STABILITY OF DEPTH AND COHEN-MACAULAYNESS OF INTEGRAL CLOSURES OF POWERS OF MONOMIAL IDEALS

LE TUAN HOA AND TRAN NAM TRUNG

Abstract. Let I be a monomial ideal I in a polynomial ring $R = k[x_1, ..., x_r]$. In this paper we give an upper bound on $\overline{\operatorname{dstab}}(I)$ in terms of r and the maximal generating degree $d(I)$ of I such that $\operatorname{depth} R/\overline{I^n}$ is constant for all $n \geqslant \overline{\operatorname{dstab}}(I)$. As an application, we classify the class of monomial ideals I such that $\overline{I^n}$ is Cohen-Macaulay for some integer $n \gg 0$.

Introduction

Let $R = k[x_1, \dots, x_r]$ be a polynomial ring over a field k and $\mathfrak{a}$ a homogeneous ideal in R. It was shown by Brodmann [2] that $\operatorname{depth} R/\mathfrak{a}^n$ is constant for $n \gg 0$. The smallest integer $m > 0$ such that $\operatorname{depth} R/\mathfrak{a}^n = \operatorname{depth} R/\mathfrak{a}^m$ for all $n \geqslant m$ is called the index of depth stability and is denoted by $\operatorname{dstab}(\mathfrak{a})$. Since the behavior of depth function $\operatorname{depth} R/\mathfrak{a}^n$ is quite mysterious (see [7, 5]), it is of great interest to bound $\operatorname{dstab}(\mathfrak{a})$ in terms of r and $\mathfrak{a}$. However, until now this problem is only solved for a few classes of monomial ideals (see, e.g., [7, 8, 20]). The bound obtained in [20] for ideals generated by square-free monomials of degree two is rather small and optimal. However, this problem is still open for a general square-free monomial ideal.

In this direction, it is also of interest to consider similar problems for other powers of $\mathfrak{a}$. In [10] together with Kimura and Terai we were able to solve the problem of bounding the index of depth stability for symbolic powers of square-free monomial ideals. In this paper we are interested in bounding the index of depth stability $\overline{\operatorname{dstab}}(\mathfrak{a})$ for integral closures, which is defined as the smallest integer $m > 0$ such that $\operatorname{depth} R/\overline{\mathfrak{a}^n} = \operatorname{depth} R/\overline{\mathfrak{a}^m}$ for all $n \geqslant m$. Like in the case of ordinary powers, $\overline{\operatorname{dstab}}(\mathfrak{a})$ is well-defined. We only consider the problem for monomial ideals I. In this context one can use geometry and convex analysis to describe the integral closures of I^n (see Definition 1.1 and some properties after it). Then one can use Takayama's formula (see Lemma 1.4) to compute the local cohomology modules of $R/\overline{I^n}$. This approach was successfully applied in several papers (see, e.g., [10, 11, 19]). In particular, one can show that in the class of monomial ideals the behavior of the function $\operatorname{depth} R/\overline{I^n}$ is much better than that of $\operatorname{depth} R/I^n$: it is "quasi-decreasing" (see Lemma 1.5) while

1991 *Mathematics Subject Classification.* 13D45, 05C90.

Key words and phrases. Depth, monomial ideal, simplicial complex, integral closure.

the function depth R/I^n can be any convergent non-negative numerical function (see [5]). Our main result is Theorem 2.3, where we can give an upper bound on $\overline{\mathrm{dstab}}(I)$ in terms of r and the maximal generating degree $d(I)$ of I for any monomial ideal I. Although our bound is very big, an example shows that an upper bound must depend on $d(I)$, and in the worst case must be an exponential function of r.

In order to bound $\overline{\mathrm{dstab}}(I)$ we have to study the index of stability for the associated primes on $R/\overline{I^n}$. This in some sense corresponds the zero depth case and was firstly done in [19]. In this paper we can improve the main result of [19] by giving an essentially better bound, see Theorem 1.7.

As an application we classify all monomial ideals such that $R/\overline{I^n}$ is a Cohen-Macaulay ring for all $n \geqslant 1$ (or for some fixed $n = n_0 \gg 0$). It turns out that only equimultiple ideals have this property, see Theorem 3.1. In the case of square-free monomial ideals, we can then derive a criterion for the Cohen-Macaulayness of $R/\overline{I^n}$ for some fixed $n \geqslant 3$, see Theorem 3.7. This criterion is exactly the one for the Cohen-Macaulayness of R/I^n given in [17, Theorem 1.2].

The paper is organized as follows. In Section 1 we study the stability of associated primes and give an upper bound on $\overline{\mathrm{astab}}(I)$ of a monomial ideal. In Section 2 we prove the main Theorem 2.3. The study of Cohen-Macaulay property of $R/\overline{I^n}$ is done in the last section.

1. Stability of associated primes

Let $R := k[x_1, \ldots, x_r]$ be a polynomial ring over a field k with the maximal homogeneous ideal $\mathfrak{m} = (x_1, \ldots, x_r)$. Throughout this paper, let I be a proper monomial ideal in R. Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ be the set of non-negative integers, real numbers and non-negative real numbers, respectively. For a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, we denote by $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$.

The *integral closure* of an arbitrary ideal $\mathfrak{a}$ of R is the set of elements x in R that satisfy an integral relation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0,$$

where $a_i \in \mathfrak{a}^i$ for $i = 1, \ldots, n$. This is an ideal and is denoted by $\overline{\mathfrak{a}}$. The integral closure of a monomial ideal I is a monomial ideal as well. We can geometrically describe $\overline{I}$ by using its Newton polyhedron.

Definition 1.1. Let I be a monomial ideal of R. We define

(1) For a subset $A \subseteq R$, the exponent set of A is $E(A) := \{\boldsymbol{\alpha} \mid \mathbf{x}^{\boldsymbol{\alpha}} \in A\} \subseteq \mathbb{N}^r$.

(2) The Newton polyhedron of I is $NP(I) := \mathrm{conv}\{E(I)\}$, the convex hull of the exponent set of I in the space $\mathbb{R}^r$.

The following results are well-known (see [14]):

$$E(\overline{I}) = NP(I) \cap \mathbb{N}^r = \{\boldsymbol{\alpha} \in \mathbb{N}^r \mid \mathbf{x}^{n\boldsymbol{\alpha}} \in I^n \text{ for some } n \geqslant 1\}. \tag{1.1}$$

$$NP(I^n) = nNP(I) = n\,\mathrm{conv}\{E(I)\} + \mathbb{R}^r_+ \text{ for all } n \geqslant 1. \tag{1.2}$$

Let $G(I)$ denote the minimal generating system of monomials of I and

$$d(I) := \max\{\alpha_1 + \cdots + \alpha_r \mid \mathbf{x}^{\boldsymbol{\alpha}} \in G(I)\},$$

the maximal generating degree of I. Let $\mathbf{e}_1, ..., \mathbf{e}_r$ be the canonical basis of $\mathbb{R}^r$. The first part of the following result is [19, Lemma 6]. It gives more precise information on the coefficients of defining equations of supporting hyperplanes of $NP(I)$.

Lemma 1.2. *The Newton polyhedron $NP(I)$ is the set of solutions of a system of inequalities of the form*

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle \geqslant b_j,\ j = 1, \ldots, q\},$$

such that each hyperplane with the equation $\langle \mathbf{a}_j, \mathbf{x}\rangle = b_j$ defines a facet of $NP(I)$, which contains s_j affinely independent points of $E(G(I))$ and is parallel to $r - s_j$ vectors of the canonical basis. Furthermore, we can choose $\mathbf{0} \neq \mathbf{a}_j \in \mathbb{N}^r, b_j \in \mathbb{N}$ for all $j = 1, ..., q$; and if we write $\mathbf{a}_j = (a_{j1}, \ldots, a_{jr})$, then

$$a_{ji} \leqslant s_j d(I)^{s_j - 1} \text{ for all } i = 1, \ldots, r,$$

where s_j is the number of non-zero coordinates of $\mathbf{a}_j$.

