i_0}$ is surjective or $I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0}$ is surjective then $\{\Lambda_i\}_{i\in I\backslash i_0}$ is a g-frame for H.

From this lemma we get the following corollary.

Corollary 2. *Suppose $\{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{W_i\}_{i\in I}$ and $\{\tilde{\Lambda}_i\}_{i\in I}$ is its canonical dual g-frame. If $\{\Lambda_i\}_{i\in I\backslash i_0}$ is a g-frame for H then* $\mathrm{Ker}(I_{W_{i_0}} - \Lambda_{i_0}\tilde{\Lambda}^*_{i_0}) = \{0\}$.

Proof. Suppose that $\mathrm{Ker}(I_{W_{i_0}} - \Lambda_{i_0}\tilde{\Lambda}^*_{i_0}) \neq \{0\}$. Since $\mathrm{Ker}(I_{W_{i_0}} - \Lambda_{i_0}\tilde{\Lambda}^*_{i_0}) = \overline{\mathrm{Ran}}(I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0})^\perp$, it follows that $\overline{\mathrm{Ran}}(I_{W_{i_0}} - \tilde{\Lambda}_{i_0}\Lambda^*_{i_0}) \neq W_{i_0}$. By Lemma 2 part 1, $\{\Lambda_i\}_{i\in I\backslash i_0}$ is not g-complete in H. Thus $\{\Lambda_i\}_{i\in I\backslash i_0}$ is not a g-frame for H. □

When $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for H with respect to $\{W_i\}_{i\in I}$, we get the following Propositions which are improvements of Lemma 2.13 in [13].

Proposition 4. *Suppose that* $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a Parseval g-frame for* H *with respect to* $\{W_i\}_{i\in I}$ *and* $\operatorname{Ker}(I_H - \Lambda_{i_0}^*\Lambda_{i_0}) = \{0\}$ *and dim* $H < \infty$*. Then* $\{\Lambda_i\}_{i\in I\setminus i_0}$ *is a g-frame for* H *with respect to* $\{W_i\}_{i\in I\setminus i_0}$*.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame, the g-frame operator S_Λ is the identity operator, $\sum\limits_{i\in I}\Lambda_i^*\Lambda_i = I_H$. Thus for all $f \in H$, we have

$$\left\langle \Lambda_{i_0}^*\Lambda_{i_0}(f), f\right\rangle = \langle f, f\rangle - \sum_{i\in I\setminus i_0}\langle \Lambda_i^*\Lambda_i f, f\rangle \le \langle f, f\rangle .$$

So $\left\|\Lambda_{i_0}^*\Lambda_{i_0}\right\| = \sup\limits_{\|f\|=1}\left\langle \Lambda_{i_0}^*\Lambda_{i_0} f, f\right\rangle \le 1$. Since the unit sphere $\{f \in H : \|f\| = 1\}$ is compact, there is $f_0 \in H, \|f_0\| = 1$ such that $\sup\limits_{\|f\|=1}\left\langle \Lambda_{i_0}^*\Lambda_{i_0} f, f\right\rangle = \left\langle \Lambda_{i_0}^*\Lambda_{i_0} f_0, f_0\right\rangle$. If $\left\|\Lambda_{i_0}^*\Lambda_{i_0}\right\| = 1$ then $1 = \left\langle \Lambda_{i_0}^*\Lambda_{i_0} f_0, f_0\right\rangle = \|\Lambda_{i_0} f_0\|^2$. Therefore,

$$0 \le \left\langle f_0 - \Lambda_{i_0}^*\Lambda_{i_0} f_0, f_0 - \Lambda_{i_0}^*\Lambda_{i_0} f_0\right\rangle = \left\|\Lambda_{i_0}^*\Lambda_{i_0} f_0\right\|^2 - 1 \le \left\|\Lambda_{i_0}^*\Lambda_{i_0}\right\|^2\|f_0\|^2 - 1 = 0.$$

So $f_0 = \Lambda_{i_0}^*\Lambda_{i_0} f_0$. Hence, $\mathrm{Ker}(\mathrm{I_H} - \Lambda_{\mathrm{i_0}}^*\Lambda_{\mathrm{i_0}}) \ne \{0\}$. Thus, $\left\|\Lambda_{i_0}^*\Lambda_{i_0}\right\| < 1$. Because $\left\|\Lambda_{i_0}\Lambda_{i_0}^*\right\| = \|\Lambda_{i_0}\|^2 = \left\|\Lambda_{i_0}^*\right\|^2 = \left\|\Lambda_{i_0}^*\Lambda_{i_0}\right\|$, we get $\left\|\Lambda_{i_0}\Lambda_{i_0}^*\right\| < 1$ and $I - \Lambda_{i_0}\Lambda_{i_0}^*$ is invertible. Since $\tilde{\Lambda}_{i_0} = \Lambda_{i_0}$, it follows that $I - \Lambda_{i_0}\tilde{\Lambda}_{i_0}^*$ is invertible. By Theorem 3 part 3, $\{\Lambda_i\}_{i\in I\setminus i_0}$ is a g-frame for H. □

One natural question is that if H is an infinite-dimensional space then whether Proposition 4 remains valid. The answer is No. Let us consider the following example.

Example 5. Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^\infty$ and consider $W_0 := l^2(\mathbb{N})$ and $\Lambda_0 : H \to l^2(\mathbb{N})$ defined by

$$\Lambda_0(f) := \left\{\left\langle f, \frac{e_2}{\sqrt{2}}\right\rangle, \left\langle f, \frac{e_3}{\sqrt{3}}\right\rangle, \left\langle f, \frac{e_3}{\sqrt{3}}\right\rangle, \left\langle f, \frac{e_4}{\sqrt{4}}\right\rangle, \left\langle f, \frac{e_4}{\sqrt{4}}\right\rangle, \left\langle f, \frac{e_4}{\sqrt{4}}\right\rangle, \ldots\right\}$$

where each vector $\left\langle f, \frac{e_i}{\sqrt{i}}\right\rangle$ is repeated $i-1$ times.

