On analyticity for Lyapunov exponents of generic bounded linear random dynamical systems

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Abstract

In this paper, we construct an open and dense set in the space of bounded linear random dynamical systems (both discrete and continuous time) equipped with the essential sup norm such that the Lyapunov exponents depend analytically on the coefficients in this set. As a consequence, analyticity for Lyapunov exponents of bounded linear random dynamical systems is a generic property.

1 Introduction

The fundamental results on Lyapunov exponents for random dynamical systems on finite dimensional spaces were first obtained in [Ose68], which is now called the Oseledets Multiplicative Ergodic Theorem, see also [Ar98].

Since the appearance of the paper [Ose68], exploring properties of Lyapunov exponents of random dynamical systems has become one of a central task in the theory of random dynamical systems. In this task, understanding the stability of Lyapunov exponents under a small perturbation has been received a lot of interests. Now, we would like to mention some publications that are related to our work in this paper:

In [Rue79], Ruelle showed that the characteristic exponents of a random matrix product over a compact base space endowed with a probability measure depends real analytically on the data of the problem. This result was generalized in [Dub08] to product of random linear operators having an invariant cone over a measurable base space.

Concerning random dynamical systems without assuming to share an invariant cone, in general Lyapunov exponents do not depend continuously on the coefficients, see [Kn90]. Using a similar technique as in [Kn90], Bochi

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gave a characterization of discontinuity for Lyapunov exponents of $\mathrm{SL}(2,\mathbb{R})$-cocycles in a unpublished preprint [Bo99]. Roughly speaking, it was showed in [Bo99] that if a $\mathrm{SL}(2,\mathbb{R})$-cocycle has a positive Lyapunov exponent and has not a dominated splitting (integral separation), then the upper Lyapunov exponent function is not continuous at this cocycle.

Although Lyapunov exponents are in general not continuous, it is proved in [Co05] that continuity of Lyapunov eponents is a generic property on the space of bounded linear cocycles. Thereafter, it is natural to investigate higher regularity property than continuity property for Lyapunov exponents of linear random dynamical systems such as Hölder continuity, smoothness and analyticity.

In this paper, we show that analyticity of Lyapunov exponents is still a generic property on the space of bounded linear random dynamical systems. As far as we are aware, this result is neither stated nor proved elesewhere and the main ingredient in the proof of this fact is to construct a suitable cone being invariant under a long enough iteration of an integrally separated linear random dynamical system. Note that generic bounded linear random dynamical systems are integrally separated, see [Co05] and [CD16]. This fact together with results in [Rue79, Dub08] deduces the conclusion that generically, the top Lyapunov exponent of random dynamical systems depend analytically on the coefficients. To derive the same conclusion for other Lyapunov exponents, we work with the generated random dynamical systems on the exterior power space.

The paper is organized as follows: In Section 2, we present a basic background on Lyapunov exponents of product of random matrices and linear random differential equations. In this section, we also recall a notion of analytic functions defined in an arbitrary Banach space. Section 3 is devoted to present the main results (Theorem 4 and Theorem 6) of this paper about genericity for analyticity of Lyapunov exponents. The Appendix section is divided into two subsections. In Subsection 4.1, we recall a basic background on the exterior power of a finite dimensional vector space. A result proved in [Cr90] about the structure of Lyapunov spectrum of induced random dynamical systems on the exterior power was also presented. Subsection 4.2 is devoted to present the work in [AGD94] and [Dub08] about the simplicity and analyticity of the top Lyapunov exponent of product of positive random matrices.

To conclude this section, we introduce notations which are used throughout the paper: For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ denote the space of all essentially bounded random matrix-valued maps $A : \Omega \to \mathbb{R}^{d\times d}$. Let $\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ be endowed with the essential sup norm, i.e.

$$|A\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)\| \qquad \text{for } A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}).$$

It is well known that $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \|\cdot\|_\infty)$ is a Banach space.

2 Preliminaries

2.1 Discrete-time bounded linear random dynamical systems

Let $(X, \mathcal{A}, m)$ be a Lebesgue probability space and $T : X \to X$ an invertible ergodic transformation preserving the probability m. Each random map $A \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ gives rise to an one-sided linear random dynamical system[1] $\Phi_A : \mathbb{N} \times X \to \mathbb{R}^{d\times d}$ via

$$\Phi_A(n, x) := \begin{cases} A(T^{n-1}x)\dots A(x), & \text{if } n > 0, \\ \text{id}, & \text{if } n = 0, \end{cases}$$

where id denotes the identity matrix in $\mathbb{R}^{d\times d}$. By virtue of the Multiplicative Ergodic Theorem for one-sided linear random dynamical systems defined on an invertible metric dynamical system, see e.g. [FGTQ15], there exist finite numbers of Lyapunov exponents $\lambda_1(A) > \dots > \lambda_{p(A)}(A)$, where $p(A) \in \{1, \dots, d\}$, and a forward invariant decomposition

$$\mathbb{R}^d = \mathcal{O}_1(A, x) \oplus \mathcal{O}_2(A, x) \oplus \dots \oplus \mathcal{O}_{p(A)}(A, x)$$

with the property that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_A(n, x)v\| = \lambda_i \quad \text{iff} \quad v \in \mathcal{O}_i(A, x) \setminus \{0\}.$$

Let $d_1(A), \dots, d_{p(A)}(A)$ denote the dimension of $\mathcal{O}_1(A, x), \dots, \mathcal{O}_{p(A)}(A, x)$, respectively. The set of pairs

$$\Sigma_{\text{Lya}}(A) := \big\{(\lambda_1(A), d_1(A)), \dots, (\lambda_{p(A)}, d_{p(A)}(A))\big\}$$

[1]In this paper, we identify a random map in $\mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ with its generated linear random dynamical system.

is called the *Lyapunov spectrum* of A. For a convenience of presentation in the rest of the paper, let $\gamma_d(A) \leq \gamma_{d-1}(A) \leq \cdots \leq \gamma_1(A)$ denote the Lyapunov exponents of A (counting with multiplicity), i.e. for $i = 1, \ldots, p(A)$

$$\gamma_j(A) = \lambda_i(A) \qquad \text{for } d_1(A) + \cdots + d_{i-1}(A) + 1 \leq j \leq d_1(A) + \cdots + d_i(A).$$

The Lyapunov spectrum is said to be *simple* if $p(A) = d$, i.e. all Oseledets subspaces $\mathcal{O}_1(A, x), \ldots, \mathcal{O}_{p(A)}(A, x)$ are of dimension 1.