Proof. The first part of the lemma is [19, Lemma 6]. Moreover, it also claims that $\mathbf{a}_j \in \mathbb{R}^r_+$ and $b \in \mathbb{R}_+$. For the second part, let H be a hyperplane which defines a facet of $NP(I)$. W.l.o.g, we may assume that H is defined by s affinely independent points $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_s \in E(G(I))$ and is parallel to $r - s$ vectors $\mathbf{e}_{s+1}, \ldots, \mathbf{e}_r$. Then the defining equation of H can be written as

$$\begin{vmatrix} x_1 & \cdots & x_s & 1 \\ \alpha_{11} & \cdots & \alpha_{1s} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{s1} & \cdots & \alpha_{ss} & 1 \end{vmatrix} = 0.$$

Expanding this determinant in the first row, we get: $a'_1 x_1 + \cdots + a'_s x_s = b'$, where a'_i are the $(1, i)$-cofactor for $i = 1, \ldots, s$ and b' is the $(1, s+1)$-cofactor of this determinant. Clearly, $a'_1, \ldots, a'_s, b' \in \mathbb{Z}$. Note that we may take $a_i = |a'_i|$ and $b = |b'|$. Expanding the determinant

$$a'_1 = \begin{vmatrix} \alpha_{12} & \cdots & \alpha_{1s} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{s2} & \cdots & \alpha_{ss} & 1 \end{vmatrix},$$

in the last column, we get

$$(-1)^{s+1}a'_1 = \begin{vmatrix} \alpha_{22} & \cdots & \alpha_{2s} \\ \alpha_{32} & \cdots & \alpha_{3s} \\ \vdots & \vdots & \vdots \\ \alpha_{s2} & \cdots & \alpha_{ss} \end{vmatrix} - \begin{vmatrix} \alpha_{12} & \cdots & \alpha_{1s} \\ \alpha_{32} & \cdots & \alpha_{3s} \\ \vdots & \vdots & \vdots \\ \alpha_{s2} & \cdots & \alpha_{ss} \end{vmatrix} + \cdots + (-1)^{s-1} \begin{vmatrix} \alpha_{12} & \cdots & \alpha_{1s} \\ \alpha_{22} & \cdots & \alpha_{2s} \\ \vdots & \vdots & \vdots \\ \alpha_{s-1,2} & \cdots & \alpha_{s-1,s} \end{vmatrix}.$$

Let $\det(c_{ij})$ be a determinant in the above sum. By Hadamard's inequality, we have

$$(\det(c_{ij}))^2 \leqslant \Pi_{i=1}^{s-1}(\Sigma_{j=1}^{s-1}|c_{ij}|^2) \leqslant \Pi_{i=1}^{s-1}(\Sigma_{j=1}^{s-1}|c_{ij}|)^2 \leqslant d(I)^{2(s-1)}.$$

Hence $a_1 = |a'_1| \leqslant sd(I)^{s-1}$. Similarly, $a_i \leqslant sd(I)^{s-1}$ for $i = 2, \ldots, r$, as required. □

The following lemma is a crucial result in the study of the stability of $\mathrm{Ass}(R/\overline{I^n})$.

Lemma 1.3. *Let I be a monomial ideal in R with $r > 2$. If $\mathfrak{m} \in \mathrm{Ass}\, R/\overline{I^s}$ for some $s \geqslant 1$, then*

$$\mathfrak{m} \in \mathrm{Ass}\, R/\overline{I^n} \text{ for all } n \geqslant (r-1)rd(I)^{r-2}.$$

Proof. Let $m := (r-1)rd(I)^{r-2}$. Since the sequence $\{\mathrm{Ass}\, R/\overline{I^n}\}_{n\geqslant 1}$ is increasing by [6, Proposition 16.3], it suffices to show that $\mathfrak{m} \in \mathrm{Ass}\, R/\overline{I^m}$.

As $\mathfrak{m} \in \mathrm{Ass}\, R/\overline{I^s}$, by [19, Lemma 13], there is a supporting hyperplane of $NP(I)$, say H, of the form $\langle \mathbf{a}, \mathbf{x}\rangle = b$ such that all coordinates of $\mathbf{a}$ are positive. By Lemma 1.2, this hyperplane passes through r affinely independent points of $E(G(I))$, say $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$. Let $J := (\mathbf{x}^{\boldsymbol{\alpha}_1}, \ldots, \mathbf{x}^{\boldsymbol{\alpha}_r})$. Clearly, H is still a supporting plane of $NP(J)$. Again by Lemma 1.2, the Newton polyhedron $NP(J)$ can be represented by a system of inequalities

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle = b_j, j = 1, \ldots, q\},$$

where $\mathbf{0} \neq \mathbf{a}_j \in \mathbb{N}^r$ and $b_j \in \mathbb{N}$. Let $H_j = \{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle = b_j\}$ for $j = 1, \ldots, q$. We may assume that q is minimal and $H_q = H$. Since J is generated by exactly r monomials and q is taken to be minimal, by Lemma 1.2, each hyperplane H_j, where $j \leqslant q-1$, must be parallel to at least one of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Hence, by the second statement of Lemma 1.2, we may assume that

$$a_{ji} \leqslant (r-1)d(J)^{r-2} \text{ for all } j \leqslant q-1 \text{ and } i \leqslant r. \tag{1.3}$$

Consider the barycenter $\boldsymbol{\alpha} := \frac{1}{r}(\boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_r)$ of the simplex $[\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r]$. Then $\boldsymbol{\alpha}$ is a relative interior point of the facet $H_q \cap NP(J)$ of $NP(J)$. Therefore, $\boldsymbol{\alpha}$ does not lie in H_j for all $j = 1, \ldots, q-1$, and so

$$\langle \mathbf{a}_j, \boldsymbol{\alpha}\rangle > b_j \text{ for all } j \leqslant q-1. \tag{1.4}$$

Next, we may assume that $a_{qr} = \min\{a_{q1}, \ldots, a_{qr}\} > 0$. Let $\boldsymbol{\beta} := m\boldsymbol{\alpha} - \mathbf{e}_r$. Then $\boldsymbol{\beta} = (r-1)d(I)^{r-2}(\boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_r) - \mathbf{e}_r \in \mathbb{Z}^r$. Since $\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_r \in H_q$ are affinely independent and $a_{q1}, ..., a_{qr} > 0$, there exists $j \leqslant r$ such that $\boldsymbol{\alpha}_{jr} > 0$, whence $\boldsymbol{\alpha}_{jr} \geqslant 1$. Hence $\boldsymbol{\beta} \in \mathbb{N}^r$. Moreover,

$$\langle \mathbf{a}_q, \boldsymbol{\beta} \rangle = m \langle \mathbf{a}_q, \boldsymbol{\alpha} \rangle - \langle \mathbf{a}_q, \mathbf{e}_r \rangle = mb_q - a_{qr} < mb_q.$$

Therefore $\boldsymbol{\beta} \notin NP(J^m)$ and also $\boldsymbol{\beta} \notin NP(I^m)$ (recall that $H = H_q$).

On the other hand, we claim that

$$\boldsymbol{\beta} + \mathbf{e}_i \in NP(J^m) \text{ for all } i = 1, \ldots, r. \tag{1.5}$$

Indeed, for $i = r$, $\boldsymbol{\beta} + \mathbf{e}_r = m\boldsymbol{\alpha} \in mNP(J) = NP(J^m)$. For $i \leqslant r - 1$, we have

$$\begin{aligned} \langle \mathbf{a}_q, \boldsymbol{\beta} + \mathbf{e}_i \rangle &= \langle \mathbf{a}_q, m\boldsymbol{\alpha} - \mathbf{e}_r + \mathbf{e}_i \rangle \\ &= m \langle \mathbf{a}_q, \boldsymbol{\alpha} \rangle - \langle \mathbf{a}_q, \mathbf{e}_r \rangle + \langle \mathbf{a}_q, \mathbf{e}_i \rangle \\ &= mb_q - a_{qr} + a_{qi} \\ &\geqslant mb_q \ \ (\text{since } a_{qr} = \min\{a_{q1}, \ldots, a_{qr}\}). \end{aligned}$$

Let $j \leqslant q-1$. Since $r\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_r \in NP(J^r) \cap \mathbb{N}^r$, by (1.4), we have $\langle \mathbf{a}_j, r\boldsymbol{\alpha} \rangle > rb_j$, which implies $\langle \mathbf{a}_j, r\boldsymbol{\alpha} \rangle \geqslant rb_j + 1$. Hence

$$\begin{aligned} \langle \mathbf{a}_j, \boldsymbol{\beta} + \mathbf{e}_i \rangle &= \langle \mathbf{a}_j, m\boldsymbol{\alpha} - \mathbf{e}_r + \mathbf{e}_i \rangle \\ &= (r-1)d(I)^{r-2} \langle \mathbf{a}_j, r\boldsymbol{\alpha} \rangle - \langle \mathbf{a}_j, \mathbf{e}_r \rangle + \langle \mathbf{a}_j, \mathbf{e}_i \rangle \\ &\geqslant (r-1)d(I)^{r-2}(rb_j + 1) - a_{jr} + a_{ji} \\ &= mb_j + ((r-1)d(I)^{r-2} - a_{jr}) + a_{ji} \\ &\geqslant mb_j \ \ (\text{by } (1.3)). \end{aligned}$$

This completes the proof of (1.5).