Let $W_i := \mathbb{C}$ for each $i \in \mathbb{N}$ and $\Lambda_i : H \to \mathbb{C}$ be defined by $\Lambda_i(f) := \left\langle f, \frac{e_i}{\sqrt{i}}\right\rangle$ for each $i \in \mathbb{N}$. We can check that the operators $\Lambda_i, i = 0, 1, 2, \ldots$, are linear bounded and

$$\sum_{i=0}^{\infty}\|\Lambda_i(f)\|^2 = \sum_{i=0}^{\infty}|\langle f, e_i\rangle|^2 = \|f\|^2 .$$

Therefore, $\{\Lambda_i\}_{i=0}^{\infty}$ is a Parseval g-frame for H.

Suppose that $f - \Lambda_0^*\Lambda_0(f) = 0$. Then $f - \sum_{i=2}^{\infty}(i-1)\left\langle f, \frac{e_i}{\sqrt{i}}\right\rangle \frac{e_i}{\sqrt{i}} = 0$. So $f \in \overline{\text{span}}\{e_i\}_{i=2}^{\infty}$. Thus, $f = \sum_{i=2}^{\infty}\langle f, e_i\rangle e_i = \sum_{i=2}^{\infty}\frac{i-1}{i}\langle f, e_i\rangle e_i$. Therefore, $\langle f, e_i\rangle = \frac{i-1}{i}\langle f, e_i\rangle$ for all $i = 2, 3, 4....$ So $\langle f, e_i\rangle = 0$ for all $i = 2, 3, 4....$ Hence, $f = 0$. Thus $\text{Ker}(I - \Lambda_{i_0}^*\Lambda_{i_0}) = \{0\}$.

However, $\{\Lambda_i\}_{i=1}^{\infty}$ is not a g-frame for H with respect to $\mathbb{C}$. Indeed,

$$\sum_{i=1}^{\infty}\|\Lambda_i(f)\|^2 = \sum_{i=1}^{\infty}\left|\left\langle f, \frac{e_i}{\sqrt{i}}\right\rangle\right|^2 = \sum_{i=1}^{\infty}\frac{1}{i}\left|\langle f, e_i\rangle\right|^2.$$

Choose $f = e_k$. Therefore, $\sum_{i=1}^{\infty}\|\Lambda_i(e_k)\|^2 = \frac{1}{k} \to 0$.

Proposition 5. *Suppose that $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame for H and $I_H - \Lambda_{i_0}^*\Lambda_{i_0}$ is surjective from H onto H. Then $\{\Lambda_i\}_{i\in I\setminus i_0}$ is a g-frame for H.*

Proof. Since $\sum_{i\in I}\Lambda_i^*\Lambda_i = I_H$, it follows that $\sum_{i\in I\setminus i_0}\Lambda_i^*\Lambda_i = I_H - \Lambda_{i_0}^*\Lambda_{i_0}$ which is the g-frame operator associated with $\{\Lambda_i\}_{i\in I\setminus i_0}$. By Proposition 2.7 in [17], $\{\Lambda_i\}_{i\in I\setminus i_0}$ is a g-frame for H. □

4 Redundancy of g-frames

4.1 Redundancy of finite g-frames

Now we introduce the notion of redundancy of g-frames. First we present the definition of the redundancy function.

Definition 8. Let $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ be a g-frame for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^{k}$. For each $f \in \mathbb{S}$, the redundancy function $\mathcal{R}_\Lambda : \mathbb{S} \to \mathbb{R}^+$ is defined by

$$\mathcal{R}_\Lambda(f) := \sum_{\{i:\Lambda_i\neq 0\}}\frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2}$$

We note that in the special case, when $\Lambda_i(f) = \langle f, f_i\rangle$ for all $i = 1, 2, 3, ..., k$ then the definition of the redundancy function $\mathcal{R}_\Lambda$ of g-frame is precisely the definition of the redundancy function $\mathcal{R}_F$ of frame $F = \{f_i\}_{i=1}^{k}$.

Since the redundancy function $\mathcal{R}_\Lambda$ is continuous on the unit sphere $\mathbb{S}$ which is compact in the finite-dimensional space H, the function attains its maximum and its minimum on $\mathbb{S}$.

Lemma 3. *If $\Lambda = \{\Lambda_i\}_{i=1}^k$ is a g-frames for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^k$ then the redundancy function $\mathcal{R}_\Lambda$ attains its maximum and its minimum on the unit sphere in H.*

Definition 9. Let $\Lambda = \{\Lambda_i\}_{i=1}^k$ be a g-frame for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^k$. The upper redundancy of Λ is defined by

$$\mathcal{R}_\Lambda^+ = \max_{f\in\mathbb{S}} \mathcal{R}_\Lambda(f),$$

and the lower redundancy of Λ by

$$\mathcal{R}_\Lambda^- := \min_{f\in\mathbb{S}} \mathcal{R}_\Lambda(f).$$

Moreover, Λ has a uniform redundancy if $\mathcal{R}_\Lambda^- = \mathcal{R}_\Lambda^+$.

When Λ is an equal norm g-frame, the upper and the lower redundancies are computed from the frame bounds.