Let $\mathrm{GL}(d)$ be the set of all invertible matrices in $\mathbb{R}^{d\times d}$. Let $\mathcal{L}^\infty(X, \mathrm{GL}(d))$ denote the space of all $A \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ such that the inverse $A^{-1}(x)$ exists for $x \in X$ and $A^{-1}(\cdot) \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$. Note that each random map $A \in \mathcal{L}^\infty(X, \mathrm{GL}(d))$ generates a two-sided linear random dynamical system which is also denoted by $\Phi_A : \mathbb{Z} \times X \to \mathrm{GL}(d)$ via

$$\Phi_A(n, x) := \begin{cases} A(T^{n-1}x) \ldots A(x), & \text{if } n > 0, \\ \mathrm{id}, & \text{if } n = 0, \\ A(T^n x)^{-1} \circ \cdots \circ \mathcal{A}(T^{-1}x)^{-1}, & \text{if } n < 0. \end{cases}$$

In the following lemma, we show that $\mathcal{L}^\infty(X, \mathrm{GL}(d))$ is an open and dense subset of $\mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$. Note that this result does not depend on the norm equipped with $\mathbb{R}^d$ and the induced norm on $\mathbb{R}^{d\times d}$. Then, in what follows let $\mathbb{R}^d$ be endowed with the standard Euclidean norm.

Lemma 1. The set $\mathcal{L}^\infty(X, \mathrm{GL}(d))$ is open and dense in the Banach space $(\mathcal{L}^\infty(X, \mathbb{R}^{d\times d}), \|\cdot\|_\infty)$.

Proof. The openess of $\mathcal{L}^\infty(X, \mathrm{GL}(d))$ in $(\mathcal{L}^\infty(X, \mathbb{R}^{d\times d}), \|\cdot\|_\infty)$ is obvious and we only need to prove the density of this set. Let $A \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ and $\varepsilon > 0$ be arbitrary. Let $A(x) = U(x)\Sigma(x)V(x)$ be a measurable singular valued decomposition of A, where $U(x)$ and $V(x)$ are measurable orthogonal matrices and $\Sigma(x) = \mathrm{diag}(\delta_1(x), \ldots, \delta_d(x))$ is a diagonal matrix. Define $\widehat{A}(x) := U(x)\widehat{\Sigma}(x)V(x)$, where $\widehat{\Sigma}(x) := \mathrm{diag}(\widehat{\delta}_1(x), \ldots, \widehat{\delta}_d(x))$ is defined by

$$\widehat{\delta}_i(x) := \begin{cases} \delta_i(x), & \text{if } |\delta_i(x)| \geq \frac{\varepsilon}{2}, \\ \frac{\varepsilon}{2}, & \text{if } |\delta_i(x)| < \frac{\varepsilon}{2}. \end{cases}$$

Then, $\|A - \widehat{A}\|_\infty = \operatorname{ess\,sup}_{x\in X} \|\Sigma(x) - \widehat{\Sigma}(x)\| \leq \varepsilon$. Furthermore, $\widehat{A}(x)$ is invertible and $\|\widehat{A}^{-1}\|_\infty = \operatorname{ess\,sup}_{x\in X} \|\widehat{A}^{-1}(x)\| \leq \frac{2}{\varepsilon}$. Hence, $\mathcal{L}^\infty(X, \mathrm{GL}(d))$ is dense in $(\mathcal{L}^\infty(X, \mathbb{R}^{d\times d}), \|\cdot\|_\infty)$ and the proof is complete. □

2.2 Bounded linear random differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue probability space and $(\theta_t)_{t\in\mathbb{R}}$ be an ergodic flow from Ω into itself preserving the probability $\mathbb{P}$. Suppose further that $(\theta_t)_{t\in\mathbb{R}}$ has no fixed point. For each $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$, we consider the corresponding linear random differential equation of the following form

$$\dot{\xi} = A(\theta_t\omega)\xi \qquad \text{for } \omega \in \Omega, t \in \mathbb{R}. \tag{1}$$

Let $\Psi_A(t,\omega)\xi$ denote the solution of (1) satisfying $\xi(0) = \xi$. The linear mapping $\Phi_A : \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}$ is a continuous random dynamical system, i.e. Ψ_A is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^{d\times d}))$ measurable and the following properties hold:

(i) $\Psi_A(0,\omega) = \mathrm{id}$,

(ii) $\Psi_A(t+s,\omega) = \Psi_A(t,\theta_s\omega)\Psi_A(s,\omega)$ for all $t, s \in \mathbb{R}$, $\omega \in \Omega$,

(iii) For each $\omega \in \Omega$, the mapping $t \mapsto \Psi_A(t,\omega)$ is continuous,

see e.g., [Ar98, Subsection 2.2]. It is well known (see e.g. [Ar98, CD16]) that $\Psi_A(t,\omega)$ satisfies the integrability condition of the Multiplicative Ergodic Theorem, i.e.

$$\log^+ \|\alpha^+(\cdot)\|, \log^+ \|\alpha^-(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}),$$

where $\log^+ x := \max(0, \log x)$ and $\alpha^+(\omega) := \sup_{0\le t\le 1} \|\Psi_A(t,\omega)\|, \alpha^-(\omega) := \sup_{-1\le t\le 0} \|\Psi_A(t,\omega)\|$. So, there exist p, where $1 \le p \le d$, non-random Lyapunov exponents $\lambda_p(A) < \lambda_{p-1} < \cdots < \lambda_1(A)$ and an invariant measurable decomposition

$$\mathbb{R}^d = \mathcal{O}_1(\omega, A) \oplus \mathcal{O}_2(\omega, A) \oplus \cdots \oplus \mathcal{O}_p(\omega, A)$$

with the property that for $k = 1, \dots, p$ the linear measurable subspace $\mathcal{O}_k(\omega, A)$ is dynamically characterized by

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Psi_A(t,\omega)v\| = \lambda_k \quad \text{iff} \quad v \in \mathcal{O}_k(\omega, A) \setminus \{0\}.$$

Analog to Subsection 2.1, let $\gamma_d(A) \le \gamma_{d-1}(A) \le \cdots \le \gamma_1(A)$ denote the Lyapunov exponents (counting with multiplicity) of (1).

2.3 Analytic functions

Let X, Y be Banach spaces. For $k \in \mathbb{N}$, a function $p : X \to Y$ is called a *continuous homogeneous polynomial of degree* k if there exists a continuous k-multilinear symmetric function $\varphi : X^k \to Y$ such that $p(x) = \varphi(x, \ldots, x)$.