Since $NP(J^m) = mNP(J) \subseteq mNP(I) = NP(I^m)$, $\boldsymbol{\beta} + \mathbf{e}_i \in NP(I^m)$, whence $\mathbf{x}^{\boldsymbol{\beta}} x_i \in \overline{I^m}$. As shown above, $\boldsymbol{\beta} \notin NP(I^m)$. Therefore, $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^m}$, as required. $\square$

A main tool in the study of the set of associated primes and the depth of rings is using local cohomology modules. In the setting of monomial ideals, one often uses a generalized version of a Hochster's formula given by Takayama in [16]. Let us recall this formula here.

Since R/I is an $\mathbb{N}^r$-graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over R. For every degree $\boldsymbol{\alpha} \in \mathbb{Z}^r$ we denote by $H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ the $\boldsymbol{\alpha}$-component of $H^i_{\mathfrak{m}}(R/I)$.

Let $\Delta(I)$ denote the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$, i.e.

$$\Delta(I) = \{\{i_1, ..., i_s\} \subseteq [r] |\ x_{i_1} \cdots x_{i_s} \notin \sqrt{I}\},$$

where $[r]$ denotes the set $\{1, 2, ..., r\}$. For every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, we define its co-support to be the set $CS_{\boldsymbol{\alpha}} := \{i \mid \alpha_i < 0\}$. For a subset F of $[r]$, let $R_F := R[x_i^{-1} \mid i \in F]$. Set

$$\Delta_{\boldsymbol{\alpha}}(I) = \{F \subseteq [r] \setminus CS_{\boldsymbol{\alpha}} |\ \mathbf{x}^{\boldsymbol{\alpha}} \notin IR_{F \cup CS_{\boldsymbol{\alpha}}}\}. \tag{1.6}$$

We set $\widetilde{H}_i(\emptyset;k)=0$ for all i, $\widetilde{H}_i(\{\emptyset\};k)=0$ for all $i\neq -1$, and $\widetilde{H}_{-1}(\{\emptyset\};k)=k$. Thanks to [4, Lemma 1.1] we may formulate Takayama's formula as follows.

Lemma 1.4. ([16, Theorem 2.2]) $\dim_k H^i_{\mathfrak m}(R/I)_{\boldsymbol\alpha} = \dim_k \widetilde{H}_{i-|CS_{\boldsymbol\alpha}|-1}(\Delta_{\boldsymbol\alpha}(I);k)$.

As an immediate consequence of this result is the following "quasi-decreasing" property of the depth function $\operatorname{depth} R/\overline{I^n}$. We don't know if this property holds for an arbitrary homogeneous ideal.

Lemma 1.5. *For any monomial ideal I of R, we have*

(1) $\operatorname{depth} R/\overline{I^m} \geqslant \operatorname{depth} R/\overline{I^{mn}}$ *for all* $m,n\geqslant 1$.

(2) $\lim_{n\to\infty}\operatorname{depth} R/\overline{I^n} = \dim R-\ell(I)$, *where $\ell(I)$ denotes the analytic spread of I.*

Proof. 1) Replacing I^m by J, it suffices to prove the statement for $m=1$. Let $t:=\operatorname{depth} R/\overline{I}$. Then we must have $H^t_{\mathfrak m}(R/\overline{I})_{\boldsymbol\alpha}\neq 0$ for some $\boldsymbol\alpha\in\mathbb{Z}^r$. By Lemma 1.4,

$$\dim_k \widetilde{H}_{t-|CS_{\boldsymbol\alpha}|-1}(\Delta_{\boldsymbol\alpha}(\overline{I});k) = \dim_k H^t_{\mathfrak m}(R/\overline{I})_{\boldsymbol\alpha}\neq 0. \tag{1.7}$$

For $n\geqslant 1$, we have $CS_{n\boldsymbol\alpha}=CS_{\boldsymbol\alpha}$ and

$$\Delta_{\boldsymbol\alpha}(\overline{I}) = \{F\in\Delta \mid \mathbf{x}^{\boldsymbol\alpha}\notin \overline{IR_{F\cup CS_{\boldsymbol\alpha}}}\} = \{F\in\Delta \mid \mathbf{x}^{n\boldsymbol\alpha}\notin \overline{(IR_{F\cup CS_{\boldsymbol\alpha}})^n}\} = \Delta_{n\boldsymbol\alpha}(\overline{I^n}).$$

The middle equality follows from (1.1). Together with Equation (1.7) and Lemma 1.4, this fact implies that

$$\dim_k H^t_{\mathfrak m}(R/\overline{I^n})_{n\boldsymbol\alpha} = \dim_k \widetilde{H}_{t-|CS_{n\boldsymbol\alpha}|-1}(\Delta_{n\boldsymbol\alpha}(\overline{I^n});k) = \dim_k \widetilde{H}_{t-|CS_{\boldsymbol\alpha}|-1}(\Delta_{\boldsymbol\alpha}(\overline{I});k)\neq 0.$$

This means $\operatorname{depth} R/\overline{I^n}\leqslant t$.

2) Let $J:=\overline{I^{r-1}}$. By [21, Theorem 7.29], J is torsion-free. Thus, by [3, Proposition 3.3], we have

$$\lim_{m\to\infty}\operatorname{depth} R/J^m = \dim R-\ell(J).$$

For each $m\geqslant 1$, by [21, Corollary 7.60], we have $J^m=\overline{I^{m(r-1)}}$. Hence

$$\lim_{n\to\infty}\operatorname{depth} R/\overline{I^n} = \lim_{m\to\infty}\operatorname{depth} R/J^m = \dim R-\ell(J).$$

Note that $\ell(J)=\ell(\overline{I^{r-1}})=\ell(I^{r-1})=\ell(I)$, so the desired equality follows. $\square$

Let F be a subset of $[r]$. Put $R[F]=k[x_i\mid i\notin F]$ and denote by $I[F]$ the ideal of $R[F]$ obtained from I by setting $x_i=1$ for all $i\in F$. Then $I[F]R=IR_F\cap R$. If $F=\{i\}$ for some $i\in[r]$, then we write $R[i]$ and $I[i]$ instead of $R[\{i\}]$ and $I[\{i\}]$ respectively.

Remark 1.6. Let I be a monomial ideal in R. Then,

(1) Using (1.1) it is easy to see that $\overline{I^n}[F] = \overline{I[F]^n}$ for any $n \geqslant 1$ (cf. [11, Lemma 4.6]).
(2) If I is Cohen-Macaulay, then $I[F]$ is Cohen-Macaulay.

We can now give an improvement of the main result, Theorem 16, in [19].

Theorem 1.7. *Let I be a monomial ideal of R and*

$$n_0(I) := \begin{cases} 1 & \text{if } \ell(I) \leqslant 2, \\ \ell(I)(\ell(I)-1)d(I)^{\ell(I)-2} & \text{if } \ell(I) > 2. \end{cases}$$

Then, $\operatorname{Ass} R/\overline{I^n} = \operatorname{Ass} R/\overline{I^{n_0(I)}}$ *for all $n \geqslant n_0(I)$.*

Proof. Fix an index $i \leqslant r$. It is well known that the analytic spread of I is equal to the minimal number of generators of a minimal reduction of I. Since $I[i]$ is obtained from I by setting $x_i = 1$, this implies that $\ell(I) \geqslant \ell(I[i])$ and $d(I[i]) \leqslant d(I)$. Hence $n_0(I[i]) \leqslant n_0(I)$. Using this remark, Remark 1.6(1) and Lemma 1.5, we can prove the theorem by induction on r. The proof is similar to that of [19, Theorem 16], so we omit details here. □

Remark. Set

$$\overline{\operatorname{astab}}(I) = \min\{m \mid \operatorname{Ass} R/\overline{I^n} = \operatorname{Ass} R/\overline{I^m} \text{ for all } n \geqslant m\}.$$

It can be called the index of stability for the associated primes of $R/\overline{I^n}$. An example given in [19, Proposition 17] shows that an upper bound on $\overline{\operatorname{astab}}(I)$ must be of the order $d(I)^{r-2}$, provided that r is fixed. The coefficient of $d(I)^{r-2}$ in the upper bound given in [19, Theorem 16] is $r2^{r-1}$.