Lemma 4. *Let $\Lambda = \{\Lambda_i\}_{i=1}^k$ be an equal norm g-frame for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^k$ with g-frame bounds A and B. Suppose $\|\Lambda_i\|^2 = c \neq 0$ for all $i = 1, 2, 3, ..., k$. Then $\mathcal{R}_\Lambda^- = \dfrac{A}{c}$ and $\mathcal{R}_\Lambda^+ = \dfrac{B}{c}$.*

Proof. It follows immediately from the definition of g-frame and the definition of the upper and lower redundancies of g-frame. □

Theorem 4. *Let $\Lambda = \{\Lambda_i\}_{i=1}^k$ be a g-frame for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^k$ with lower and upper g-frame bounds A and B, respectively. Then the following statements hold:*

(1) $0 < \mathcal{R}_\Lambda^- \leq \mathcal{R}_\Lambda^+ < \infty$.

(2) The unit norm g-frame Λ is an A-tight if and only if $\mathcal{R}_\Lambda^- = \mathcal{R}_\Lambda^+ = A$.

(3) Additivity. For each orthonormal basis $\mathcal{E} = \{\Theta_j\}_{j=1}^l$ for H with respect to $\{H_j\}_{j=1}^l$,

$$\mathcal{R}_{\Lambda\cup\mathcal{E}}^\pm = \mathcal{R}_\Lambda^\pm + 1.$$

Moreover, for each g-frame $\Phi = \{\Phi_j\}_{j=1}^l$ for H with respect to $\{H_j\}_{j=1}^l$, we have

$$\mathcal{R}_{\Lambda\cup\Phi}^- \geq \mathcal{R}_\Lambda^- + \mathcal{R}_\Phi^-, \;\; \mathcal{R}_{\Lambda\cup\Phi}^+ \leq \mathcal{R}_\Lambda^+ + \mathcal{R}_\Phi^+.$$

In particular, if Λ and Φ have uniform redundancy then

$$\mathcal{R}_{\Lambda\cup\Phi}^- = \mathcal{R}_\Phi + \mathcal{R}_\Lambda = \mathcal{R}_{\Lambda\cup\Phi}^+.$$

(4) *Invariance. Redundancy is invariant under application of a unitary operator U on H, i.e.,*

$$\mathcal{R}^{\pm}_{\Lambda U} = \mathcal{R}^{\pm}_{\Lambda},$$

where $\Lambda U := \{\Lambda_i U\}_{i=1}^{k}$.

Redundancy is invariant under scaling, i.e.,

$$\mathcal{R}^{\pm}_{\{c_i\Lambda_i\}_{i=1}^{k}} = \mathcal{R}^{\pm}_{\Lambda},$$

and under permutations, i.e.,

$$\mathcal{R}^{\pm}_{\Lambda_\sigma} = \mathcal{R}^{\pm}_{\Lambda},$$

where $\Lambda_\sigma := \{\Lambda_{\sigma(i)}\}$ *where* σ *is a permutation of* $\{1, 2, 3, ..., k\}$.

Proof. (1) It is obvious that $\mathcal{R}^{-}_{\Lambda} \leq \mathcal{R}^{+}_{\Lambda}$. By Lemma 3, $\mathcal{R}_{\Lambda}$ attains its minimum at some $f_0 \in \mathbb{S}$ and its maximum at some $f_1 \in \mathbb{S}$. So $f_0 \neq 0$ and

$$\mathcal{R}^{-}_{\Lambda} = \mathcal{R}_{\Lambda}(f_0) = \sum_{\{i:\Lambda_i \neq 0\}} \frac{\|\Lambda_i(f_0)\|^2}{\|\Lambda_i\|^2}.$$

Since $0 < \dfrac{A\|f_0\|^2}{\max\limits_{i=1,2,...,k}\{\|\Lambda_i\|^2\}} \leq \sum\limits_{\{i:\Lambda_i \neq 0\}} \dfrac{\|\Lambda_i(f_0)\|^2}{\|\Lambda_i\|^2}$, it follows that $\mathcal{R}^{-}_{\Lambda} > 0$. We have

$$\mathcal{R}^{+}_{\Lambda} = \mathcal{R}_{\Lambda}(f_1) = \sum_{\{i:\Lambda_i \neq 0\}} \frac{\|\Lambda_i(f_1)\|^2}{\|\Lambda_i\|^2} < \infty.$$

(2) If the unit norm g-frame Λ is an A-tight then by Lemma 4, $\mathcal{R}^{\pm}_{\Lambda} = A$.

Conversely, assume that $\mathcal{R}^{\pm}_{\Lambda} = A$. Then $\mathcal{R}_{\Lambda}(f) = A$ for all $f \in \mathbb{S}$. Therefore

$$A = \sum_{i=1}^{k} \frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2} = \sum_{i=1}^{k} \|\Lambda_i(f)\|^2.$$

Hence, Λ is a tight g-frame with the frame bound A.

(3) By definition, for $f \in \mathbb{S}$,

$$\mathcal{R}_{\Lambda\cup\mathcal{E}}(f) = \sum_{\{i:\Lambda_i \neq 0\}} \frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2} + \sum_{\{j:\Theta_j \neq 0\}} \frac{\|\Theta_j(f)\|^2}{\|\Theta_j\|^2} = \mathcal{R}_{\Lambda}(f) + \|f\|^2.$$

Thus,

$$\mathcal{R}^{\pm}_{\Lambda\cup\mathcal{E}} = \mathcal{R}^{\pm}_{\Lambda} + 1.$$

Moreover,

$$\begin{aligned}
\mathcal{R}^-_{\Lambda\cup\Phi} &= \min_{f\in\mathbb{S}} \mathcal{R}_{\Lambda\cup\Phi}(f) \\
&\geq \min_{f\in\mathbb{S}} \sum_{\{i:\Lambda_i\neq 0\}} \frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2} + \min_{f\in\mathbb{S}} \sum_{\{j:\Phi_j\neq 0\}} \frac{\|\Phi_j(f)\|^2}{\|\Phi_j\|^2} \\
&= \mathcal{R}^-_\Lambda + \mathcal{R}^-_\Phi.
\end{aligned}$$

Similarly, $\mathcal{R}^+_{\Lambda\cup\Phi} \leq \mathcal{R}^+_\Lambda + \mathcal{R}^+_\Phi$.