A function $f : X \to Y$ is said to be *real analytic* at $x \in X$ if there exists $\delta > 0$ and a sequence of continuous homogeneous polynomials $(p_k)_{k\geq 0}$, where $p_k : X \to Y$ is of degree k, such that $\sum_{k=0}^{\infty} \|p_k\|\delta^k < \infty$ and

$$f(x + \Delta) = \sum_{k=0}^{\infty} p_k(\Delta) \qquad \text{for } \Delta \in X \text{ with } \|\Delta\| \leq \delta.$$

In the following lemma, we collect some fundamental properties of analytic functions which are used in the next section.

Lemma 2. The following statements hold:

(i) Let X, Y be Banach spaces. Then, the composition, the summation and the subtraction of two analytic functions from X to Y are again analytic.

(ii) Let X, Y be Banach spaces. A bounded linear operator $T : X \to Y$ is analytic.

(iii) Let $f, g : \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ be analytic functions. Then, the product function $fg : \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), A \mapsto fg(A)$ defined by $fg(A)(\omega) := f(A)(\omega)g(A)(\omega)$, is also analytic.

Proof. We can find a proof of (i) in e.g. [Whit65]. Meanwhile, the statements (ii) and (iii) follow directly from the definition of analytic function. □

3 Analyticity of Lyapunov exponents

3.1 Discrete-time linear random dynamical systems

This subsection is devoted to study discrete-time bounded linear random dynamical systems introduced in Subsection 2.1. The main result in this subsection is to show that generically Lyapunov exponents of discrete-time bounded linear random dynamical systems depends analytically on the generators. To achieve this result, we show that in the set of integrally separated bounded linear random dynamical systems the Lyapunov exponents depends analytically on the generator. This assertion together with the result on the genericity of integral separation in the space of bounded invertible random matrices (see [Co05]) and Lemma 1 implies that analyticity of Lyapunov exponents is a generic property.

Recall that a bounded linear random dynamical system $A \in L^\infty(X, \mathrm{GL}(d))$ is said to be *integrally separated* if there exist $K, \alpha > 0$ and an invariant measurable decomposition $\mathbb{R}^d = \oplus_{i=1}^d E_i(x)$, where $E_i(x)$ is a linear subspace of dimension 1, such that for all $n \in \mathbb{N}$ and m-a.e. $x \in X$ the following inequality

$$\frac{\|\Phi_A(n,x)u\|}{\|u\|} \geq K e^{\alpha n} \frac{\|\Phi_A(n,x)v\|}{\|v\|} \tag{2}$$

holds for $u \in \oplus_{j=1}^i E_j(x) \setminus \{0\}, v \in \oplus_{j=i+1}^d E_j(x) \setminus \{0\}$, where $i = 1, \dots, d-1$.

In what follows, we introduce and prove some fundamental properties of a specific cone in the exterior power space $\Lambda^k \mathbb{R}^d$. This cone is a key object in the proof of the main result in which we show that the generated random dynamical system of any integrally separated bounded linear random dynamical system on $\Lambda^k \mathbb{R}^d$ after a long enough iteration and a random change of coordinate preserves this cone.

Lemma 3. For each $k \in \{1, \dots, d\}$, we define a subset $\mathcal{C}_k \subset \Lambda^k \mathbb{R}^d$ by

$$\mathcal{C}_k := \Big\{ \sum_{1 \leq i_1 < \cdots < i_k \leq d} : \alpha_{i_1 \dots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} : \alpha_{1 \dots k} = \max_{1 \leq i_1 < \cdots < i_k \leq d} |\alpha_{i_1 \dots i_k}| \Big\}. \tag{3}$$

Then, the following statements hold:

(i) $\mathcal{C}_k$ is a closed proper convex cone.

(ii) The dual cone $\mathcal{C}_k'$ of $\mathcal{C}_k$ is given by

$$\mathcal{C}_k' := \Big\{ \sum_{1 \leq i_1 < \cdots < i_k \leq d} : \alpha_{i_1 \dots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} : \alpha_{1 \dots k} \geq \frac{1}{2} \sum_{1 \leq i_1 < \cdots < i_k \leq d} |\alpha_{i_1 \dots i_k}| \Big\}.$$

Consequently, $e_1 \wedge \cdots \wedge e_k$ is an interior point of both $\mathcal{C}_k$ and $\mathcal{C}_k'$. More, precisely, we have

$$B(e_1 \wedge \cdots \wedge e_k, \frac{1}{3}) \subset \mathcal{C}_k \quad \text{and} \quad B(e_1 \wedge \cdots \wedge e_k, \frac{1}{3C_d^k}) \subset \mathcal{C}_k^{'}, \tag{4}$$

where $C_d^k := \frac{d!}{k!(d-k)!}$.

Proof. The assertion (i) follows directly from the definition of $\mathcal{C}_k$ and the space $\Lambda^k\mathbb{R}^d$, see Subsection 4.1. To prove (ii), we compute the dual cone $\mathcal{C}_k^{'}$ of $\mathcal{C}_k$. For this purpose, let $v = \sum_{1\leq i_1<\cdots<i_k\leq d} \alpha_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \in \mathcal{C}_k^{'}$ be arbitrary. Let $x = \sum_{1\leq i_1<\cdots<i_k\leq d} \beta_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $\beta_{1\ldots k} = 1$ and for $1 \leq i_1 < \cdots < i_k \leq d$ with $(i_1, \ldots, i_k) \neq (1, \ldots k)$

$$\beta_{i_1\ldots i_k} = \begin{cases} 1 & \text{if } \alpha_{i1\ldots i_k} \geq 0, \\ -1 & \text{if } \alpha_{i1\ldots i_k} < 0. \end{cases}$$

Obviously, $x \in \mathcal{C}_k$ and a direct computation yields that

$$\langle x, v\rangle = \alpha_{1\ldots k} + \sum_{1\leq i_1<\cdots<i_k\leq d,(i_1,\ldots,i_k)\neq(1,\ldots,k)} |\alpha_{i_1\ldots i_k}|.$$