2. Stability of Depth

In this section we study the stability index of the depth function $\operatorname{depth} R/\overline{I^n}$. It is clear that a simplicial complex Δ is defined by the set of its maximal faces, say $F_1, ..., F_s$. In this case we write $\Delta = \langle F_1, ..., F_s\rangle$. Keeping the notations in Lemma 1.2, we set $\operatorname{supp}(\mathbf{a}_j) := \{i \mid a_{ji} \neq 0\}$. We can describe $\Delta_{\boldsymbol{\alpha}}(\overline{I^n})$ as follows.

Lemma 2.1. *For any $\boldsymbol{\alpha} \in \mathbb{N}^r$ and $n \geqslant 1$, we have*

$$\Delta_{\boldsymbol{\alpha}}(\overline{I^n}) = \langle [r] \setminus \operatorname{supp}(\mathbf{a}_j) \mid j \in \{1, \dots, q\} \text{ and } \langle \mathbf{a}_j, \boldsymbol{\alpha}\rangle < nb_j\rangle\,.$$

Proof. Let $F \in \Delta_{\boldsymbol{\alpha}}(\overline{I^n})$. We may assume that $F = \{s+1, \dots, r\}$ for some $0 \leqslant s \leqslant r$. By Lemma 1.2 and (1.2), we can deduce that $NP(I^n)$ is the set of solutions of the system

$$\{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle \geqslant nb_j,\ j = 1, \dots, q\}.$$

Since $CS_{\boldsymbol{\alpha}} = \emptyset$, $\mathbf{x}^{\boldsymbol{\alpha}} \notin \overline{I^n}R_F$ if and only if $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\gamma}} \notin \overline{I^n}$ for any monomial $\mathbf{x}^{\boldsymbol{\gamma}} \in k[x_{s+1}, \ldots, x_r]$. Taking $\mathbf{x}^{\boldsymbol{\gamma}} = x_{s+1}^m \cdots x_r^m$, where

$$m > \max\{nb_j - \langle \mathbf{a}_j, \boldsymbol{\alpha}\rangle \mid \ 1 \le j \leqslant q\},$$

is fixed, it implies that there is $1 \le p \leqslant q$ such that

$$\langle \mathbf{a}_p, \boldsymbol{\alpha} + m(\mathbf{e}_{s+1} + \mathbf{e}_r)\rangle < nb_p.$$

Assume that there is $i \geqslant s+1$ such that $a_{pi} > 0$. Then

$$\langle \mathbf{a}_p, \boldsymbol{\alpha} + m(\mathbf{e}_{s+1} + \mathbf{e}_r)\rangle = \langle \mathbf{a}_p, \boldsymbol{\alpha}\rangle + m(a_{p(s+1)} + \cdots a_{pr}) \geqslant \langle \mathbf{a}_p, \boldsymbol{\alpha}\rangle + m > nb_p,$$

a contradiction. Hence $a_{p(s+1)} = \cdots = a_{pr} = 0$, whence $F \subseteq [r] \setminus \operatorname{supp}(\mathbf{a}_p)$. Then $\langle \mathbf{a}_p, \boldsymbol{\alpha}\rangle = \langle \mathbf{a}_p, \boldsymbol{\alpha} + m(\mathbf{e}_{s+1} + \mathbf{e}_r)\rangle < nb_p$.

Conversely, assume that there is $j \leqslant q$ such that $F \subseteq [r] \setminus \operatorname{supp}(\mathbf{a}_j)$, i.e. $a_{j(s+1)} = \cdots = a_{jr} = 0$, and $\langle \mathbf{a}_j, \boldsymbol{\alpha}\rangle < nb_j$. Then for all monomials $\mathbf{x}^{\boldsymbol{\gamma}} \in k[x_{s+1}, \ldots, x_r]$, we have $\langle \mathbf{a}_j, \boldsymbol{\alpha} + \boldsymbol{\gamma}\rangle = \langle \mathbf{a}_j, \boldsymbol{\alpha}\rangle < nb_j$. By (1.1) and Lemma 1.2, this implies $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\gamma}} \notin \overline{I^n}$. From (1.6), we get that $F \in \Delta_{\boldsymbol{\alpha}}(\overline{I^n})$. This completes the proof of the lemma. □

The following lemma is the main step in the proof of Theorem 2.3

Lemma 2.2. *Let $m \geqslant 1$ and $t := \operatorname{depth} R/\overline{I^m}$. Assume that $H^t_{\mathfrak{m}}(R/\overline{I^m})_{\boldsymbol{\beta}} \neq 0$ for some $\boldsymbol{\beta} \in \mathbb{N}^r$. If $r \geqslant 3$, then*

$$\operatorname{depth} R/\overline{I^n} \leqslant t \quad \textit{for all } n \geqslant r(r^2-1)r^{r/2}(r-1)^r d(I)^{(r-2)(r+1)}.$$

Proof. For simplicity, set $n^* := r(r^2-1)r^{r/2}(r-1)^r d(I)^{(r-2)(r+1)}$. We keep the notations in Lemma 1.2.

Assume that $\operatorname{supp}(\mathbf{a}_j) = [r]$ for some $1 \leqslant j \leqslant q$. By [19, Lemma 14] we have $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^n}$ for all $n \gg 0$. By Lemma 1.3, it yields $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^n}$ for all $n \geqslant n^*$. Thus, $\operatorname{depth} R/\overline{I^n} = 0$ for $n \geqslant n^*$, and the lemma holds in this case.

We now assume that $\operatorname{supp}(\mathbf{a}_j) \neq [r]$ for all $j = 1, \ldots, q$, i.e. the number of non-zero coordinates of $\mathbf{a}_j$ is strictly less than r. By Lemma 1.2, we have

$$a_{ji} \leqslant (r-1)d(I)^{r-2} \ \text{ for } i = 1, \ldots, r. \tag{2.1}$$

Assume that

$$\langle \mathbf{a}_j, \boldsymbol{\beta}\rangle < b_j \text{ for } j = 1, \ldots, p,$$

and

$$\langle \mathbf{a}_j, \boldsymbol{\beta}\rangle \geqslant b_j \text{ for } j = p+1, \ldots, q,$$

for some $0 \leqslant p \leqslant q$. Then, by Lemma 2.1,

$$\Delta_{\boldsymbol{\beta}}(\overline{I^m}) = \langle [r] \setminus \operatorname{supp}(\mathbf{a}_j) \mid j = 1, \ldots, p\rangle\,.$$

By Lemma 1.4, we have

$$\dim_k \widetilde{H}_{t-1}(\Delta_{\boldsymbol{\beta}}(\overline{I^m}); k) = \dim_k H^t_{\mathfrak{m}}(R/\overline{I^m})_{\boldsymbol{\beta}} \neq 0.$$

Hence $\Delta_{\boldsymbol{\beta}}(\overline{I^m})$ is not acyclic. In particular, $p \geqslant 1$. For each $n \geqslant 1$, put

$$\Gamma(\overline{I^n}) := \{\boldsymbol{\alpha} \in \mathbb{N}^r \mid \Delta_{\boldsymbol{\alpha}}(\overline{I^n}) = \Delta_{\boldsymbol{\beta}}(\overline{I^m})\},$$

and

$$(2.2) \quad C_n := \{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle < nb_j, \langle \mathbf{a}_l, \mathbf{x}\rangle \geqslant nb_l \text{ for } j \leqslant p;\ p+1 \leq l \leqslant q\} \subseteq \mathbb{R}^r_+.$$

It is clear that $C_n = nC_1$. By Lemma 2.1, $C_n \cap \mathbb{N}^r \subseteq \Gamma(\overline{I^n})$.

Assume that $C_n \cap \mathbb{N}^r \neq \emptyset$. Then for any $\boldsymbol{\alpha} \in C_n \cap \mathbb{N}^r$, by Lemma 1.4, we have

$$\dim_k H^t_{\mathfrak{m}}(R/\overline{I^n}))_{\boldsymbol{\alpha}} = \dim_k \widetilde{H}_{t-1}(\Delta_{\boldsymbol{\alpha}}(\overline{I^n}); k) = \dim_k \widetilde{H}_{t-1}(\Delta_{\boldsymbol{\beta}}(\overline{I^m}); k) \neq 0,$$

whence $\operatorname{depth} R/\overline{I^n} \leqslant t$.

Thus, in order to complete the proof of the lemma, it remains to show that $C_n \cap \mathbb{N}^r \neq \emptyset$ for any $n \geqslant n^*$. Fix such an integer n.