In particular, suppose Λ and Φ have uniform redundancy. On the one hand,

$$\mathcal{R}^-_{\Lambda\cup\Phi} \geq \mathcal{R}^-_\Lambda + \mathcal{R}^-_\Phi = \mathcal{R}^+_\Lambda + \mathcal{R}^+_\Phi \geq \mathcal{R}^+_{\Lambda\cup\Phi}.$$

On the other hand, $\mathcal{R}^-_{\Lambda\cup\Phi} \leq \mathcal{R}^+_{\Lambda\cup\Phi}$. So the statement follows.

(4) Let $\Theta_i = \Lambda_i U$ and $\Theta = \{\Theta_i\}_{i=1}^k$. Then for $f \in \mathbb{S}$,

$$\mathcal{R}_\Theta(f) = \sum_{\{i:\Lambda_i U\neq 0\}} \frac{\|\Lambda_i U(f)\|^2}{\|\Lambda_i U\|^2} = \sum_{\{i:\Lambda_i U\neq 0\}} \frac{\|\Lambda_i y\|^2}{\|\Lambda_i U\|^2}$$

where $y = U(f)$. Since $\|y\| = 1$ and $\|\Lambda_i U\| = \sup_{f\in\mathbb{S}} \|\Lambda_i U(f)\| = \sup_{y\in\mathbb{S}} \|\Lambda_i y\| = \|\Lambda_i\|$, it follows that $\mathcal{R}_\Theta(f) = \mathcal{R}_\Lambda(y)$. Hence, $\mathcal{R}^\pm_{\Lambda U} = \mathcal{R}^\pm_\Lambda$.

For $f \in \mathbb{S}$,

$$\mathcal{R}_{\{c_i\Lambda_i\}_{i=1}^k}(f) = \sum_{\{i:\Lambda_i\neq 0\}} \frac{\|c_i\Lambda_i(f)\|^2}{\|c_i\Lambda_i\|^2} = \sum_{\{i:\Lambda_i\neq 0\}} \frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2} = \mathcal{R}_\Lambda(f).$$

Therefore,

$$\mathcal{R}^\pm_{\{c_i\Lambda_i\}_{i=1}^k} = \mathcal{R}^\pm_\Lambda.$$

It is obvious that redundancy is invariant under permutations. $\square$

Corollary 3. *If $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for H then $\mathcal{R}^\pm_\Lambda = 1$ and $\|\Lambda_i\| = 1$ for all i.*

Proof. By definition of g-orthonormal basis, Λ_i^* is isometric for every i. So $\|\Lambda_i^*\| = 1$ for every i. Hence $\|\Lambda_i\| = 1$ for every i. Therefore, by Theorem 4 part (2) we get $\mathcal{R}^\pm_\Lambda = 1$. $\square$

A natural question is that whether the converse statement is true. The answer is No. Let us consider the following example.

Example 6. Let H be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Put $W_i = \operatorname{span}\{e_i, e_{i+1}\}$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ we define a bounded linear operator $\Lambda_i : H \to W_i$ as follows: $\Lambda_i(f) = \langle f, e_i \rangle e_i$ for all $f \in H$. We can check that $\sum\limits_{i=1}^{\infty} \|\Lambda_i(f)\|^2 = \|f\|^2$ for all $f \in H$. We have

$$\|\Lambda_i\| = \sup_{\|f\|=1} \|\Lambda_i(f)\| = \sup_{\|f\|=1} |\langle f, e_i \rangle| \leq 1.$$

Choose $f = e_i$. Then $|\langle f, e_i \rangle| = 1$. Thus, $\|\Lambda_i\| = 1$. Hence, $\mathcal{R}_{\Lambda}^{\pm} = 1$.

Suppose that $g_i = c_i e_i + c_{i+1} e_{i+1} \in H_i$. Then

$$\langle \Lambda_i^* g_i, f \rangle = \langle g_i, \Lambda_i(f) \rangle = \langle c_i e_i + c_{i+1} e_{i+1}, \langle f, e_i \rangle e_i \rangle = \langle c_i e_i, f \rangle, \forall f \in H.$$

So $\Lambda_i^* g_i = c_i e_i$. Thus, $u_{i_1} := \Lambda_i^* e_i = e_i, u_{i_2} := \Lambda_i^* e_{i+1} = 0$ for all $i \in \mathbb{N}$. Since $\{u_{i_j}\}_{i \in \mathbb{N}, j \in \{1,2\}}$ is a Parseval frame for H but it is not an orthonormal basis for H. By Theorem 2, $\{\Lambda_i\}_{i \in \mathbb{N}}$ is not a g-orthonormal basis.

Lemma 5. *Let $\mathbb{G}$ be the set of g-frames for a finite-dimensional real or complex Hilbert space H. Then the relation $\sim$ on $\mathbb{G}$ defined by*

$$\Lambda \sim \Phi \Leftrightarrow \mathcal{R}_{\Lambda} = \mathcal{R}_{\Phi}$$

is an equivalent relation on $\mathbb{G}$.

