SOME PROPERTIES OF MAXIMAL PLURISUBHARMONIC FUNCTION

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Abstract. The purpose of this paper is to provide some properties of maximal plurisubharmonic functions in bounded domains in $\mathbb{C}^n$.

Contents

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Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain ($n \geq 2$). A function $u \in PSH(\Omega)$ is called maximal if for every open set $G \Subset \Omega$, and for each upper semicontinuous function v on $\bar{G}$ such that $v \in PSH(G)$ and $v|_{\partial G} \leq u|_{\partial G}$, we have $v \leq u$. There are some equivalent descriptions of maximality which have been presented in [Sad81], [Kli91].

For the convenience, we denote

$$MPSH(\Omega) = \{u \in PSH(\Omega) | u \text{ is maximal } \},$$

$$MPSH_{loc}(\Omega) = \{u \in PSH(\Omega) | \forall z \in \Omega : u \in MPSH(U) \text{ for some } z \in U \Subset \Omega\}.$$

The following question was given by Blocki [Blo04], [DGZ16]:

Question 0.1. *Is $MPSH(\Omega)$ equal to $MPSH_{loc}(\Omega)$? (or is maximality a local notion?)*

Denote by $D(\Omega)$ the domain of definition of Monge-Ampère operator in Ω. By [Blo04], a function $u \in D(\Omega)$ is maximal iff $(dd^c u)^n = 0$. Moreover, it follows from [Blo06] that to belong to the class D is a local property. Then

$$MPSH(\Omega) \cap D(\Omega) = MPSH_{loc}(\Omega) \cap D(\Omega).$$

In the general case, the question 0.1 is still open. It raises another question

Question 0.2. *What information can be obtained from the condition $u \in MPSH_{loc}(\Omega)$?*

In this paper, we study on some subclass of $MPSH(\Omega)$ and use results on it to show some properties of the class $MPSH_{loc}(\Omega)$.

Our main results are following:

Theorem 0.3. *If $u, v \in MPSH_{loc}(\Omega)$ and $\chi : \mathbb{R} \to \mathbb{R}$ is a convex non-decreasing function then $(z, w) \mapsto \chi(u(z) + v(w)) \in MPSH(\Omega \times \Omega)$.*

Theorem 0.4. *Let u be a negative maximal plurisubharmonic function in Ω and let $U, \tilde{U}$ be open subset of Ω such that $U \Subset \tilde{U} \Subset \Omega$. Assume that $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ is decreasing to u in $\tilde{U}$. Then*

$$(1) \qquad \int\limits_U |u_j|^{-a} (dd^c u_j)^n \overset{j\to\infty}{\longrightarrow} 0, \forall a > n - 1.$$

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1. A class of maximal plurisubharmonic functions

We say that a function $u \in PSH^-(\Omega)$ has M_1 property iff for every open set $U \Subset \Omega$, there are $u_j \in PSH^-(U) \cap C(U)$ such that u_j is decreasing to u in U and

$$(2) \qquad \lim_{j\to\infty} \left(\int\limits_{U\cap\{u_j>-t\}} (dd^c u_j)^n + \int\limits_{U\cap\{u_j>-t\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1} \right) = 0,$$

for any $t > 0$. We denote by $M_1PSH(\Omega)$ the set of negative plurisubharmonic functions in Ω satisfying M_1 property.
If $\chi : \mathbb{R} \to \mathbb{R}$ is a convex non-decreasing function, we denote $MPSH_\chi(\Omega)$ the set of negative plurisubharmonic functions in Ω such that $\chi(u) \in MPSH(\Omega)$.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u \in PSH^-(\Omega)$. Then the following conditions are equivalent*

(i) $u \in M_1PSH(\Omega)$.

(ii) $u \in MPSH_\chi(\Omega)$ *for any convex non-decreasing function* $\chi : \mathbb{R} \to \mathbb{R}$.

(iii) *For any open sets $U, \tilde{U}$ such that $U \Subset \tilde{U} \Subset \Omega$, for any $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ such that u_j is decreasing to u in $\tilde{U}$, we have*

$$\lim_{j\to\infty}\left(\int_U |u_j|^{-a}(dd^c u_j)^n + \int_U |u_j|^{-a-1} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}\right) = 0,$$

for all $a > n-1$.

In particular, M_1 property is a local notion and $M_1PSH(\Omega) \subset MPSH(\Omega)$.