Since $\boldsymbol{\beta} \in C_m = mC_1$, $C_1 \neq \emptyset$. First, we prove that C_1 is bounded in $\mathbb{R}^r$. Assume that $a_{ji} = 0$ for some $1 \leqslant i \leqslant r$ and for all $j = 1, \ldots, p$. Then, for any $s \gg 0$, by Formula (2.2) we get that $\boldsymbol{\beta} + s\mathbf{e}_i \in C_m$, which implies $\Delta_{\boldsymbol{\beta}+s\mathbf{e}_i}(\overline{I^m}) = \Delta_{\boldsymbol{\beta}}(\overline{I^m})$. Again by Lemma 1.4, we have

$$\dim_k H^t_{\mathfrak{m}}(R/\overline{I^m})_{\boldsymbol{\beta}+s\mathbf{e}_i} = \dim_k \widetilde{H}_{t-1}(\Delta_{\boldsymbol{\beta}+s\mathbf{e}_i}(\overline{I^m}); k) = \dim_k \widetilde{H}_{t-1}(\Delta_{\boldsymbol{\beta}}(\overline{I^m}); k) \neq 0.$$

This contradicts the Artiness of $H^t_{\mathfrak{m}}(R/\overline{I^m})$. Hence, for each $i \leqslant r$, there is $j_i \leqslant p$ such that $a_{j_i i} \geqslant 1$. Let $\mathbf{y} = (y_1, \ldots, y_r)$ be an arbitrary point of $C_1 \subseteq \mathbb{R}^r_+$. Then for each $i \leqslant r$, we have $y_i \leqslant a_{j_i i} y_i \leqslant \langle \mathbf{a}_{j_i}, \mathbf{y}\rangle < b_{j_i}$. This implies that C_1 is bounded and so is C_n.

Let $\overline{C}_n$ be the closure of C_n in $\mathbb{R}^r$ with respect to the usual Euclidean topology. Then $\overline{C_n}$ is bounded as well. Moreover,

$$(2.3) \quad \overline{C}_n = \{\mathbf{x} \in \mathbb{R}^r \mid \langle \mathbf{a}_j, \mathbf{x}\rangle \geqslant nb_j, \langle \mathbf{a}_l, \mathbf{x}\rangle \leqslant nb_l \text{ for } j \leqslant p;\ p+1 \leq l \leqslant q\} \subseteq \mathbb{R}^r_+,$$

and hence $\overline{C}_n$ is a polytope.

We next claim that $\overline{C}_1$ is full dimensional. Indeed, for any $\mathbf{y} \in C_1$, by Formula (2.2) we can choose a real number $\varepsilon > 0$ such that for all real numbers $\varepsilon_1, \ldots, \varepsilon_r$ with $0 \leqslant \varepsilon_1, \ldots, \varepsilon_r \leqslant \varepsilon$, we have $\mathbf{y} + \varepsilon_1\mathbf{e}_1 + \cdots + \varepsilon_r\mathbf{e}_r \in C_1$. This means that the parallelotope $[y_1, y_1+\varepsilon] \times \cdots \times [y_r, y_r+\varepsilon] \subseteq C_1$, and thus $\overline{C}_1$ is full dimensional in $\mathbb{R}^r$, as claimed.

Since the polytope $\overline{C}_1$ is full dimensional, by the Decomposition Theorem for polyhedra (see [15, Corollary 7.1.b]), we can find $r+1$ vertices, say $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_r$, of the polytope $\overline{C}_1$, which are affinely independent. Let $\boldsymbol{\alpha} = \frac{1}{r+1}(\boldsymbol{\alpha}_0 + \cdots + \boldsymbol{\alpha}_r)$ be the barycenter of the r-simplex $[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r] \subseteq \overline{C_1}$.

For each $i \leqslant r$, set $\lambda_i = \lceil \alpha_i \rceil - \alpha_i \geqslant 0$, where $\lceil \alpha_i \rceil$ is the least integer which is bigger than or equal α_i. Then $\lambda_1 + \cdots + \lambda_r < r$ and $\boldsymbol{\gamma} := n\boldsymbol{\alpha} + \lambda_1 \mathbf{e}_1 + \cdots + \lambda_r \mathbf{e}_r \in \mathbb{N}^r$. In order to show $C_n \cap \mathbb{N}^r \neq \emptyset$, it suffices to show that $\boldsymbol{\gamma} \in C_n$.

Since $\boldsymbol{\alpha} \in \overline{C_1}$, by Formula (2.3), we have

$$\langle \mathbf{a}_l, \boldsymbol{\gamma} \rangle = \langle \mathbf{a}_l, n\boldsymbol{\alpha} \rangle + \left\langle \mathbf{a}_l, \sum_{i=1}^{r} \lambda_i \mathbf{e}_i \right\rangle \geqslant n \langle \mathbf{a}_l, \boldsymbol{\alpha} \rangle \geqslant n b_l,$$

for all $l = p+1, \ldots, q$.

Now, fix an index $j \leqslant p$. Since $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$ are affinely independent in $\mathbb{R}^r$, there is at least one point not lying in the hyperplane $\langle \mathbf{a}_j, \mathbf{x} \rangle = b_j$. We may assume that $\boldsymbol{\alpha}_0$ is such a point. From Formula (2.3) we then have

$$\langle \mathbf{a}_j, \boldsymbol{\alpha}_0 \rangle < b_j \ \text{ and } \ \langle \mathbf{a}_j, \boldsymbol{\alpha}_i \rangle \leqslant b_j \text{ for all } i = 1, \ldots, r. \tag{2.4}$$

Since $\boldsymbol{\alpha}_0$ is a vertex of the polytope $\overline{C}_1$, by [15, Formula 23 in Page 104], $\boldsymbol{\alpha}_0$ can be represented as the unique solution of a system of linear equations of the form:

$$\langle \mathbf{a}_h, \mathbf{x} \rangle = b_h \ \text{ for all } h \in S \subseteq \{1, \ldots, q\},$$

where $|S| = r$. By Cramer's rule we have $\alpha_{0i} = \delta_i/\delta$ for all $i \leqslant r$, where $\delta, \ \delta_1, \ldots, \ \delta_r \in \mathbb{N}$ and δ is the absolute value of the determinant of this system of linear equations. In particular, $\delta\boldsymbol{\alpha}_0 \in \mathbb{N}^r$. Using the inequalities (2.1) and Hadamard's inequality applied to δ, we get

$$\delta \leqslant r^{r/2}(r-1)^r d(I)^{r(r-2)}. \tag{2.5}$$

By (2.4) we have $\langle \mathbf{a}_j, \delta\boldsymbol{\alpha}_0 \rangle < \delta b_j$, whence $\langle \mathbf{a}_j, \delta\boldsymbol{\alpha}_0 \rangle \leqslant \delta b_j - 1$ because $\mathbf{a}_j, \ \delta\boldsymbol{\alpha}_0 \in \mathbb{N}^r$. Let $c = n/(r+1)\delta$, then by (2.5), $c \geqslant r(r-1)d(I)^{r-2}$. We then have

$$\begin{aligned}
\langle \mathbf{a}_j, n\boldsymbol{\alpha} \rangle &= c\delta \langle \mathbf{a}_j, (r+1)\boldsymbol{\alpha} \rangle \\
&= c\delta \sum_{i=0}^{r} \langle \mathbf{a}_j, \boldsymbol{\alpha}_i \rangle \\
&= c \langle \mathbf{a}_j, \delta\boldsymbol{\alpha}_0 \rangle + c\delta \sum_{i=1}^{r} \langle \mathbf{a}_j, \boldsymbol{\alpha}_i \rangle \\
&\leqslant c(\delta b_j - 1) + rc\delta b_j \\
&= nb_j - c.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle \mathbf{a}_j, \boldsymbol{\gamma} \rangle &= \langle \mathbf{a}_j, n\boldsymbol{\alpha} \rangle + \left\langle \mathbf{a}_j, \sum_{i=1}^{r} \lambda_i \mathbf{e}_i \right\rangle \\
&= \langle \mathbf{a}_j, n\boldsymbol{\alpha} \rangle + \sum_{i=1}^{r} \lambda_i a_{ji} \\
&< nb_j - c + r(r-1)d(I)^{r-2} \quad \text{(by (2.1) and } \sum \lambda_i < r) \\
&\leqslant nb_j.
\end{aligned}$$

So $\langle \mathbf{a}_j, \boldsymbol{\gamma} \rangle \leqslant nb_j - 1$, for all $j \leqslant p$. This means that $\boldsymbol{\gamma} \in C_n$, as required. □

We are now in position to prove the main result of this section.

Theorem 2.3. *Let I be a monomial ideal of R. Let*

$$n_1(I) := \begin{cases} 1 & \text{if } r \leqslant 2, \\ r(r^2-1)r^{r/2}(r-1)^r d(I)^{(r-2)(r+1)} & \text{if } r > 2. \end{cases}$$

Then, $\operatorname{depth} R/\overline{I^n} = \dim R - \ell(I)$ *for all* $n \geqslant n_1(I)$.