Proof. The proof is obvious. □

For a g-frames $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ in H, we denote $\tilde{S}_{\Lambda} = \sum\limits_{\{i:\Lambda_i \neq 0\}} \dfrac{\Lambda_i^* \Lambda_i}{\|\Lambda_i\|^2}$. We denote the associated quadratic form by $\mathcal{Q}_{\Lambda}(f) = \left\langle \tilde{S}_{\Lambda}(f), f \right\rangle$ and note that $\mathcal{Q}_{\Lambda}$ extend $\mathcal{R}_{\Lambda}$ to all $f \in H$.

By using the same arguments as in the proof of Corollary 3.3 in [4] we can prove the following corollary.

Corollary 4. *If Φ, Λ are two g-frames for a finite-dimensional real or complex Hilbert space H then the following statements are equivalent:*

(1) $\mathcal{R}_{\Lambda} = \mathcal{R}_{\Phi}$ on $\mathbb{S}$.

(2) $\tilde{S}_{\Lambda} = \tilde{S}_{\Phi}$ on H.

Proposition 6. *Let $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ be a g-frame for a finite-dimensional real or complex Hilbert space H and T be an invertible operator on H. Then*

$$(k(T))^{-2} \mathcal{R}_{\Lambda}^{\pm} \leq \mathcal{R}_{\Lambda T}^{\pm} \leq (k(T))^2 \mathcal{R}_{\Lambda}^{\pm},$$

where $\Lambda T := \{\Lambda_i T\}_{i=1}^{k}, k(T) := \|T\| \, \|T^{-1}\|$.

In particular, if we denote $\Phi = \{\Lambda_i S_\Lambda^{-1/2}\}_{i=1}^k$, where S_Λ denotes the g-frame operator associated with Λ, then

$$(k(S_\Lambda))^{-1}\mathcal{R}_\Lambda^{\pm} \leq \mathcal{R}_\Phi^{\pm} \leq (k(S_\Lambda))\mathcal{R}_\Lambda^{\pm}.$$

Proof. Assume without loss of generality that $\Lambda_i \neq 0$ for all i. Since for each $f \in \mathbb{S}$,

$$\mathcal{R}_{\Lambda T}(f) = \sum_{i=1}^k \frac{\|\Lambda_i T f\|^2}{\|\Lambda_i T\|^2} \geq \sum_{i=1}^k \frac{\|\Lambda_i T f\|^2}{\|\Lambda_i\|^2 \|T\|^2}$$

we have

$$\begin{aligned}
\mathcal{R}_{\Lambda T}^{+} = \max_{f\in\mathbb{S}} \mathcal{R}_{\Lambda T}(f) \geq \max_{f\in\mathbb{S}} \sum_{i=1}^k \frac{\|\Lambda_i T f\|^2}{\|\Lambda_i\|^2 \|T\|^2} &= \frac{1}{\|T\|^2} \max_{f\in\mathbb{S}} \|Tf\|^2 \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&\geq \frac{1}{\|T\|^2} \max_{f\in\mathbb{S}} \frac{\|f\|^2}{\|T^{-1}\|^2} \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&= \frac{1}{\|T\|^2 \|T^{-1}\|^2} \max_{f\in\mathbb{S}} \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&= (k(T))^{-2}\mathcal{R}_\Lambda^{+}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{R}_{\Lambda T}^{-} = \min_{f\in\mathbb{S}} \mathcal{R}_{\Lambda T}(f) \geq \min_{f\in\mathbb{S}} \sum_{i=1}^k \frac{\|\Lambda_i T f\|^2}{\|\Lambda_i\|^2 \|T\|^2} &= \frac{1}{\|T\|^2} \min_{f\in\mathbb{S}} \|Tf\|^2 \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&\geq \frac{1}{\|T\|^2} \min_{f\in\mathbb{S}} \frac{\|f\|^2}{\|T^{-1}\|^2} \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&= \frac{1}{\|T\|^2 \|T^{-1}\|^2} \min_{f\in\mathbb{S}} \mathcal{R}_\Lambda\left(\frac{Tf}{\|Tf\|}\right) \\
&= (k(T))^{-2}\mathcal{R}_\Lambda^{-}.
\end{aligned}$$

We have $\mathcal{R}_\Lambda^{\pm} = \mathcal{R}_{\Lambda T T^{-1}}^{\pm} \geq (k(T^{-1}))^{-2}\mathcal{R}_{\Lambda T}^{\pm} = (k(T))^{-2}\mathcal{R}_{\Lambda T}^{\pm}$. Therefore, $(k(T))^{-2}\mathcal{R}_\Lambda^{\pm} \leq \mathcal{R}_{\Lambda T}^{\pm} \leq (k(T))^2\mathcal{R}_\Lambda^{\pm}$.

Since $\Phi_i = \Lambda_i S_\Lambda^{-1/2}$ for every i and $\left\|S_\Lambda^{\pm 1/2}\right\| = \left\|S_\Lambda^{\pm 1}\right\|^{1/2}$ and let $T = S_\Lambda^{-1/2}$, we get the result immediately. □

Corollary 5. *Let $\Lambda = \{\Lambda_i\}_{i=1}^k$ be a g-frame for a finite-dimensional real or complex Hilbert space H and $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i=1}^k$, where $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, be the canonical* *dual g-frame of* Λ. *Then*

$$(k(S_\Lambda))^{-2}\mathcal{R}^{\pm}_{\Lambda} \leq \mathcal{R}^{\pm}_{\tilde{\Lambda}} \leq (k(S_\Lambda))^{2}\mathcal{R}^{\pm}_{\Lambda},$$

where $k(S_\Lambda) := \|S_\Lambda\| \left\|S_\Lambda^{-1}\right\|$.