Proof. $(iii \Rightarrow i)$: Obvious.
$(i \Rightarrow ii)$:
Assume that $U \Subset \tilde{U} \Subset \Omega$. Let $u_j \in PSH^-(U) \cap C(U)$ such that u_j is decreasing to u in U and the condition (2) is satisfied.

If χ is smooth and χ is constant in some interval $(-\infty, -m)$ then

$$\begin{aligned}(dd^c\chi(u_j))^n &= (\chi'(u_j))^n(dd^c u_j)^n + n\chi''(u_j)(\chi'(u_j))^{n-1} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}\\ &\leq C\mathbf{1}_{\{u_j > -t\}}(dd^c u_j)^n + C\mathbf{1}_{\{u_j > -t\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1},\end{aligned}$$

where $C, t > 0$ depend only on χ. Hence

$$\int_U (dd^c\chi(u_j))^n \overset{j\to\infty}{\longrightarrow} 0.$$

Then, by [Sad81] (see also [Ceg09]), $\chi(u)$ is maximal on U for any open set $U \Subset \Omega$. Thus $\chi(u) \in MPSH(\Omega)$.
In the general case, for any convex non-decreasing function χ, we can find $\chi_l \searrow \chi$ such that χ_l is smooth, convex and $\chi|_{(-\infty,-m)} = const$ for some m. By above argument, $\chi_l \in MPSH(\Omega)$ for any $l \in \mathbb{N}$. Hence $\chi(u) \in MPSH(\Omega)$.
$(ii \Rightarrow iii)$:
For any $0 < \alpha < \frac{1}{n}$, the function

$$\Phi_\alpha(t) = -(-t)^\alpha$$

is convex and non-decreasing in $\mathbb{R}^-$. Assume that u satisfies (ii), we have $\Phi_\alpha \in MPSH(\Omega)$.

By [Bed93] (see also [Blo09]), for any $0 < \alpha < \frac{1}{n}$, we have $\Phi_\alpha(u) \in D(\Omega)$. Then, for any $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ such that u_j is decreasing to u in $\tilde{U}$, we have

$$\int_U (dd^c\Phi_\alpha(u_j))^n \overset{j\to\infty}{\longrightarrow} 0, \forall 0 < \alpha < \tfrac{1}{n},$$

and it implies (iii).

Finally, by using $(i \Leftrightarrow iii)$, we conclude that M_1 property is a local notion. □

The following proposition is an immediately corollary of Theorem 1.1

Proposition 1.2. *Let Ω be a bounded domain in $\mathbb{C}^n$.*

(i) *If $u \in M_1PSH(\Omega)$ then $\chi(u) \in M_1PSH(\Omega)$ for any convex non-decreasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$.*

(ii) *If $u_j \in M_1PSH(\Omega)$ and u_j is decreasing to u then $u \in M_1PSH(\Omega)$.*

(iii) *Let $u \in PSH^-(\Omega) \cap C^2(\Omega \setminus F)$, where $F = \{z : u(z) = -\infty\}$ is closed. If*

$$(dd^c u)^n = du \wedge d^c u \wedge (dd^c u)^{n-1} = 0$$

in $\Omega \setminus F$ then $u \in M_1PSH(\Omega)$.

In some special cases, we can easily check M_1 property by the following criteria

Proposition 1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth convex increasing function such that $\chi''(t) > 0$ for any $t \in \mathbb{R}$. Assume also that χ is lower bounded. If $u \in PSH^-(\Omega)$ and $\chi(u) \in MPSH(\Omega)$ then $u \in M_1PSH(\Omega)$.*

Proof. Let $U \Subset \tilde{U} \Subset \Omega$ and $u_j \in PSH(\tilde{U}) \cap C(\tilde{U})$ such that u_j is decreasing to u. Then

$$dd^c(\chi(u_j)) = \chi'(u_j) dd^c u_j + \chi''(u_j) du_j \wedge d^c u_j$$

and

$$(dd^c \chi(u_j))^n = (\chi'(u_j))^n (dd^c u_j)^n + n\chi''(u_j)(\chi'(u_j))^{n-1} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}.$$

For any $t > 0$, there exists $C > 0$ depending only on t and χ such that

$$(dd^c \chi(u_j))^n \geq C\mathbf{1}_{\{u_j > -t\}} (dd^c u_j)^n + C\mathbf{1}_{\{u_j > -t\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}. \tag{3}$$