Proof. We prove the theorem by induction on r. If $r \leqslant 2$, and $I \neq 0$, then $\operatorname{depth} R/\overline{I^n} = 0, 1$, and $\operatorname{depth} R/\overline{I^n} = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass}(R/\overline{I^n})$. Since $\operatorname{Ass}(R/\overline{I^n})$ is constant for all $n \geqslant 1$ in this case (by [12, Proposition 16]), we get

$$\operatorname{depth} R/\overline{I^n} = \operatorname{depth} R/\overline{I} \quad \text{for all } n \geqslant 1,$$

and the theorem follows from Lemma 1.5(2).

Assume that $r \geqslant 3$. By virtue of Lemma 1.5(2) and symmetry, it suffices to show that

$$\operatorname{depth} R/\overline{I^n} \leqslant \operatorname{depth} R/\overline{I^m}, \tag{2.6}$$

for any $m, n \geqslant n_1(I)$. Let $t := \operatorname{depth} R/\overline{I^m}$. As $H^t_{\mathfrak{m}}(R/\overline{I^m}) \neq 0$, by Lemma 1.4, there is $\boldsymbol{\beta} \in \mathbb{Z}^r$ such that

$$\dim_k \widetilde{H}_{t-|CS_{\boldsymbol{\beta}}|-1}(\Delta_{\boldsymbol{\beta}}(\overline{I^m}); k) = H^t_{\mathfrak{m}}(R/\overline{I^m})_{\boldsymbol{\beta}} \neq 0. \tag{2.7}$$

In particular, $\Delta_{\boldsymbol{\beta}}(\overline{I^m}) \neq \emptyset$. If $CS_{\boldsymbol{\beta}} = \emptyset$, i.e., $\boldsymbol{\beta} \in \mathbb{N}^r$, then (2.6) follows from Lemma 2.2.

We now assume that $CS_{\boldsymbol{\beta}} \neq \emptyset$. Without loss of generality, we may assume that $CS_{\boldsymbol{\beta}} = \{s+1, \ldots, r\}$ for some integer $0 \leqslant s \leqslant r$. If $s = 0$, i.e. $CS_{\boldsymbol{\beta}} = [r]$, then $\Delta_{\boldsymbol{\beta}}(\overline{I^m}) = \{\emptyset\}$. By Lemma 1.4, it follows that $H^0_{\mathfrak{m}}(R/\overline{I^m}) = 0$, which is equivalent to $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^m}$. Since $r \geqslant 3$, $n \geqslant n_1(I) > (r-1)rd(I)^{r-2}$, $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^n}$ by Lemma 1.3. Hence, $\operatorname{depth} R/\overline{I^n} = 0 = \operatorname{depth} R/\overline{I^m}$ in this case.

Assume that $s \geqslant 1$. Let $R' := k[x_1, \ldots, x_s] = R[\{s+1, ..., r\}]$ (in the notation before Remark 1.6) and $I' := I[\{s+1, ..., r\}] \subseteq R'$. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$ and $\boldsymbol{\beta}' := (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s$. Then, by Formula (1.6), $\Delta_{\boldsymbol{\beta}'}(\overline{I'^m}) = \Delta_{\boldsymbol{\beta}}(\overline{I^m})$. Let $\mathfrak{n} := (x_1, \ldots, x_s)$ be the maximal homogeneous ideal of R'. Using (2.7) and Lemma 1.4 we obtain

$$\begin{aligned}\dim_k H_{\mathfrak{n}}^{t-|CS_{\boldsymbol{\beta}}|}(R'/\overline{I'^m})_{\boldsymbol{\beta}'} &= \dim_k \widetilde{H}_{t-|CS_{\boldsymbol{\beta}}|-1}(\Delta_{\boldsymbol{\beta}'}(\overline{I'^m}); k)\\ &= \dim_k \widetilde{H}_{t-|CS_{\boldsymbol{\beta}}|-1}(\Delta_{\boldsymbol{\beta}}(\overline{I^m}); k) \neq 0.\end{aligned}$$

Hence $I' \neq R'$ and $\operatorname{depth} R'/\overline{I'^m} \leqslant t - |CS_{\boldsymbol{\beta}}|$, or equivalently $\operatorname{depth} R/\overline{I^m} \geqslant |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^m}$. On the other hand, by [10, Lemma 1.3], we have $\operatorname{depth} R/\overline{I^m} \leqslant |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^m}$ and $\operatorname{depth} R/\overline{I^n} \leqslant |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^n}$. Hence

$$\operatorname{depth} R/\overline{I^m} = |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^m},$$

and (noticing that $n_1(I') \leqslant n_1(I)$, since $d(I') \leqslant d(I)$ and $s \leqslant r$)

$$\begin{aligned}\operatorname{depth} R/\overline{I^n} &\leqslant |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^n}\\ &= |CS_{\boldsymbol{\beta}}| + \operatorname{depth} R'/\overline{I'^m} \quad \text{(by the induction hypothesis)}\\ &= \operatorname{depth} R/\overline{I^m}.\end{aligned}$$

□

Remark. Set

$$\overline{\operatorname{dstab}}(I) = \min\{m| \ \operatorname{depth} R/\overline{I^n} = \operatorname{depth} R/\overline{I^m} \text{ for all } n \geqslant m\}.$$

One can call it the index of depth stability for integral closures. Then Theorem 2.3 says that $\overline{\operatorname{dstab}}(I) \leq n_1(I)$. It seems that this bound is too big. However, an example given in [19, Proposition 17] shows that an upper bound on $\overline{\operatorname{dstab}}(I)$ must be at least of the order $d(I)^{r-2}$.

3. Cohen-Macaulay property

In this section we apply results in previous sections to study the Cohen-Macaulayness of integeral closures of powers of monomial ideals. We say that I is equimultiple if $\ell(I) = \operatorname{ht}(I)$. Note that, by [1, Theorem 2.3], we can compute $\ell(I)$ in terms of geometry of $NP(I)$.

$$\ell(I) = \max\{\dim F + 1 \mid F \text{ is a compact face of } NP(I)\}.$$

Therefore, the condition I being equimultiple is independent on the characteristic of the base field k.

Theorem 3.1. *Let I be a monomial ideal of R. The following conditions are equivalent*

(1) *$R/\overline{I^n}$ is a Cohen-Macaulay ring for all $n \geqslant 1$,*

(2) $R/\overline{I^n}$ *is a Cohen-Macaulay ring for some* $n \geqslant n_1(I)$*, where* $n_1(I)$ *is defined in Theorem 2.3,*
(3) I *is an equimultiple ideal of* R*.*

Proof. If $R/\overline{I^n}$ is a Cohen-Macaulay ring for some $n \geq n_1(I)$, then by Theorem 2.3 we have $\dim R/\overline{I^n} = \operatorname{depth} R/\overline{I^n} = \dim R - \ell(I)$. On the other hand, $\dim R/\overline{I^n} = \dim R/I = \dim R - \operatorname{ht}(I)$. Hence, $\ell(I) = \operatorname{ht}(I)$.

Conversely, assume that $\ell(I) = \operatorname{ht}(I)$. Then,

$$\operatorname{depth} R/\overline{I} \leqslant \dim R/\overline{I} = \dim R/I = \dim R - \operatorname{ht}(I) = \dim R - \ell(I).$$

For all $n \geqslant 1$, by Lemma 1.5 applied to $m \gg 0$, we have

$$\dim R - \ell(I) \geqslant \operatorname{depth} R/\overline{I} \geqslant \operatorname{depth} R/\overline{I^n} \geqslant \operatorname{depth} R/\overline{I^{mn}} = \dim R - \ell(I).$$

Hence,

$$\operatorname{depth} R/\overline{I^n} = \dim R - \ell(I) = \dim R - \operatorname{ht}(I) = \dim R/I = \dim R/\overline{I^n},$$

which means that $R/\overline{I^n}$ is a Cohen-Macaulay ring. □

In the rest of this section we will improve the above theorem for the class of square-free monomial ideals. We need some auxiliary results.