Corollary 6. *Let* $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ *be a tight g-frame for a finite-dimensional real or complex Hilbert space* H *and* $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i=1}^{k}$ *be the canonical dual g-frame of* Λ. *Then for each* $f \in \mathbb{S}$, $\mathcal{R}_\Lambda(f) = \mathcal{R}_{\tilde{\Lambda}}(f)$ *and hence,* $\mathcal{R}^{\pm}_{\Lambda} = \mathcal{R}^{\pm}_{\tilde{\Lambda}}$.

Proof. Since Λ is a tight g-frame, $S_\Lambda = cI$ for $c > 0$. So $\tilde{\Lambda}_i = \frac{1}{c}\Lambda_i$. The conclusions follow immediately. $\square$

The following results are similar to the results for frames of subspaces (see [18]).

When Λ is a general equal norm g-frame, the bounds for the ratio between $\mathcal{R}_\Lambda(f)$ and $\mathcal{R}_{\tilde{\Lambda}}(f)$ can be given as follows. Suppose that $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ is a g-frame with bounds A and B. Then the canonical dual g-frame $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i=1}^{k}$ has bounds $1/B$ and $1/A$ (see [19]). Suppose that Λ and $\tilde{\Lambda}$ are equal norm, $\|\Lambda_i\| = c > 0$ for all i and $\left\|\tilde{\Lambda}_i\right\| = d > 0$ for all i. Then for each $f \in \mathbb{S}$,

$$\frac{A\|f\|^2}{c^2} \leq \mathcal{R}_\Lambda(f) = \frac{\sum\limits_{i=1}^{k}\|\Lambda_i(f)\|^2}{c^2} \leq \frac{B\|f\|^2}{c^2}$$

and

$$\frac{1}{B}\frac{\|f\|^2}{d^2} \leq \mathcal{R}_{\tilde{\Lambda}}(f) = \frac{\sum\limits_{i=1}^{k}\left\|\tilde{\Lambda}_i(f)\right\|^2}{d^2} \leq \frac{1}{A}\frac{\|f\|^2}{d^2}.$$

Therefore,

$$A^2\frac{d^2}{c^2} \leq \frac{\mathcal{R}_\Lambda(f)}{\mathcal{R}_{\tilde{\Lambda}}(f)} \leq B^2\frac{d^2}{c^2}.$$

Let $\Phi = \{\Phi_i\}_{i=1}^{k}$ be an alternate dual of the g-frame $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ and $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i=1}^{k}$ be the canonical dual of Λ. Suppose that $\tilde{\Lambda}$ and Φ are equal norm, i.e. $\left\|\tilde{\Lambda}_i\right\| = c$ and $\|\Phi_i\| = d$ for some $c, d > 0$ for all i. Then for each $f \in \mathbb{S}$,

$$\mathcal{R}_{\tilde{\Lambda}}(f) = \sum_{i=1}^{k}\frac{\left\|\tilde{\Lambda}_i(f)\right\|^2}{\left\|\tilde{\Lambda}_i\right\|^2} = \sum_{i=1}^{k}\frac{\left\|\tilde{\Lambda}_i(f)\right\|^2}{c^2}$$

and

$$\mathcal{R}_\Phi(f) = \sum_{i=1}^{k} \frac{\|\Phi_i(f)\|^2}{\|\Phi_i\|^2} = \sum_{i=1}^{k} \frac{\|\Phi_i(f)\|^2}{d^2}.$$

By Proposition 2, $\sum\limits_{i=1}^{k} \left\|\tilde{\Lambda}_i(f)\right\|^2 \leq \sum\limits_{i=1}^{k} \|\Phi_i(f)\|^2$. Thus, $\mathcal{R}_{\tilde{\Lambda}}(f) \leq \left(\dfrac{d}{c}\right)^2 \mathcal{R}_\Phi(f)$.

Now we consider two examples which are similar to the examples in [4] for finite frames and in [18] for frames of subspaces.

Let $\Lambda = \{\Lambda_i\}_{i=1}^{k}$ be a g-orthonormal basis for a finite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i=1}^{k}$. We consider

$$\Theta = \{\Lambda_1, ..., \Lambda_1, \Lambda_2, ..., \Lambda_k\}$$

where Λ_1 occurs $k+1$ times. Then $\mathcal{R}_\Theta^+ = k+1$ and $\mathcal{R}_\Theta^- = 1$ when Λ_1 is not a unitary operator. This can be seen as follows. Let $f \in \mathbb{S}$. By Corollary 3, $\|\Lambda_i\| = 1$ for every $i = 1, 2, ..., k$. Therefore,

$$\mathcal{R}_\Theta(f) = k\,\|\Lambda_1(f)\|^2 + \sum_{i=1}^{k} \|\Lambda_i(f)\|^2 = k\,\|\Lambda_1(f)\|^2 + 1.$$

Hence, $\mathcal{R}_\Theta^+ = \max\limits_{f \in \mathbb{S}} \mathcal{R}_\Theta(f) = k+1$. If Λ_1 is not a unitary operator then there exists $f \in \mathbb{S}$ such that $\Lambda_1(f) = 0$ because if otherwise, Λ_1 is injective. So $\overline{\mathrm{Ran}(\Lambda_1^*)} = H$. Since by definition of g-orthonormal basis, Λ_1^* is an isometric, Λ_1^* have a closed range. So Λ_1^* is a unitary operator. It follows that Λ_1 is a unitary operator. We get a contradiction. Hence, $\mathcal{R}_\Theta^- = \min\limits_{f \in \mathbb{S}} \mathcal{R}_\Theta(f) = 1$.