Note that $\chi(u) \in D(\Omega) \cap MPSH(\Omega)$. Hence

$$\lim_{j\to\infty} \int\limits_U (dd^c \chi(u_j))^n = 0. \tag{4}$$

Combining (3) and (4), we have

$$\lim_{j\to\infty} \left(\int\limits_{U\cap\{u_j > -t\}} (dd^c u_j)^n + \int\limits_{U\cap\{u_j>-t\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1} \right) = 0.$$

Thus $u \in M_1PSH(\Omega)$. □

Example 1.4. *(i) If u is a negative plurisubharmonic function in $\Omega \subset \mathbb{C}^n$ depending only on $n-1$ variables then u has M_1 property.*
(ii) If $f : \Omega \to \mathbb{C}^n$ is a holomorphic mapping of rank $< n$ then $(dd^c|f|^2)^n = 0$ (see, for example, in [Ras98]*). Then, by Proposition 1.3, $\log|f| \in M_1PSH(\Omega)$ if it is negative in Ω.*

Question 1.5. *Does Proposition 1.3 still hold if the assumption "χ is lower bounded" is removed from it?*

2. Proof of the main theorems

2.1. **Proof of Theorem 0.3.** Without loss of generality, we can assume that $u, v \in PSH^-(\Omega)$.

If $u, v \in MPSH_{loc}(\Omega)$ then for any $z_0, w_0 \in \Omega$, there are hyperconvex domains $U, \tilde{U}, V, \tilde{V}$ such that $z_0 \in U \Subset \tilde{U} \Subset \Omega$, $w_0 \in V \Subset \tilde{V} \Subset \Omega$, $u \in MPSH(\tilde{U})$ and $v \in MPSH(\tilde{V})$. We need to show that $u(z) + v(w)$ have M_1 property in $U \times V$.

Let $u_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$ and $v_j \in PSH^-(\tilde{V}) \cap C(\tilde{V})$ such that u_j is decreasing to u in $\tilde{U}$ and v_j is decreasing to v in $\tilde{V}$. By [Wal68], there are $\tilde{u}_j \in PSH^-(\tilde{U}) \cap C(\tilde{U})$, $\tilde{v}_j \in PSH^-(\tilde{V}) \cap C(\tilde{V})$ such that

$$\begin{cases} \tilde{u_j} = u_j & \text{in } \tilde{U} \setminus U, \\ \tilde{v_j} = v_j & \text{in } \tilde{V} \setminus V, \\ (dd^c \tilde{u}_j)^n = 0 & \text{in } U, \\ (dd^c \tilde{v}_j)^n = 0 & \text{in } V. \end{cases}$$

By the maximality of u and v, we conclude that $\tilde{u}_j$ is decreasing to u in $\tilde{U}$ and $\tilde{v}_j$ is decreasing to v in $\tilde{V}$. In $U \times V$, we have

$$(dd^c(\tilde{u}_j(z) + \tilde{v}_j(w)))^{2n} = C_{2n}^n (dd^c \tilde{u}_j)_z^n \wedge (dd^c \tilde{v}_j)_w^n = 0$$

$$d(\tilde{u}_j(z) + \tilde{v}_j(w)) \wedge d^c(\tilde{u}_j(z) + \tilde{v}_j(w)) \wedge (dd^c(\tilde{u}_j(z) + \tilde{v}_j(w)))^{2n-1}$$

$$= C_{2n-1}^{n-1} d_z \tilde{u}_j \wedge d_z^c \tilde{u}_j \wedge (dd^c \tilde{u}_j)_z^{n-1} \wedge (dd^c \tilde{v}_j)_w^n + C_{2n-1}^{n-1} d_w \tilde{v}_j \wedge d_w^c \tilde{v}_j \wedge (dd^c \tilde{v}_j)_w^{n-1} \wedge (dd^c \tilde{u}_j)_z^n$$

$$= 0.$$

Then $u(z) + v(w)$ has M_1 property in $U \times V$. By Theorem 1.1, M_1 property is a local notion. Hence $u(z) + v(w) \in M_1 PSH(\Omega \times \Omega)$. And it implies that $u(z) + v(w) \in MPSH_\chi(\Omega \times \Omega)$ for any convex non-decreasing function χ.