Lemma 3.2. *Let* I *be an unmixed monomial ideal and* $I = Q_1 \cap \cdots \cap Q_s$ *be an irredundant primary decomposition of* I*. Assume that* $\overline{I^n}$ *is unmixed for some* $n \geqslant 1$*. Then*

$$\overline{I^n} = \overline{Q_1^n} \cap \cdots \cap \overline{Q_s^n}.$$

Proof. We prove by induction on s and on r. If $s = 1$ (which also includes the case $r = 1$) there is nothing to prove. Assume that $s \geqslant 2$ and $r \geqslant 2$. Since I is unmixed, $\sqrt{Q_j} \neq \mathfrak{m}$ for all $j \leqslant s$. For each $i \leqslant r$, let

$$A_i := \{j \leqslant s \mid x_i \notin \sqrt{Q_j}\}.$$

If there is $j_0 \leq s$ such that $j_0 \notin \cup_{i=1}^r A_i$, then $x_1, ..., x_r \in \sqrt{Q_{j_0}}$, a contradiction. Hence $\cup_{i=1}^r A_i = [s]$ and $I = \bigcap_{i=1}^r (\cap_{j \in A_i} Q_j))$, where we set $\cap_{j \in A_i} Q_j = R$ if $A_i = \emptyset$. Moreover, using Remark 1.6(1) and the induction hypothesis on r, we may assume that $|A_i| < s$ for all $i \leqslant r$.

It is well-known that one can get a primary decomposition of a monomial ideal $\mathfrak{a} \subset R$ by repeated application of the formula $(B, uv) = (B, u) \cap (B, v)$, where B is a set of monomials and u, v are monomials having no common variable. Based on this fact, it is immediate to see that one can get a primary decomposition of $\mathfrak{a}[i]R$ from that of $\mathfrak{a}$ by deleting those primary components whose associated prime ideals contain x_i (recall that $\mathfrak{a}[i]$ is obtained from $G(I)$ by setting $x_i = 1$).

Using this remark we see that $I[i]$ is an unmixed ideal for all $i \leqslant r$. Moreover $I[i] = \cap_{j \in A_i} Q_j$ and $I = \cap_{i=1}^r I[i]$.

By Remark 1.6(1) $\overline{I[i]^n R} = (\overline{I^n})[i]R$ and $\overline{I^n} = \cap_{i=1}^r \overline{I[i]^n R}$. Since $\overline{I^n}$ is unmixed, by the above remark, all $\overline{I[i]^n R}$ are unmixed ideals. Since $|A_i| < s$, by the induction hypothesis on s, we also get $\overline{I[i]^n R} = \cap_{j \in A_i} \overline{Q_j^n}$.

Let $J := \overline{Q_1^n} \cap \cdots \cap \overline{Q_s^n}$. This is an unmixed ideal. Hence, as shown above, $J = \bigcap_{i=1}^r J[i]R$ and $J[i]R = \cap_{j \in A_i} \overline{Q_j^n} = \overline{I[i]^n R}$. Then $J = \cap_{i=1}^r \overline{I[i]^n R} = \overline{I^n}$, as required. □

Lemma 3.3. *Let y be a new variable and $S = R[y] = k[x_1, ..., x_r, y]$. Then, for every $n \geqslant 1$ we have*

(1) $\overline{(I, y)^n} = \sum_{i=0}^n y^i \overline{I^{n-i}} S$.

(2) $\overline{(I, y)^n}$ *is Cohen-Macaulay if and only if $\overline{I^i}$ is Cohen-Macaulay for all $i \leqslant n$.*

Proof. (1) The inclusion $\sum_{i=0}^n y^i \overline{I^{n-i}} S \subseteq \overline{(I, y)^n}$ follows from the fact that $\overline{I_1} \cdot \overline{I_2} \subseteq \overline{I_1 I_2}$ for all ideals I_1 and I_2 in S.

In order to prove the reverse inclusion, let $G(I) = \{\mathbf{x}^{\boldsymbol{\alpha}_1}, \ldots, \mathbf{x}^{\boldsymbol{\alpha}_s}\}$. Set $\boldsymbol{\alpha}_j^* = (\boldsymbol{\alpha}_j, 0) \in \mathbb{N}^{r+1}$ and let $\mathbf{e}_{r+1}$ be the $(r+1)$-th unit vector of $\mathbb{R}^{r+1}$. Assume that $(\boldsymbol{\alpha}, \beta) \in NP((I, y)^n) \cap \mathbb{N}^{r+1}$. From (1.2) we see that there are non-negative numbers $a_1, ..., a_s, b$ such that $\sum_{j=1}^r a_j + b \geqslant n$ and $(\boldsymbol{\alpha}, \beta) = \sum_{j=1}^s a_j \boldsymbol{\alpha}_j^* + b\mathbf{e}_{r+1}$. Then $b = \beta \in \mathbb{N}$. If $b \geqslant n$, then $\mathbf{x}^{\boldsymbol{\alpha}} y^\beta \in y^n S$. Assume that $b < n$. Then $\sum_{j=1}^r a_j \geqslant n - b > 0$ and $\boldsymbol{\alpha} = \sum_{j=1}^s a_j \boldsymbol{\alpha}_j \in NP(I^{n-b})$. Hence, by (1.1), $\mathbf{x}^{\boldsymbol{\alpha}} y^\beta \in y^m \overline{I^{n-m}} S$. In both cases, $\mathbf{x}^{\boldsymbol{\alpha}} y^\beta \in \sum_{i=0}^n y^i \overline{I^{n-i}} S$, i.e., $\overline{(I, y)^n} \subseteq \sum_{i=0}^n y^i \overline{I^{n-i}} S$.

(2) From (1) we deduce that

$$S/\overline{(I, y)^n} \cong R/\overline{I} \oplus R/\overline{I^2} \oplus \cdots \oplus R/\overline{I^n}$$

as R-modules, and the conclusion follows. □

From now on, let I be an ideal generated by square-free monomials. Such an ideal is often called a Stanley-Reisner ideal and is associated to the simplicial complex $\Delta := \Delta(I)$. In this case we also denote I by I_Δ. Note that we do not require that Δ contains all vertices $\{i\}$, $i \leqslant r$. Recall that for a face $F \in \Delta$, the link of F is defined by

$$\text{lk}_\Delta(F) = \{G \subseteq [r] \setminus F | \ F \cup G \in \Delta\}.$$

We simply write $\text{lk}_\Delta i$ for $\text{lk}_\Delta\{i\}$.

Corollary 3.4. *If $\overline{I_\Delta^n}$ is Cohen-Macaulay, then so is $\overline{I_{\text{lk}_\Delta i}^n}$ for every vertex i of Δ.*

Proof. Let $S = k[x_j \mid j \neq i]$. Then $R = S[x_i]$. Let $J = I_\Delta R[x_i^{-1}] \cap S$. We have $\overline{J^n} = \overline{I_\Delta^n} R[x_i^{-1}] \cap S$, whence $\overline{J^n}$ is Cohen-Macaulay.

Denote by $V(\Delta)$ the set of vertices of a simplicial complex Δ. Let $Y = \{x_j \mid j \notin V(\operatorname{lk}_\Delta i) \text{ and } j \neq i\}$. By [17, Lemma 2.1] we have

$$I_\Delta R[x_i^{-1}] = (I_{\operatorname{lk}_\Delta i}, Y)R[x_i^{-1}].$$

It follows that $J = (I_{\operatorname{lk}_\Delta i}, Y)$ as ideals in S. By Lemma 3.3 we conclude that $\overline{I^n_{\operatorname{lk}_\Delta i}}$ is Cohen-Macaulay. □

Note that I_Δ is a complete intersection if and only if any two of its minimal monomial generators have no common variable. Recall that $\dim \Delta = \max\{|F| \mid F \in \Delta\} - 1$.

Lemma 3.5. *Assume that* $\dim \Delta = 0$ *and that* $\overline{I^n_\Delta}$ *is Cohen-Macaulay for some* $n \geqslant 2$. *Then,* I_Δ *is a complete intersection. Moreover,* Δ *has at most two vertices.*

Proof. Since $\dim \Delta = 0$, we may assume that $\Delta = \langle \{1\}, \dots, \{s\} \rangle$ for some $s \leqslant r$. Then

$$I_\Delta = (x_{s+1}, ..., x_r) + \cap_{i=1}^s (x_1, \dots, \widehat{x_i}, \dots, x_s) = (x_{s+1}, ..., x_r; x_i x_j \mid 1 \leqslant i < j \leqslant s).$$

If $s \leqslant 2$, then I_Δ is a complete intersection. It remains to show that $s \leqslant 2$. Assume on the contrary that $s \geqslant 3$. Since

$$\overline{I^2_\Delta} = \overline{(x_{s+1}, ..., x_r, x_i x_j \mid 1 \leqslant i < j \leqslant s)^2},$$

we can check that $\mathfrak{m} = \overline{I^2} \colon (x_1 x_2 x_3)$, but $x_1 x_2 x_3 \notin I^2_\Delta$. Hence $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^2}$. Using the non-decreasing property of the sets associated prime ideals of integral closures of powers of an ideal (see [6, Proposition 16.3]), we have $\mathfrak{m} \in \operatorname{Ass} R/\overline{I^n}$. Hence $\overline{I^n_\Delta}$ is not Cohen-Macaulay, a contradiction. □

Lemma 3.6. *Assume that* $\dim \Delta = 1$ *and that* $\overline{I^n_\Delta}$ *is Cohen-Macaulay for some* $n \geqslant 3$. *Then,* I_Δ *is a complete intersection.*

Proof. Since $\overline{I^n_\Delta}$ is Cohen-Macaulay and $\sqrt{\overline{I^n_\Delta}} = I_\Delta$, Δ is Cohen-Macaulay by [9, Theorem 2.6]. Since $\dim \Delta = 1$, this property implies that Δ is connected. In particular, every facet of Δ has exactly two vertices, and we can regard Δ as a connected graph without isolated vertices. We may assume that $V(\Delta) = [s]$ for some $s \leqslant r$.