Let $\Phi = \{\Lambda_1, \Lambda_1, \Lambda_2, \Lambda_2, ..., \Lambda_k, \Lambda_k\}$, where each $\Lambda_i, i = 1, 2, 3, ..., k$, occurs 2 times. Then Φ possesses a uniform redundancy, $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = 2$.

4.2 Redundancy of infinite g-frames

Now we consider the redundancy of infinite g-frames in an infinite-dimensional Hilbert spaces. Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-frame for H. Similarly, for each $f \in H$, we define the redundancy function $\mathcal{R}_\Lambda : \mathbb{S} \to \mathbb{R}^+$ by

$$\mathcal{R}_\Lambda(f) := \sum_{\{i \in I : \Lambda_i \neq 0\}} \frac{\|\Lambda_i(f)\|^2}{\|\Lambda_i\|^2}.$$

In contrast to the finite case, this redundancy function may not assume its maximum or minimum on the unit sphere and in general both the max and min of this function could be infinite.

Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H. Then the upper redundancy of Λ is defined by

$$\mathcal{R}^+_\Lambda = \sup_{f\in\mathbb{S}} \mathcal{R}_\Lambda(f)$$

and the lower redundancy of Λ by

$$\mathcal{R}^-_\Lambda = \inf_{f\in\mathbb{S}} \mathcal{R}_\Lambda(f).$$

Now we can verify that Theorem 4 holds in the infinite-dimensional setting.

Theorem 5. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for an infinite-dimensional real or complex Hilbert space H with respect to $\{W_i\}_{i\in I}$ with lower and upper g-frame bounds A and B, respectively. We assume that $\mathcal{R}^+_\Lambda < \infty$. Then the following statements hold:*

(1) $0 < \mathcal{R}^-_\Lambda \leq \mathcal{R}^+_\Lambda < \infty$.

(2) The unit norm g-frame Λ is an A-tight if and only if $\mathcal{R}^-_\Lambda = \mathcal{R}^+_\Lambda = A$.

(3) Additivity. For each orthonormal basis $\mathcal{E} = \{\Theta_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$,

$$\mathcal{R}^\pm_{\Lambda\cup\mathcal{E}} = \mathcal{R}^\pm_\Lambda + 1.$$

Moreover, for each g-frame Φ for H with respect to $\{H_j\}_{j\in J}$, we have

$$\mathcal{R}^-_{\Lambda\cup\Phi} \geq \mathcal{R}^-_\Lambda + \mathcal{R}^-_\Phi, \ \ \mathcal{R}^+_{\Lambda\cup\Phi} \leq \mathcal{R}^+_\Lambda + \mathcal{R}^+_\Phi.$$

In particular, if Λ and Φ have uniform redundancy then

$$\mathcal{R}^-_{\Lambda\cup\Phi} = \mathcal{R}_\Phi + \mathcal{R}_\Lambda = \mathcal{R}^+_{\Lambda\cup\Phi}.$$

(4) Invariance. Redundancy is invariant under application of a unitary operator U on H, i.e.,

$$\mathcal{R}^\pm_{\Lambda U} = \mathcal{R}^\pm_\Lambda,$$

where $\Lambda U := \{\Lambda_i U\}_{i\in I}$.

Redundancy is invariant under scaling, i.e.,

$$\mathcal{R}^\pm_{\{c_i\Lambda_i\}_{i\in I}} = \mathcal{R}^\pm_\Lambda,$$

and under permutations, i.e.,

$$\mathcal{R}^\pm_{\Lambda_\sigma} = \mathcal{R}^\pm_\Lambda,$$

where $\Lambda_\sigma := \{\Lambda_{\sigma(i)}\}$ where σ is a permutation of I.

Proof. For (1), we need only check that $0 < \mathcal{R}_\Lambda^-$. Suppose the contrary $\mathcal{R}_\Lambda^- = 0$. Then there exists a sequence $\{f_k\}$ in $\mathbb{S}$ such that $\sum\limits_{\{i\in I:\Lambda_i\neq 0\}} \dfrac{\|\Lambda_i(f_k)\|^2}{\|\Lambda_i\|^2}$ goes to 0 as $k \to \infty$. Since $\sum_{i\in I} \|\Lambda_i(f)\|^2 \leq B\|f\|^2$ for every $f \in H$, it follows that $\|\Lambda_i(f)\|^2 \leq B\|f\|^2$ for every $i \in I$ and for every $f \in H$. Therefore, $\|\Lambda_i\| \leq \sqrt{B}$ for every $i \in I$. Hence,

$$\frac{A}{B} = \frac{A\,\|f_k\|^2}{B} \leq \sum_{i\in I} \frac{\|\Lambda_i(f_k)\|^2}{B} = \sum_{\{i\in I:\Lambda_i\neq 0\}} \frac{\|\Lambda_i(f_k)\|^2}{B} \leq \sum_{\{i\in I:\Lambda_i\neq 0\}} \frac{\|\Lambda_i(f_k)\|^2}{\|\Lambda_i\|^2}$$

which goes to 0 as $k \to \infty$. The contradiction implies that $0 < \mathcal{R}_\Lambda^-$.

Other properties are proved similarly to the finite case. □

5 Excess of g-frames

Now we introduce a notion of excess of g-frames and show that some basic properties of excess of frames still remain valid for g-frames.

Definition 10. Let Λ be a g-frame for H with respect to $\{W_i\}_{i\in I}$ with analysis operator T_Λ. The excess of Λ is defined as $e(\Lambda) = \dim(\mathrm{RanT}_\Lambda)^\perp$.

Proposition 7. *Let Λ and Φ be two similar g-frames. Then $e(\Lambda) = e(\Phi)$.*

Proof. It follows immediately by Proposition 3. □

As in the case of ordinary frames, we have the following result for g-Riesz bases.