2.2. **Proof of Theorem 0.4.** Let $v = |z_1|^2 + ... + |z_{n-1}|^2 + x_n + y_n - M$, where $M = \sup_\Omega(|z|^2 + |x_n| + |y_n|)$. Then $v \in MPSH(\Omega)$. By Theorem 0.3, $\chi(u(z) + v(w)) \in MPSH(\Omega \times \Omega)$ for any convex non-decreasing function χ.

By [Bed93],[Blo09], for any $0 < \alpha < \frac{1}{2n}$, we have $\Phi_\alpha(u(z) + v(w)) \in D(\Omega \times \Omega)$, where Φ_α is defined as in the proof of Theorem 1.1.
Then

$$\int_{U \times U} (dd^c \Phi(u_j(z) + v(w)))^{2n} \overset{j \to \infty}{\longrightarrow} 0,$$

for any $0 < \alpha < \frac{1}{2n}$. Hence

$$\int_U |u_j|^{-2n-1+2n\alpha} (dd^c u_j)^n \overset{j \to \infty}{\longrightarrow} 0, \forall 0 < \alpha < \frac{1}{2n}. \tag{5}$$

Moreover, $\Phi_\beta(u) \in D(\Omega)$ for any $0 < \beta < \frac{1}{n}$. Then, for any $0 < \beta < \frac{1}{n}$, there is $C_\beta > 0$ such that

$$\int_U (dd^c \Phi_\beta(u_j))^n \leq C_\beta, \forall j > 0.$$

Hence

$$\int_U |u_j|^{-n+n\beta}(dd^c u_j)^n \leq C_\beta, \forall j > 0, \forall 0 < \beta < \frac{1}{n}. \tag{6}$$

Combining (5), (6) and using Hölder inequality, we obtain (1).

3. Relation between some class of maximal plurisubharmonic functions

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $u \in PSH(\Omega)$. If there exists a sequence of convex non-decreasing functions $\chi^m : \mathbb{R} \to \mathbb{R}$ such that

- χ^m is lower bounded for every m,
- χ^m is decreasing to Id as $m \to \infty$,
- $(dd^c \chi^m(u))^n \overset{m\to\infty}{\longrightarrow} 0$ in the weak sense,

then, by [Sad81], u is maximal. We are interested in the following question

Question 3.1. *If u is maximal, does there exist a sequence of convex non-decreasing function χ^m satisfying above conditions? If it exists, how to find it?*

In this section we discuss about relation between some class of maximal plurisubharmonic functions in a bounded domain Ω in $\mathbb{C}^n$. It can be seen as the first step in approaching Question 3.1.

Assume that $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex function such that $\chi|_{(-\infty,-2)} = -1$, $\chi|_{(0,\infty)} = Id_{(0,\infty)}$ and $\chi''(-1) > 0$. We denote

$M_2PSH(\Omega) = \{u \in PSH^-(\Omega) | (dd^c \max\{u, -k\})^n \overset{weak}{\longrightarrow} 0 \text{ as } k \to \infty\}$,
$M_3PSH(\Omega) = \{u \in PSH^-(\Omega) | (dd^c \chi_k(u))^n \overset{weak}{\longrightarrow} 0 \text{ as } k \to \infty\}$,
$M_4PSH(\Omega) = \{u \in PSH^-(\Omega) | (dd^c \log(e^u + \frac{1}{k}))^n \overset{weak}{\longrightarrow} 0 \text{ as } k \to \infty\}$,

where $\chi_k(t) = \chi(t + k) - k$. The main result of this section is following

Theorem 3.2. *$M_2PSH(\Omega) \subset M_3PSH(\Omega)$ and $M_3PSH(\Omega) = M_4PSH(\Omega)$.*

First, we introduce some characteristics of $M_2PSH(\Omega)$, $M_3PSH(\Omega)$ and $M_4PSH(\Omega)$.