Since $\langle x, v\rangle \geq 0$ it follows that $\alpha_{1\ldots k} \geq \frac{1}{2}\sum_{1\leq i_1<\cdots<i_k\leq d} |\alpha_{i_1\ldots i_k}|$. Conversely, let $v = \sum_{1\leq i_1<\cdots<i_k\leq d} \alpha_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$ satisfy that $\alpha_{1\ldots k} \geq \frac{1}{2}\sum_{1\leq i_1<\cdots<i_k\leq d} |\alpha_{i_1\ldots i_k}|$. For any $x = \sum_{1\leq i_1<\cdots<i_k\leq d} \beta_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \in \mathcal{C}_k$, we have

$$\begin{aligned} \langle x, v\rangle &= \sum_{1\leq i_1<\cdots<i_k\leq d} \alpha_{i_1\ldots i_k}\beta_{i_1\ldots i_k} \\ &\geq \alpha_{1\ldots k}\beta_{1\ldots k} - \sum_{1\leq i_1<\cdots<i_k\leq d,(i_1,\ldots,i_k)\neq(1,\ldots k)} |\alpha_{i_1\ldots i_k}\beta_{i_1\ldots i_k}| \\ &\geq \beta_{1\ldots k}\left(\alpha_{1\ldots k} - \sum_{1\leq i_1<\cdots<i_k\leq d,(i_1,\ldots,i_k)\neq(1,\ldots k)} |\alpha_{i_1\ldots i_k}|\right), \end{aligned}$$

where we use the fact that $\beta_{1\ldots k} = \max_{1\leq i_1<\cdots<i_k\leq d} |\beta_{i_1\ldots i_k}|$ to obtain the preceding inequality. Therefore, $\langle x, v\rangle \geq 0$ and the assertion (ii) is verified. In the remaining part of the proof, we show (4). For this purpose, let $v = \sum_{1\leq i_1<\cdots<i_k\leq d} \alpha_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$ satisfy that $\|e_1 \wedge \cdots \wedge e_k - v\|_{\Lambda^k\mathbb{R}^d} \leq \frac{1}{3}$. Thus, from

$$\|e_1 \wedge \cdots \wedge e_k - v\|^2_{\Lambda^k\mathbb{R}^d} = (\alpha_{1\ldots k} - 1)^2 + \sum_{1\leq i_1<\cdots<i_k\leq d,(i_1,\ldots,i_k)\neq(1,\ldots k)} \alpha^2_{i_1\ldots i_k}$$

we derive that

$$\frac{2}{3} \le \alpha_{1\dots k} \quad \text{and} \quad \max_{1\le i_1<\dots<i_k\le d, (i_1,\dots,i_k)\neq(1,\dots,k)} |\alpha_{i_1\dots i_k}| \le \frac{1}{3}. \tag{5}$$

Consequently, $v \in \mathcal{C}_k$ and thus $B(e_1 \wedge \dots \wedge e_k, \frac{1}{3}) \subset \mathcal{C}_k$. To prove the remaining assertion in (4), let $v = \sum_{1\le i_1<\dots<i_k\le d} \alpha_{i_1\dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}$ satisfy that $\|e_1 \wedge \dots \wedge e_k - v\|_{\Lambda^k\mathbb{R}^d} \le \frac{1}{3C_d^k}$. Analog to the arguments yielding (5), we have

$$1 - \frac{1}{3C_d^k} \le \alpha_{1\dots k} \quad \text{and} \quad \max_{1\le i_1<\dots<i_k\le d, (i_1,\dots,i_k)\neq(1,\dots,k)} |\alpha_{i_1\dots i_k}| \le \frac{1}{3C_d^k},$$

which implies that $\alpha_{1\dots k} \ge \sum_{1\le i_1<\dots<i_k\le d, (i_1,\dots,i_k)\neq(1,\dots,k)} |\alpha_{i_1\dots i_k}|$. Thus, $v \in \mathcal{C}_k'$ and the proof is complete. □

Theorem 4 (Analyticity for Lyapunov exponents of generic discrete-time linear random dynamical systems)**.** Let $A \in L^\infty(X, \mathrm{GL}(d))$ be an integrally separated bounded linear random dynamical system. Then, for $i = 1, \dots, d$ the map $\gamma_i(\cdot)$ is analytic at A. As a consequence, analyticity of Lyapunov exponents is a generic property in the Banach space $(L^\infty(X, \mathbb{R}^{d\times d}), \|\cdot\|_\infty)$.

Proof. Let $\mathbb{R}^d = \oplus_{i=1}^d E_i(x)$ be the invariant decomposition of A such that inequality (2) holds for $K, \alpha > 0$ and m-a.e. $x \in X$. For $i = 1, \dots, d$, choose and fix a measurable unit vector $u_i(x) \in E_i(x)$. By invariance of $E_i(x)$, we have

$$A(x)u_i(x) = a_i(x)u_i(Tx) \qquad \text{for } i = 1, \dots, d \text{ and } x \in X,$$

where $a_i : X \to \mathbb{R}$ is measurable. Since $A \in L^\infty(X, \mathrm{GL}(d))$, there exists $M_1 > 0$ such that

$$|a_i(x)|, \frac{1}{|a_i(x)|} \le M_1 \qquad \text{for } i = 1, \dots, d \text{ and } x \in X \tag{6}$$

and by (2)

$$\frac{|a_i(T^{n-1}x)\dots a_i(x)|}{|a_{i+1}(T^{n-1}x)\dots a_{i+1}(x)|} \ge Ke^{\alpha n} \quad \text{for } i = 1, \dots, d-1, n \in \mathbb{N}, x \in X. \tag{7}$$

Define $L : X \to \mathrm{GL}(d)$ by

$$L(x)e_i := u_i(x) \qquad \text{for } i = 1, \dots, d \text{ and } x \in X,$$

where $e_1, \dots, e_d$ is the standard Eucledian orthogonal basic of $\mathbb{R}^d$. From boundedness of A, A^{-1} and (2), for $i = 1, \dots, d$ the angle between $E_i(x)$ and $\oplus_{j=1, j\neq i}^d E_j(x)$ is uniformly bounded away from zero, see e.g. [Co05]. Consequently, there exists $M_2 > 0$ such that

$$\|L(x)\|, \|L^{-1}(x)\| \leq M_2 \qquad \text{for } x \in X. \tag{8}$$

Define a linear operator $\mathcal{Q}_L : L^\infty(X, \mathbb{R}^{d\times d}) \to L^\infty(X, \mathbb{R}^{d\times d})$ by

$$\mathcal{Q}_L M(x) := L^{-1}(Tx) M(x) L(x).$$

Then, $\mathcal{Q}_L$ is invertible and from (8), we have $\|\mathcal{Q}_L\|_\infty, \|\mathcal{Q}_L^{-1}\|_\infty \leq M_2^2$. Therefore, to complete the proof it is sufficient to show that for $i = 1, \dots, d$ the map $\gamma_i(\cdot)$ is analytic at $D \in \mathcal{L}^\infty(X, \mathrm{GL}(d))$ which is defined by

$$D(x) := L^{-1}(Tx) A(x) L(x) = \mathrm{diag}(a_1(x), \dots, a_d(x)).$$

The proof of this fact is divided into two steps:

Step 1: Choose and fix an arbitrary $k \in \{1, \dots, d\}$. Let N be a positive integer such that $Ke^{\alpha N} \geq 2$. Define a random map $D^{(N)} \in \mathcal{L}^\infty(X, \mathrm{GL}(d))$ by

$$D^{(N)}(x) := D(T^{N-1}x) \dots D(x). \tag{9}$$

Let $\Lambda^k D^{(N)}$ denote the generated random map of $D^{(N)}$ in $\mathcal{L}^\infty(X, \mathcal{L}(\Lambda^k \mathbb{R}^d))$. In this step, we show that $\Lambda^k D^{(N)}$ preserves the cone $\mathcal{C}_k$ defined as in (3), i.e.