Proposition 8. *Λ is a g-Riesz basis for H with respect to $\{W_i\}_{i\in I}$ if and only if*

$$e(\Lambda) = 0.$$

Proof. By [1], Λ is a g-Riesz basis for H with respect to $\{W_i\}_{i\in I}$ if and only if

$$\mathrm{Ran}(T_\Lambda) = \left(\sum_{i\in I} \oplus W_i\right)_{l_2},$$

which is equivalent to $e(\Lambda) = 0$. □

Similar to Proposition 5.5 in [3] for frames, we obtain the following result for g-frames.

Proposition 9. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{W_i\}_{i\in I}$ with the canonical dual g-frame $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i\in I}$. Then the excess of Λ is*

$$e(\Lambda) = \sum_{i\in I}\left(\mathrm{dim}W_i - \mathrm{trace}\tilde{\Lambda}_i\Lambda_i^*\right)$$

Proof. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i and for each $i \in I, j \in J_i$, let E_{ij} be an element of $\left(\sum_{i\in I}\oplus W_i\right)_{l_2}$ defined by

$$(E_{ij})_k = \begin{cases} e_{ij} & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

It is obvious that $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $\left(\sum_{i\in I}\oplus W_i\right)_{l_2}$.

By definition, $e(\Lambda) = \mathrm{dim}(\mathrm{RanT}_\Lambda)^\perp$. The orthogonal projection of $\left(\sum_{i\in I}\oplus W_i\right)_{l_2}$ onto $(\mathrm{Ran}\ T_\Lambda)^\perp$ is given by $P = I - T_\Lambda S_\Lambda^{-1} T_\Lambda^*$ where S_Λ is the g-frame operator for Λ. Since $T_\Lambda^*(E_{ij}) = \Lambda_i^*(e_{ij})$, we have

$$\begin{aligned}
e(\Lambda) = \mathrm{dim}(\mathrm{RanT}_\Lambda)^\perp &= \mathrm{trace(P)} \\
&= \sum_{i\in I, j\in J_i} \langle E_{ij}, PE_{ij}\rangle \\
&= \sum_{i\in I, j\in J_i} \left\langle E_{ij}, E_{ij} - T_\Lambda S_\Lambda^{-1} T_\Lambda^* E_{ij}\right\rangle \\
&= \sum_{i\in I, j\in J_i} \left(1 - \left\langle T_\Lambda^* E_{ij}, S_\Lambda^{-1} T_\Lambda^* E_{ij}\right\rangle\right) \\
&= \sum_{i\in I}\left(\mathrm{dim}W_i - \sum_{j\in J_i}\left\langle \Lambda_i^*(e_{ij}), S_\Lambda^{-1}\Lambda_i^*(e_{ij})\right\rangle\right) \\
&= \sum_{i\in I}\left(\mathrm{dim}W_i - \sum_{j\in J_i}\left\langle \Lambda_i^*(e_{ij}), \tilde{\Lambda}_i^*(e_{ij})\right\rangle\right) \\
&= \sum_{i\in I}\left(\mathrm{dim}W_i - \mathrm{trace}\tilde{\Lambda}_i\Lambda_i^*\right).
\end{aligned}$$

□

We note that if for each $i \in I$, $\Lambda_i : H \to \mathbb{C}$ is given by $\Lambda_i(f) = \langle f, f_i\rangle$ where $\{f_i\}_{i\in I}$ is a frame for H then Proposition 9 is precisely Proposition 5.5 in [3] for frames.

We can characterize a g-frame Λ by the sequences $\{u_{i,j}\}_{i\in I, j\in J_i}$ defined as in Theorem 2. One natural question is that whether the excess of Λ is equal to the excess of every associated sequence $\{u_{i,j}\}_{i\in I, j\in J_i}$. The answer is Yes.

Proposition 10. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for H with respect to $\{W_i\}_{i\in I}$ and $u_{i,j} := \Lambda_i^*(e_{i,j}), i \in I, j \in J_i$, where $\{e_{i,j}\}_{i\in I, j\in J_i}$ be an orthonormal basis for W_i. Then the excess of g-frame Λ is equal to the excess of the sequence $\{u_{i,j}\}_{i\in I, j\in J_i}$.*

Proof. We denote the excess and the analysis operator of the sequence $\{u_{i,j}\}_{i\in I, j\in J_i}$ by $e(U)$ and T_U, respectively. It is proved in [3] that $e(U) = \dim(\mathrm{Ker}(T_U^*))$. But $\dim(\mathrm{Ker}(T_U^*)) = \dim(\mathrm{Ran}T_U)^\perp$. So $e(U) = \dim(\mathrm{Ran}T_U)^\perp$. By definitions, for any $f \in H$ we have $T_U(f) = \{\langle f, u_{i,j}\rangle\}_{i\in I, j\in J_i}$, $T_\Lambda(f) = \{\Lambda_i(f)\}_{i\in I}$. Since $\langle f, u_{i,j}\rangle = \langle \Lambda_i(f), e_{i,j}\rangle$, it follows that $\Lambda_i(f) = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}$. Therefore, $\dim(\mathrm{Ran}T_\Lambda) = \dim(\mathrm{Ran}T_U)$. Thus,
$\dim(\mathrm{Ran}T_\Lambda)^\perp = \dim(\mathrm{Ran}T_U)^\perp$ and $e(\Lambda) = e(U)$. □

6 Acknowledgements

This work was supported by Vietnam National Foundation for Science and Technology Development [grant number 101.02 - 2014.51], to which, the author would like to express many thanks. The author also thanks Professor Nguyen Minh Chuong for carefully reading and useful comments which improved the manuscript.

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