Proposition 3.3. *Let $u \in PSH^-(\Omega)$. Assume that $u_j \in PSH^-(\Omega) \cap C(\Omega)$ is decreasing to u. Then the following conditions are equivalent*

(i) $u \in M_2PSH(\Omega)$.
(ii) *If $U \Subset \Omega$ then for every $M > 0$*

$$\int_{U\cap\{u_j>-M\}} (dd^c u_j)^n \overset{j\to\infty}{\longrightarrow} 0, \tag{7}$$

and for any $\epsilon > 0$, there is $k_0 > 0$ such that

$$\forall k \geq k_0, \exists l_0 > 0 : \limsup_{j\to\infty} \int_{U\cap\{-k-\frac{1}{l}<u_j<-k+\frac{1}{l}\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1} < \frac{\epsilon}{l}, \forall l > l_0. \tag{8}$$

Proof. For any $k, l > 0$, we denote

$$\chi_{k,l}(t) = \frac{1}{l}\chi((t + k - \frac{1}{l})l) - k + \frac{1}{l}.$$

Then $\chi_{k,l}(u)$ converges to $\max\{u, -k\}$ as $l \to \infty$. Moreover

$$\chi_{k,l}(u) \geq \max\{u, -k\} + O(\frac{1}{l}).$$

Hence, by [Ceg09],

$$(dd^c\chi_{k,l}(u))^n \stackrel{l\to\infty}{\longrightarrow} (dd^c \max\{u, -k\})^n \tag{9}$$

in the weak sense.

For any $j > 0$, we have

$$(dd^c\chi_{k,l}(u_j))^n = (\chi'((u_j + k - \frac{1}{l})l))^n(dd^cu_j)^n + nl\chi''((u_j + k - \frac{1}{l})l)(\chi'((u_j + k - \frac{1}{l})l))^{n-1}du_j \wedge d^cu_j \wedge (dd^cu_j)^{n-1}.$$

Then, there are $C_1, C_2, \delta > 0$ depending only on χ such that

$$(dd^c\chi_{k,l}(u_j))^n \leq C_1\mathbf{1}_{\{u_j>-k-\frac{1}{l}\}}(dd^cu_j)^n + C_1l\mathbf{1}_{\{-k-\frac{1}{l}<u_j<-k+\frac{1}{l}\}}du_j \wedge d^cu_j \wedge (dd^cu_j)^{n-1},$$

and

$$(dd^c\chi_{k,l}(u_j))^n \geq C_2\mathbf{1}_{\{u_j>-k+\frac{1}{l}\}}(dd^cu_j)^n + C_2l\mathbf{1}_{\{-k-\frac{\delta}{l}<u_j<-k+\frac{\delta}{l}\}}du_j \wedge d^cu_j \wedge (dd^cu_j)^{n-1}.$$

Moreover,

$$(dd^c\chi_{k,l}(u_j))^n \stackrel{j\to\infty}{\longrightarrow} (dd^c\chi_{k,l}(u))^n$$

in the weak sense.

Then (ii) is equivalent to

$$\limsup_{l\to\infty} \int_U (dd^c\chi_{k,l}(u))^n \stackrel{k\to\infty}{\longrightarrow} 0,$$

for any $U \Subset \Omega$.

Hence, by (9), we conclude that (i) is equivalent to (ii). □

Proposition 3.4. *Let $u \in PSH^-(\Omega)$. Assume that $u_j \in PSH^-(\Omega) \cap C(\Omega)$ is decreasing to u. Then the following conditions are equivalent*

(i) $u \in M_3PSH(\Omega)$.
(ii) *If $U \Subset \Omega$ then*

$$\int\limits_{U\cap\{u_j>-M\}} (dd^cu_j)^n \stackrel{j\to\infty}{\longrightarrow} 0, \tag{10}$$

and

$$\limsup_{j\to\infty} \int\limits_{U\cap\{-k-1<u_j<-k\}} du_j \wedge d^cu_j \wedge (dd^cu_j)^{n-1} \stackrel{k\to\infty}{\longrightarrow} 0. \tag{11}$$

The proof of Proposition 3.4 is similar to the proof of Proposition 3.3. We leave the details to the reader.

In [Sad12], Sadullaev has proved a characteristic of $M_4PSH(\Omega)$:

Theorem 3.5. *Let $u \in PSH^-(\Omega)$. Denote $v = e^u$. Then $u \in M_4PSH(\Omega)$ iff*

$$v(dd^cv)^n - ndv\wedge d^cv\wedge (dd^cv)^{n-1}=0,$$

and

$$\lim_{t\to 0}\frac{1}{t^n}\int\limits_{U\cap\{v<t\}}(dd^cv)^n=0,$$

for any $U\Subset\Omega$.

Using Theorem 3.5, it is easy to show that

Proposition 3.6. *Let* $u\in PSH^-(\Omega)$. *Assume that* $u_j\in PSH^-(\Omega)\cap C(\Omega)$ *is decreasing to* u. *Then the following conditions are equivalent*

(i) $u\in M_4PSH(\Omega)$.