$$\Lambda^k D^{(N)}(x) \mathcal{C}_k \subset \big(\mathcal{C}_k \cup (-\mathcal{C}_k)\big) \quad \text{for } x \in X \tag{10}$$

and we also show that there exists $R > 0$ such that

$$d_{\mathcal{C}_k}\big(\Lambda^k D^{(N)}(x) v, e_1 \wedge \dots \wedge e_k\big) < R \quad \text{for } x \in X, v \in \mathcal{C}_k \setminus \{0\} \text{ with } \|v\|_{\Lambda^k \mathbb{R}^d} = 1. \tag{11}$$

Choose and fix an arbitrary $x \in X$. By linearity of $D^{(N)}(x)$, to prove (10) it is sufficient to show that

$$\Lambda^k D^{(N)}(x) v \in \big(\mathcal{C}_k \cup (-\mathcal{C}_k)\big) \qquad \text{for } v \in \mathcal{C}_k \setminus \{0\} \text{ with } \|v\|_{\Lambda^k \mathbb{R}^d} = 1.$$

Let $v = \sum_{1 \leq i_1 < \dots < i_k \leq d} : \alpha_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k} \in \mathcal{C}_k \setminus \{0\}$ satisfy that $\|v\|_{\Lambda^k \mathbb{R}^d} = 1$. By definition of $\|\cdot\|_{\Lambda^k \mathbb{R}^d}$, see Subsection 4.1, we have

$$\|v\|^2_{\Lambda^k \mathbb{R}^d} = \sum_{1 \leq i_1 < \dots < i_k \leq d} \alpha^2_{i_1 \dots i_k} = 1,$$

which together with the fact that $\alpha_{1\dots k} = \max_{1\leq i_1<\dots<i_k\leq d}|\alpha_{i_1\dots i_k}|$ implies that $\frac{1}{\sqrt{C_d^k}} \leq \alpha_{1\dots k} \leq 1$, where $C_d^k := \frac{d!}{k!(d-k)!}$. By definition of $D^{(N)}$ and the generated random dynamical system $\Lambda^k D^{(N)}$ on $\Lambda^k\mathbb{R}^d$, see Subsection 4.1, we have

$$\begin{aligned}\Lambda^k D^{(N)}(x)v &= \sum_{1\leq i_1<\dots<i_k\leq d} \alpha_{i_1\dots i_k}\Lambda^k D^{(N)}(x)e_{i_1}\wedge\dots\wedge e_{i_k}\\ &= \sum_{1\leq i_1<\dots<i_k\leq d} \Big(\alpha_{i_1\dots i_k}\prod_{j=1}^k a_{i_j}(T^{N-1}x)\dots a_{i_j}(x)\Big)e_{i_1}\wedge\dots\wedge e_{i_k}.\end{aligned}$$

Using (7), we obtain that

$$\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)| \geq Ke^{\alpha N}\prod_{j=1}^k |a_{i_j}(T^{N-1}x)\dots a_{i_j}(x)|$$

for any index set $(i_1,\dots,i_k)\neq(1,\dots,k)$. This together with the fact that $\alpha_{1\dots k} = \max_{1\leq i_1<\dots<i_k\leq d}|\alpha_{i_1\dots i_k}|$ and $Ke^{\alpha N}\geq 2$ implies that for any index set $(i_1,\dots,i_k)\neq(1,\dots,k)$

$$\alpha_{1\dots k}\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)| \geq 2|\alpha_{i_1\dots i_k}|\prod_{j=1}^k |a_{i_j}(T^{N-1}x)\dots a_{i_j}(x)|. \quad (12)$$

Consequently, $\Lambda^k D^{(N)}(x)v \in \big(\mathcal{C}_k\cup(-\mathcal{C}_k)\big)$ and (10) is proved. Furthermore, from (12) and a direct computation yields that

$$\begin{aligned} 2\alpha_{1\dots k}\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)|e_1\wedge\dots\wedge e_k - \Lambda^k D^{(N)}(x)v &\in \mathcal{C}_k,\\ \frac{2}{\alpha_{1\dots k}\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)|}\Lambda^k D^{(N)}(x)v - e_1\wedge\dots\wedge e_k &\in \mathcal{C}_k.\end{aligned}$$

Thus, the distance (with respect to the Hilbert metric on the cone $\mathcal{C}_k$) between two vectors $\Lambda^k D^{(N)}(x)v$ and $e_1\wedge\dots\wedge e_k$ can be estimated as follows

$$\begin{aligned} d_{\mathcal{C}_k}(e_1\wedge\dots\wedge e_k, \Lambda^k D^{(N)}(x)v) \leq\ & \log\Big(2\alpha_{1\dots k}\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)|\Big) +\\ & \log\big(\frac{2}{\alpha_{1\dots k}\prod_{j=1}^k |a_j(T^{N-1}x)\dots a_j(x)|}\big)\end{aligned}$$

This together with (6) and the fact that $\frac{1}{\sqrt{C_d^k}} \leq \alpha_{1\dots k} \leq 1$ shows (11) with

$$R := \log(2M_1^k) + \log(2\sqrt{C_d^k}M_1^k).$$

Step 2: So far we have proved that for any $k \in \{1, \dots, d\}$ the random linear map $\Lambda^k D^{(N)}$ preserves the cone $\mathcal{C}_k$. On the other hand, by Lemma 3 the cone $\mathcal{C}_k$ satisfies conditions (C1) and (C2) of Theorem 8 in the Appendix with the measurable mappings $c : X \to \mathcal{C}_k$ and $c^{'} : X \to \mathcal{C}_k^{'}$ defined by

$$c(x) = c^{'}(x) := e_1 \wedge \cdots \wedge e_k \quad \text{for } x \in X.$$

To apply Theorem 8 to the random map $\Lambda^k D^{(N)}$ over $(X, \mathcal{A}, m, T^N)$, we only need to check the ergodicity of the transformation T^N. In what follows, we consider two separated cases and our aim is to show that the map $\gamma_1(\cdot) + \cdots + \gamma_k(\cdot)$ is analytic at D, where $k \in \{1, \dots, d\}$.