If $s = 2$, then Δ is just an edge, and $I_\Delta = (x_3, ..., x_r)$. Assume that $s \geqslant 3$. For each vertex i of Δ, by Corollary 3.4 and Lemma 3.5, $\operatorname{lk}_\Delta(i)$ is either one vertex or consists of exactly two vertices. Consequently, Δ is either a path or a cycle.

For each edge $\{i, j\}$ of Δ , set $P_{ij} = (x_{s+1}, ..., x_r; x_l \mid 1 \leq l \leqslant s;\ l \neq i \text{ and } l \neq j)$. Then, $I_\Delta = \cap_{\{i,j\} \in \Delta} P_{ij}$. Since $\overline{I^n_\Delta}$ is unmixed, by Lemma 3.2 we have

$$\overline{I^n_\Delta} = \cap_{\{i,j\} \in \Delta} \overline{P^n_{ij}} = \cap_{\{i,j\} \in \Delta} P^n_{ij} = I^{(n)}_\Delta,$$

where $I_\Delta^{(n)}$ is the n-th symbolic power of I_Δ. This implies that $I_\Delta^{(n)}$ is Cohen-Macaulay. By [13, Theorem 2.4], every pair of disjoint edges of Δ is contained in a cycle of length 4. Since Δ is either a path or a cycle, we conclude that Δ is either a path of length two, or a cycle of length 3 or 4. Hence, either $I_\Delta = (x_4, ..., x_r; x_1x_3),\ (x_4, ..., x_r; x_1x_2x_3)$ or $I_\Delta = (x_5, ..., x_r; x_1x_3, x_2x_4)$ - all are complete intersections. $\square$

We can now improve Theorem 3.1 for square-free monomial ideals by giving an exact description of all square-free monomial ideals I_Δ such that $\overline{I_\Delta^n}$ is Cohen-Macaulay for some $n \geqslant 3$.

Theorem 3.7. *Let Δ be a simplicial complex. Then the following conditions are equivalent:*

(1) *$\overline{I_\Delta^n}$ is Cohen-Macaulay for every $n \geqslant 1$;*
(2) *$\overline{I_\Delta^n}$ is Cohen-Macaulay for some $n \geqslant 3$;*
(3) *I_Δ is a complete intersection;*
(4) *I_Δ is an equimultipe ideal.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear. (4) $\Rightarrow$ (1) follows from Theorem 3.1.

It remains to prove that (2) $\Rightarrow$ (3). The following proof is similar to that of [17, Theorem 4.3]. We prove the implication by induction on $\dim \Delta$. The case $\dim \Delta \leqslant 1$ follows from Lemmas 3.5 and 3.6. Assume that $\dim \Delta \geqslant 2$. Since $\overline{I_\Delta^n}$ is Cohen-Macaulay and $\sqrt{\overline{I_\Delta^n}} = I_\Delta$, I_Δ is Cohen-Macaulay by [9, Theorem 2.6]. In particular, Δ is connected. On the other hand, by Corollary 3.4, $\overline{I_{\mathrm{lk}_\Delta i}^n}$ is Cohen-Macaulay for all $i \leqslant s$, where w.l.o.g. we assume $V(\Delta) = [s]$. By the induction hypothesis, $I_{\mathrm{lk}_\Delta} i$ is a complete intersection. Since Δ is connected, this implies that I_Δ is a complete intersection complex by [18, Theorem 1.5]. $\square$

Note that [17, Theorem 1.2] states that the last condition in Theorem 3.7 is also equivalent to the Cohen-Macaulayness of R/I^n for all $n \geqslant 1$ (or for some fixed $n \geqslant 3$).

The property that the Cohen-Macaulay property of $\overline{I^n}$ for some $n \geqslant 3$ forces that for all n is very specific for square-free monomial ideals. For an arbitrary monomial ideal, the picture is much more complicate, as shown by the following example.

Example 3.8. Let $d \geqslant 3$ and $I = (x^d, xy^{d-2}z, y^{d-1}z) \subset R = k[x, y, z]$. Then $(x, y, z) \in \mathrm{Ass}(R/\overline{I^n})$ if and only if $n \geqslant d$ (see the example in [19, Page 54]). Since $\dim R/I = 1$, it follows that:

(1) $R/\overline{I^n}$ is Cohen-Macaulay for each $n = 1, \ldots, d-1$;
(2) $R/\overline{I^n}$ is not Cohen-Macaulay for any $n \geqslant d$.

Note that $\mathrm{ht}(I) = 2$ and $\ell(I) = 3$ in this case.

Acknowledgment. This work is partially supported by NAFOSTED (Vietnam) under the grant number 101.04-2015.02.

References

[1] C. Bivia-Ausina, *The analytic spread of monomial ideals*, Comm. Algebra, **31** (2003), 3487-3496.

[2] M. P. Brodmann, *The Asymptotic Nature of the Analytic Spread*, Math. Proc. Cambridge Philos Soc. **86** (1979), 35-39.

[3] D. Eisenbud and C. Huneke, *Cohen-Macaulay Rees Algebras and their Specializations*, J. Algebra **81** (1983) 202-224.

[4] D. H. Giang and L. T. Hoa, *On local cohomology of a tetrahedral curve*, Acta Math. Vietnam., **35** (2010), 229-241.

[5] H. T. Ha, D. H. Nguyen, N. V. Trung and T. N. Trung, *Symbolic powers of sums of ideals*, Preprint ArXiv:1702.01766.

[6] M. Herrmann, S. Ikeda, U. Orbanz, Equimultiplicity and Blowing up, Springer-Verlag, 1988.

[7] J. Herzog and T. Hibi, *The Depth of Powers of an Ideal*, J. Algebra **291** (2005), 534-550.

[8] J. Herzog and A. A. Qureshi, *Persistence and stability properties of powers of ideals*, J. Pure Appl. Algebra **219** (2015), 530-542.

[9] J. Herzog, Y. Takayama and N. Terai, *On the radical of a monomial ideal*, Arch. Math. **85** (2005), 397-408.

[10] L. T. Hoa, K. Kimura, N. Terai and T. N. Trung, *Stability of depths of symbolic powers of Stanley-Reisner ideals*. J. Algebra **473** (2017), 307-323.

[11] L. T. Hoa and T. N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Cambridge Philos Soc. **149** (2010), 1-18.

[12] S. McAdam, P. Eakin, *The asymptotic Ass*, J. Algebra **61** (1979), 71-81.

[13] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of powers of two-dimensional square-free monomial ideals*, J. Algebra **322** (2009), 4219–4227.

[14] L. Reid, L. G. Roberts and M. A. Vitulli, *Some results on normal homogeneous ideals*, Comm. Algebra **31**(2003), 4485–4506.

[15] A. Schrijver, Theory of linear and integer programming, John Wiley & Sons, 1998.

[16] Y. Takayama, *Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327-344.

[17] N. Terai and N. V. Trung, *Cohen-Macaulayness of large powers of Stanley-Reisner ideals*, Adv. Math. **229** (2012), no. 2, 711-730.

[18] N. Terai and K. Yoshida, *Locally complete intersection Stanley-Reisner ideals*. Illinois J. Math. **53** (2009), no. 2, 413-429.

[19] T. N. Trung, *Stability of associated primes of integral closures of monomial ideals*, J. Combin. Ser. A., **116**(2009), 44-54.

[20] T. N. Trung, *Stability of depths of powers of edge ideals*, J. Algebra **452** (2016), 157-187.

[21] W. Vasconcelos. Integral closure: Rees algebras, multiplicities, algorithms. Springer Monographs in Mathematics. Springer, New York, 2005.

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, 10307 Hanoi, Viet Nam
E-mail address: `lthoa@math.ac.vn, tntrung@math.ac.vn`