(ii) *If* $U\Subset\Omega$ *then*

$$\int\limits_{U\cap\{u_j>-M\}}(dd^cu_j)^n\stackrel{j\to\infty}{\longrightarrow}0, \tag{12}$$

and

$$\limsup_{j\to\infty}\int\limits_{U\cap\{u_j<-k\}}e^{n(u_j+k)}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}\stackrel{k\to\infty}{\longrightarrow}0. \tag{13}$$

Now, by using Propositions 3.3, 3.4 and 3.6, we will prove Theorem 3.2.

Proof of Theorem 3.2. Let $u\in PSH^-(\Omega)$. By replacing Ω by an exhaustive sequence of relative compact subsets of Ω, we can assume that there exists a sequence $u_j\in PSH^-(\Omega)\cap C(\Omega)$ such that u_j decreasing to u in Ω.

Assume that $u\in M_2PSH(\Omega)$. By Proposition 3.3, for any $\epsilon>0$, there exist $k_0>0$ such that

$$\forall k\geq k_0, \exists l_0>0: \limsup_{j\to\infty}\int\limits_{U\cap\{-k-\frac{1}{l}<u_j<-k+\frac{1}{l}\}}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}<\frac{\epsilon}{l}, \forall l>l_0.$$

Let $k>k_0$. By Besicovitch's covering theorem [Matt95], there are $k_1,l_1,...,k_m,l_m...>0$ such that

$$\limsup_{j\to\infty}\int\limits_{U\cap\{-k_m-\frac{1}{l_m}<u_j<-k_m+\frac{1}{l_m}\}}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}<\frac{\epsilon}{l_m}, \forall m>0,$$

and

$$\mathbf{1}_{(-k-1,-k)}\leq\sum_m\mathbf{1}_{(-k_m-\frac{1}{l_m},-k_m+\frac{1}{l_m})}=\sum_m\frac{2}{l_m}\leq C,$$

where $C>0$ is a universal constant. Hence

$$\limsup_{j\to\infty}\int\limits_{U\cap\{-k-1<u_j<-k\}}du_j\wedge d^cu_j\wedge(dd^cu_j)^{n-1}\leq C\epsilon.$$

By Proposition 3.4, we get $u\in M_3PSH(\Omega)$.

Thus $M_2PSH(\Omega)\subset M_3PSH(\Omega)$.

Now, assume that u is an arbitrary element of $M_3PSH(\Omega)$. By Proposition 3.4, for any $\epsilon>0$, there exists $k_0>0$ such that

$$\limsup_{j\to\infty} \int\limits_{U\cap\{-k-1<u_j<-k\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1} < \epsilon.$$

for every $k > k_0$. Then

$$\limsup_{j\to\infty} \int\limits_{U\cap\{u_j<-k\}} e^{n(u_j+k)} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\leq \sum_{l=0}^{\infty} \limsup_{j\to\infty} \int\limits_{U\cap\{-k-l-1<u_j<-k-l\}} e^{n(u_j+k)} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\leq \sum_{l=0}^{\infty} \limsup_{j\to\infty} \int\limits_{U\cap\{-k-l-1<u_j<-k-l\}} e^{-nl} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\leq \epsilon \sum_{l=0}^{\infty} e^{-nl},$$

for any $k > k_0$. Hence

$$\limsup_{j\to\infty} \int\limits_{U\cap\{u_j<-k\}} e^{n(u_j+k)} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1} \xrightarrow{k\to\infty} 0.$$

By Proposition 3.6, we get $u \in M_4PSH(\Omega)$.

Thus $M_3PSH(\Omega) \subset M_4PSH(\Omega)$.

Conversely, if $u \in M_4PSH(\Omega)$ then

$$\limsup_{j\to\infty} \int\limits_{U\cap\{-k-1<u_j<-k\}} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\leq e^n \limsup_{j\to\infty} \int\limits_{U\cap\{-k-1<u_j<-k\}} e^{n(u_j+k)} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\leq e^n \limsup_{j\to\infty} \int\limits_{U\cap\{u_j<-k\}} e^{n(u_j+k)} du_j \wedge d^c u_j \wedge (dd^c u_j)^{n-1}$$
$$\xrightarrow{k\to\infty} 0.$$

Hence $M_4PSH(\Omega) \subset M_3PSH(\Omega)$.

The proof is completed. □

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