Case 1: T^N is an ergodic transformation from X into itself. In this case, using the Multiplicative Ergodic Theorem, the Lyapunov exponents of the random dynamical system generated by $M \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ over $(X, \mathcal{A}, m, T^N)$, denoted by $\gamma_d^{(N)}(M) \leq \cdots \leq \gamma_1^{(N)}(M)$, are well-defined. Define a map $\pi : \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ by

$$\pi(M)(x) := M(T^{N-1}x)\dots M(x).$$

Note that π is a product of bounded linear operators $\pi_i : \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}), M \mapsto \pi_i(M)$ with $\pi_i(M)(x) := M(T^i x)$ and $i = 1, \dots, N-1$. Thus, by Lemma 2 (ii) and (iii), the function π is analytic. Furthermore, by definition of $\gamma_i^{(N)}$ we have

$$\gamma_i(M) = \frac{1}{N}\gamma_i^{(N)}(\pi(M)) \qquad \text{for } i = 1, \dots, d \text{ and } M \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}).$$

In view of Theorem 7, we have for all $\Delta \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$

$$\sum_{i=1}^k \gamma_i(D+\Delta) = \frac{1}{N}\sum_{i=1}^k \gamma_i^{(N)}(\pi(D+\Delta)) = \frac{1}{N}\gamma_1^{(N)}(\Lambda^k \pi(D+\Delta)). \tag{13}$$

By (9), we have $\pi(D) = D^{(N)}$. Using achievements proved in **Step 1** and Theorem 8, the map $\gamma_1^{(N)}$ is analytic in a neighborhood of $\Lambda^k\pi(D)$. Hence, by Lemma 2 and (13), the map $\sum_{i=1}^k \gamma_i(\cdot)$ is analytic at D so the analyticity of functions $\gamma_i(\cdot)$, $i = 1, \dots, d$, at D is proved.

Case 2: T^N is not ergodic. Since all statements above still work if N is replaced by a larger integer number. Hence, we can assume additionally that N is a prime number. Let W be a measurable set satisfying that $m(W) \in (0,1)$ and $W = T^N W$. We prove by induction that for $k = 1, \dots, N$ there exists a measurable set W_k such that

$$T^N W_k = W_k \quad \text{and} \quad m(W_k) \in (0,1), m\big(W_k \cap \bigcup_{i=1}^{k-1} T^i W_k\big) = 0. \tag{14}$$

Obviously, (14) holds for $k = 1$ with $W_1 := W$ and suppose that it holds for some $k \in \{1, \dots, N-1\}$ with a measurable set W_k. By ergodicity of T and the fact that $T\big(\bigcap_{i=1}^{N-1} T^i(W_k)\big) = \bigcap_{i=1}^{N-1} T^{i+1}(W_k) = \bigcap_{i=1}^{N-1} T^i(W_k)$, we have $m(\bigcap_{i=1}^{N-1} T^i(W_k)) = 0$. Since N is a prime number and $T^N W_k = W_k$ it follows that

$$\bigcap_{i=1}^{N-1} T^i(W_k) = \bigcap_{i=1}^{N-1} T^{ik}(W_k).$$

Then, let $\ell \in \{1, \dots, N-1\}$ be the smallest integer such that $m\big(W_k \cap \bigcap_{i=1}^{\ell} T^{ik} W_k\big) = 0$. Hence, a measurable set $W_{k+1} := W_k \cap \bigcap_{i=1}^{\ell-1} T^{ik} W_k$ is also invariant under T^N and satisfies (14) for $k+1$. For the measurable set $\widehat{W} := W_N$ we have $T^N \widehat{W} = \widehat{W}$ and the sets $\widehat{W}, \dots, T^{N-1}\widehat{W}$ are pairwise disjoint. From $T(\bigcup_{i=1}^{N-1} T^i \widehat{W}) = \bigcup_{i=1}^{N-1} T^i \widehat{W}$ and ergodicity of T, we have $m\big(\bigcup_{i=1}^{N-1} T^i \widehat{W}\big) = 1$. On the other hand, for any subset $U \subset \widehat{W}$ with positive measure and $m(\widehat{W} \setminus U) > 0$ we have $m\big(\bigcup_{i=1}^{N-1} T^i U\big) < 1$. This implies together with ergodicity of T that $m\big(\bigcup_{i=1}^{N-1} T^i U \Delta \bigcup_{i=2}^{N} T^i U\big) > 0$. Consequently, $m(T^N U \Delta U) \neq 0$ and therefore the restriction of the map T^N on the set $\widehat{W}$ is ergodic. Thus, using a similar statement as in *Case 1* to the induced random dynamical systems on $(\widehat{W}, \mathcal{A}, m, T^N)$ completes the proof. □

3.2 Bounded linear random differential equations

Our aim in this subsection is to study the analyticity of Lyapunov exponents of bounded linear random differential equations over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. For this purpose, we need the following preparation. Firstly, we recall a notion of flow built under a function:

Definition 5 (Flow built under a function)**.** Let $(X, \mathcal{A}, m)$ be a probability space and $T : X \to X$ be an invertible measurable transformation preserving the probability m. Let $f : X \to \mathbb{R}_{\geq 0}$ be a measurable function with $\int_X f(x)\, dm(x) < \infty$. The flow built under the function (X, T, m, f) is defined with the following ingredients:

(i) *The base space* (B, σ, μ): The set B is defined by $B := \{(x, s) \in X \times \mathbb{R}_{\geq 0} : x \in X, 0 \leq s < f(x)\}$. Then, B is a measurable set of the measurable space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$. Let σ and μ denote, respectively, the restriction of the sigma algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ and the product measure $m \times \lambda$ on the measurable set B, where λ is the Lebesgue measure on $\mathbb{R}_{\geq 0}$.

(ii) *The flow* $(S_t)_{t \in \mathbb{R}}$: For each $t \in \mathbb{R}_{\geq 0}$, the map $S_t : B \to B$ is defined by

$$S_t(x, s) := (T^{k-1}x, t + s - \sum_{j=1}^{k-1} f(T^j x)),$$

where k is the smallest positive integer satisfying

$$\sum_{j=1}^{k-1} f(T^j x) \leq t + s < \sum_{j=1}^{k} f(T^j x),$$

and for each $t < 0$, $S_t := (S_{-t})^{-1}$.

According to the representation theory of ergodic flow (see e.g. [Am41, CFS82]), there exists a Lebesgue probability space $(X, \mathcal{A}, m)$ and an ergodic transformation $T : X \to X$ preserving the probability m and a measurable function $f : X \to \mathbb{R}_{\geq 0}$ which is bounded from 0 and ∞ such that the flow $(S_t)_{t \in \mathbb{R}}$ built under a function (X, T, m, f) is isomorphic to the flow $(\theta_t)_{t \in \mathbb{R}}$, i.e. there exists a measure preserving bijective transformation $H : \Omega \to B$, where $B = \{(x, s) : 0 \leq s < f(x)\}$, such that

$$H \circ \theta_t(\omega) = S_t \circ H(\omega) \qquad \text{for all } \omega \in \Omega. \tag{15}$$

Since f is bounded from 0 and ∞, it follows that

$$\overline{c} := \operatorname*{ess\,sup}_{x \in X} f(x) \in (0, \infty), \quad \underline{c} := \operatorname*{ess\,inf}_{x \in X} f(x) \in (0, \infty). \tag{16}$$

Finally, since $(X, \mathcal{A}, m)$ is a Lebesgue probability space, $(X, \mathcal{F}, m)$ is of one of the following cases:

Case A: $(X, \mathcal{A}, m)$ is isomorphic to $([0,1], \lambda)$, where λ is the standard Lebesgue probability on $[0,1]$.

Case B: $(X, \mathcal{A}, m)$ is isomorphic to a probability space $[0,s] \cup \{x_1, x_2, \ldots, x_k\}$, where $s = 1 - \sum_{i=1}^{k} p_i$ with p_i is the probability of $\{x_i\}$ and k can be equal to ∞. Note that in this case, the base space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is isomorphic to $(\mathcal{S}^1, \mathcal{B}(\mathcal{S}^1), \lambda, (R_t)_{t\in\mathbb{R}})$, where $\mathcal{S}^1 := \{e^{2\pi i z} : z \in [0,1]\}$ is the unit circle, λ is the standard Lebesgue probability on $\mathcal{S}^1$ and $R_t : \mathcal{S}^1 \to \mathcal{S}^1$ is the rotation map defined by $R_t(e^{2\pi i z}) = e^{2\pi i (z+t)}$, see [CD16, Remark 1].

Theorem 6 (Analyticity for Lyapunov exponents of generic bounded linear random differential equations)**.** For each $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$, let $\gamma_d(A) \leq \cdots \leq \gamma_1(A)$ denote the Lyapunov exponents of the linear random differential equation

$$\dot{\xi} = A(\theta_t\omega)\xi. \tag{17}$$

Then, the following statements hold:

(i) Suppose that **Case A** holds. Then, there exists an open and dense set $\mathcal{R} \subset \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ such that the functions $\gamma_1(\cdot), \ldots, \gamma_d(\cdot)$ are analytic at $A \in \mathcal{R}$.

(ii) Suppose that **Case B** holds. Then, the functions $\gamma_1(\cdot), \ldots, \gamma_d(\cdot)$ are analytic at all $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$.

Proof. Define a function $\mathcal{T} : \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ by

$$\mathcal{T}(A)(x) := \Psi_A(f(x), H^{-1}(x,0)) \qquad \text{for all } x \in X, \tag{18}$$

where Φ_A denote the linear random dynamical system generated by (17) and H is defined as in (15). By variation of constants formula, we have for all $\Delta \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$

$$\begin{aligned}
\mathcal{T}(A+\Delta)(x) &= \Psi_{A+\Delta}(f(x), H^{-1}(x,0)) \\
&= \Psi_A(f(x), H^{-1}(x,0)) + \\
\int_0^{f(x)} &\Psi_A(f(x)-s, \theta_s H^{-1}(x,0))\Delta(\theta_s H^{-1}(x,0))\Psi_A(s, H^{-1}(x,0))\, ds \\
&= \mathcal{T}(A)(x) + \mathcal{L}_A(\Delta)(x),
\end{aligned}$$

where the operator $\mathcal{L}_A : \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ is defined by

$$\Delta \mapsto \int_0^{f(x)} \Psi_A(f(x)-s, \theta_s H^{-1}(x,0))\Delta(\theta_s H^{-1}(x,0))\Psi_A(s, H^{-1}(x,0))\, ds.$$

Obviously, $\mathcal{L}_A$ is a linear operator and from the boundedness of A and (16), the operator $\mathcal{L}_A$ is bounded. Consequently, by Lemma 2 (ii), the function $\mathcal{T}$ is analytic. In the remaining part of the proof, we consider two separated cases:

(i) Suppose that **Case A** holds. As is proved in [CD16, Theorem 3.2], there exists an open and dense set $\mathcal{R} \subset \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ such that for all $A \in \mathcal{R}$, the generated random dynamical system Ψ_A and hence also $\mathcal{T}(A)$ are integrally separated. This together with analyticity of $\mathcal{T}$, Theorem 4 and Lemma 2 completes the proof of this part.

(ii) Suppose that **Case B** holds. In this case, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ can be identified with $(\mathcal{S}^1, \mathcal{B}(\mathcal{S}^1), \lambda, (R_t)_{t\in\mathbb{R}})$. Choose and fix an arbitrary $\omega_0 \in \mathcal{S}^1$ and consider a map $\mathcal{P} : \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}) \to \mathbb{R}^{d\times d}$ defined by $\mathcal{P}A := \Psi_A(1, \omega_0)$. This is clear that the Lyapunov exponents of the linear random dynamical systems generated by

$$\dot{\xi} = A(\theta_t\omega)\xi$$

coincides with the set of the modulus of the eigenvalues of $\mathcal{P}A$. Analog to the proof of the analyticity of the map $\mathcal{T}$ above, the map $\mathcal{P}$ is also analytic. This together with the fact that the eigenvalues of matrices depend analytically on the matrices (see e.g. [Ka80]) proves (ii). The proof is complete. □

4 Appendix

Throughout this section, let $(X, \mathcal{A}, m)$ be a probability space and $T : X \to X$ is an ergodic transformation preserving the probability m.

4.1 Exterior powers

For $1 \le k \le d$, let $\Lambda^k\mathbb{R}^d$, the *k-fold exterior power* of $\mathbb{R}^d$, be the vector space of alternating k-linear forms on the dual space $\mathbb{R}^d$. The space $\Lambda^k\mathbb{R}^d$ can be identified with the set of formal expressions $\sum_{i=1}^m c_i(u_1^{(i)} \wedge \cdots \wedge u_k^{(i)})$ with $m \in \mathbb{N}$, $c_i \in \mathbb{R}$ and $u_j^{(i)} \in \mathbb{R}^d$ if we do computations with the following conventions:

1. *Adddition*:

$$u_1 \wedge \cdots \wedge (u_j + \widehat{u}_j) \wedge \cdots \wedge u_k = u_1 \wedge \cdots \wedge u_j \wedge \cdots \wedge u_k + u_1 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_k,$$

2. *Scalar multiplication*: $u_1 \wedge \cdots \wedge cu_j \wedge \cdots \wedge u_k = cu_1 \wedge \cdots \wedge u_j \wedge \cdots \wedge u_k$,

3. for any permutation π of $\{1, \ldots, k\}$

$$u_{\pi(1)} \wedge \cdots \wedge u_{\pi(k)} = \operatorname{sign}(\pi) u_1 \wedge \cdots \wedge u_k.$$

The canonical inner product on $\mathbb{R}^d$ induces an inner product $\langle \cdot, \cdot \rangle$ on $\Lambda^k \mathbb{R}^d$ via

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle := \det(\langle u_i, v_j \rangle)_{k \times k}.$$

Let $\{e_i\}_{i=1\ldots,d}$ denote the standard Eucledian orthogonal basic of $\mathbb{R}^d$. For each $k \in \{1, \ldots, d\}$, we define

$$e_{i_1 \ldots i_k} := e_{i_1} \wedge \cdots \wedge e_{i_k} \qquad \text{for } 1 \le i_1 < \cdots < i_k \le d. \tag{19}$$

Then,

$$\{e_{i_1 \ldots i_k} : 1 \le i_1 < \cdots < i_k \le d\}$$

is an orthogonal basis of $\Lambda^k \mathbb{R}^d$.

A random map $A : X \to \mathbb{R}^{d \times d}$ generates a random map $\Lambda^k A : X \to \mathcal{L}(\Lambda^k \mathbb{R}^d)$ defined by

$$\Lambda^k A(x)(u_1 \wedge \cdots \wedge u_k) := A(x)u_1 \wedge \cdots \wedge A(x)u_k,$$

where $\mathcal{L}(\Lambda^k \mathbb{R}^d)$ denotes the space of linear operators from $\Lambda^k \mathbb{R}^d$ into itself. The generated linear random dynamical system by $\Lambda^k A$ on $\Lambda^k \mathbb{R}^d$, which is denoted by $\Lambda^k \Phi_A$, is called *the k fold exterior of* Φ_A.

Theorem 7 (Lyapunov exponents of induced linear RDS on the exterior power)**.** Let $\gamma_d(A) \le \gamma_{d-1}(A) \le \cdots \le \gamma_1(A)$ denote the Lyapunov exponents (counting with multiplicity) of the linear RDS Φ_A generated by A. Then, the Lyapunov exponents (counting with multiplicity) of $\Lambda^k \Phi_A$ are all sums of k of the exponents of Φ_A:

$$\{\gamma_{i_1}(A) + \cdots + \gamma_{i_k}(A) : 1 \le i_1 < \cdots < i_k \le d\}.$$

Proof. See [Cr90, Remark 3.4]. □

4.2 Positive linear random dynamical systems

A subset $\mathcal{C} \subset \mathbb{R}^d$ is called a *proper cone* if $tv \in \mathcal{C}$ for all $t \geq 0, v \in \mathcal{C}$ and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. For a closed proper convex cone $\mathcal{C}$, the dual cone $\mathcal{C}^{'}$ is defined by

$$\mathcal{C}^{'} := \{x \in \mathbb{R}^d : \langle x, v\rangle \geq 0 \quad \text{for all } v \in \mathcal{C}\}.$$

Recall that the *Hilbert metric* $d_{\mathcal{C}}$ on $\mathcal{C} \setminus \{0\}$ is defined by

$$d_{\mathcal{C}}(x,y) := \log(\beta_{\mathcal{C}}(x,y)) + \log(\beta_{\mathcal{C}}(y,x)) \qquad \text{for } x, y \in \mathcal{C} \setminus \{0\}, \tag{20}$$

where $\beta_{\mathcal{C}}(x,y) := \inf\{t > 0 : tx - y \in \mathcal{C}\}$. In what follows, the Hilbert metric $d_{\mathcal{C}}$ will be understood in a wider sense that $d_{\mathcal{C}}(\pm x, \pm y) := d_{\mathcal{C}}(x,y)$ for all $(x,y) \in \mathcal{C}$.

Theorem 8. Let $(\mathcal{C}_x)_{x \in X}$ be a family of closed proper convex cones satisfying the following conditions:

(C1) there is a measurable mapping $c : X \to \mathbb{R}^d$ such that $\|c(x)\| = 1$ and $B(c(x), r) \subset \mathcal{C}_x$ with r is independent of x,

(C2) there is a measurable mapping $c^{'} : X \to \mathbb{R}^d$ such that $\|c^{'}(x)\| = 1$ and $B(c^{'}(x), r^{'}) \subset \mathcal{C}^{'}_x$ with $r^{'}$ is independent of x.

Let $A \in L^\infty(X, \mathrm{GL}(d))$ satisfy that $A(x)\mathcal{C}_x \subset \mathcal{C}_{Tx} \cup (-\mathcal{C}_{Tx})$ and there exists $R < \infty$ such that

$$d_{\mathcal{C}_{Tx}}(A(x)v, c(Tx)) \leq R \qquad \text{for all } x \in X, v \in \mathcal{C}_x \setminus \{0\}.$$

Then, the following statements hold:

(i) The Oseledets subspace corresponding to the top Lyapunov exponent $\lambda_1(A)$ is of dimensional one,

(ii) The map $\lambda_1(\cdot)$ is analytic at A.

Proof. For each $x \in X$, let

$$r(x) := \begin{cases} 1, & \text{if } A(x)\mathcal{C}_x \subset \mathcal{C}_{Tx}, \\ -1, & \text{if } A(x)\mathcal{C}_x \subset -\mathcal{C}_{Tx}. \end{cases}$$

By properties of measurable functions c and $c^{'}$ in (C1) and (C2), we have

$$r(x) = \begin{cases} 1, & \text{if } \langle A(x)c(x), c^{'}(Tx)\rangle > 0, \\ -1, & \text{if } \langle A(x)c(x), c^{'}(Tx)\rangle < 0, \end{cases}$$

which implies that r is measurable. Now, we consider a linear operator $\mathcal{T} : \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}) \to \mathcal{L}^\infty(X, \mathbb{R}^{d\times d})$ defined by

$$\mathcal{T}M(x) = r(x)M(x) \quad \text{for all } M \in \mathcal{L}^\infty(X, \mathbb{R}^{d\times d}).$$

Obviously, $\lambda_1(\mathcal{T}M) = \lambda_1(M)$ and the Oseledets subspaces of M and $\mathcal{T}M$ coincide. Hence, it is sufficient to prove (i) and (ii) for $\mathcal{T}A$. By definition of r, the random matrix $\mathcal{T}A$ preserves the family of cones $\mathcal{C}_x$ in the sense that $\mathcal{T}A(x)\mathcal{C}_x \subset \mathcal{C}_{Tx}$. Hence, using results in [AGD94] yields (i). Meanwhile, the assertion (ii) is obtained by applying the result in [Dub08]. The proof is complete. □

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