A Perron-type theorem for fractional differential systems

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Abstract

In this paper, we prove a Perron-type theorem for fractional differential systems. More precisely, we obtain a necessary and sufficient condition for a system of linear inhomogeneous fractional differential equations to have at least one bounded solution for every bounded inhomogeneity.

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1 Introduction

In recent years, fractional differential equations have attracted increasing interest due to their varied applications on various fields of science and engineering, see e.g., [BK15, Di04, Pod99, SKM93]. Several results on asymptotic behavior of fractional differential equations are published (e.g., on Linear theory [M96, CST14], Stability theory for nonlinear systems [ASS07, CDST16], Stable manifolds [CSST16], Stability theory for perturbed linear systems [CST16],...). However, the qualitative theory of fractional differential equations is still in its infancy. One of the reasons for this fact might

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be that these equations do not generate semigroups and the well-developed theory to ordinary differential equations cannot be applied directly.

Consider the inhomogenneous system of the order $\alpha \in (0,1)$ involving *Caputo derivative*

$$ {}^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(t), \quad x(0) = x_0 \in \mathbb{R}^d, \tag{1} $$

where $t \in [0,\infty)$, $A \in \mathbb{R}^{d\times d}$ and $f : [0,\infty) \to \mathbb{R}^d$.

Motivated by Perron's work, an interesting question arises here: what is the necessary and sufficient condition on A for which (1) has at least one bounded solution for any bounded continuous vector-valued function f? In the case of ordinary differential equations ($\alpha = 1$), the answer is known: the matrix A is hyperbolic (see [Cop78, Proposition 3, p. 22]). However, for the fractional case the question is still open.

Note that in Matignon [M96], the author gives a necessary and sufficient condition of the matrix A such that for any external force f and any initial condition x_0, the solution x of (1) is bounded.

In this paper, we will give a Perron-type theorem for fractional differential systems saying that the inhomogeneous system (1) has at least one bounded solution for any bounded continuous function f if and only if the matrix A satisfies a (fractional) hyperbolic condition

$$ \sigma(A) \cap \left\{ \lambda \in \mathbb{C} : \lambda = 0 \ \text{ or } \ |\arg(\lambda)| = \frac{\alpha\pi}{2} \right\} = \emptyset, \tag{2} $$

where $\sigma(A)$ is the set of all eigenvalues of the matrix A. This result is a natural analog of the known theorem of the theory of ordinary differential equations. Our approach is as follows. First, we transform the matrix A of the system (1) into its Jordan normal form to obtain a simpler system. Next, using the variation of constants formula and a procedure of substitution, we describe explicitly bounded solutions. Finally, by estimating Mittag-Leffler functions, we show the asymptotic behavior of solutions which enable us to describe the set of unbounded solutions of (1) when the matrix does not satisfy the hyperbolic condition.

The paper is organized as follows. In Section 2, we present some basics of fractional calculus and some preliminary results related to Mittag-Leffler functions. In Section 3, we state and prove the main result of the paper (Theorem 5).

To conclude the introductory section, we fix some notation which will be used later. Let $\mathbb{R}$, $\mathbb{C}$ be the set of all real numbers and complex numbers, respectively. Denote by $\mathbb{R}_{\geq 0}$ the set of all nonnegative real numbers. For a Banach space $(X, \|\cdot\|)$, we define $(C_b(\mathbb{R}_{\geq 0}; X), \|\cdot\|_\infty)$ as the space of all continuous function $\xi : \mathbb{R}_{\geq 0} \to X$ such that

$$\|\xi\|_\infty := \sup_{t\geq 0} \|\xi(t)\| < \infty.$$

For any $\lambda \in \mathbb{C} \setminus \{0\}$, we define its argument to be in the interval $-\pi < \arg(\lambda) \leq \pi$ and $\Re\lambda$, $\Im\lambda$ the real part, the imaginary part of the complex number λ, respectively. For $\alpha \in (0, 1)$, we define the sets

$$\Lambda_\alpha^u := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \frac{\alpha\pi}{2}\right\}, \tag{3}$$

$$\Lambda_\alpha^s := \left\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2}\right\}. \tag{4}$$

2 Preliminaries

2.1 Inhomogeneous linear fractional differential equations

For $\alpha > 0$, $[a, b] \subset \mathbb{R}$ and $x : [a, b] \to \mathbb{R}$ is a measurable function such that $\int_a^b |x(\tau)|\, d\tau < \infty$, the Riemann–Liouville integral operator of order α is defined by

$$(I_{a+}^\alpha x)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau, \quad t \in (a, b],$$

where the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1} \exp(-\tau)\, d\tau.$$

The *Caputo fractional derivative* ${}^C D_{a+}^\alpha x$ of a function $x \in C^m([a, b])$ is defined by

$$({}^C D_{a+}^\alpha x)(t) := (I_{a+}^{m-\alpha} D^m x)(t), \qquad \forall t \in [a, b],$$

where $D^m = \frac{d^m}{dt^m}$ is the usual m^{th}-order derivative and $m := \lceil \alpha \rceil$ is the smallest integer larger or equal to α, see, e.g., [Pod99, p. 79]. While the Caputo fractional derivative of a d-dimensional vector function $x(t) = (x_1(t), \cdots, x_d(t))^{\mathrm{T}}$ is defined component-wise as

$$({}^C D_{a+}^\alpha x)(t) := ({}^C D_{a+}^\alpha x_1(t), \cdots, {}^C D_{a+}^\alpha x_d(t))^{\mathrm{T}}.$$

Throughout this paper, we consider the initial value problem:

$$
{}^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(t), \qquad x(0) = \xi \in \mathbb{R}^d \tag{5}
$$

with $\alpha \in (0,1)$ and $f : [0,\infty) \to \mathbb{R}^d$ is a continuous function. It is well known that the initial problem (5) has a unique solution defined on the whole interval $[0,\infty)$, see, e.g., [Di04, Theorem 6.8]. An explicit formula of this solution is given by using *Mittag-Leffler functions* which are defined as

$$
E_{\alpha,\beta}(M) := \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \qquad E_{\alpha}(M) := E_{\alpha,1}(M), \quad \forall M \in \mathbb{C}^{d\times d},
$$

where $\beta \in \mathbb{R}$.

Theorem 1 (Variation of constants formula for fractional differential equations)**.** Let $\xi \in \mathbb{R}^d$ and $\varphi(\cdot,\xi)$ denote the solution of the initial problem (5). Then the following (variation of constants) formula holds

$$
\varphi(t,\xi) = E_{\alpha}(t^{\alpha}A)\,\xi + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^{\alpha}A) f(\tau)\, d\tau, \quad \forall t \geq 0.
$$

Proof. Use the same arguments as in the proof of [CT17, Lemma 2]; see also [CT17, Remark 4]. □

2.2 Some useful properties of Mittag-Leffler functions

To investigate the asymptotic behavior of the solutions to linear fractional differential equations, it is important to know the behavior of Mittag-Leffler functions. Hence, we next introduce some basic properties of these functions. To save the length of the paper we give only sketch of the proofs of the results presented in this subsection.

Lemma 2. Let $\lambda \in \mathbb{C}$ be arbitrary. There exist a positive real number $m(\alpha,\lambda)$ such that for every $t \geq 1$ the following estimations hold:

(i) if $\lambda \in \Lambda^u_\alpha$ then

$$
\left| E_{\alpha}(\lambda t^{\alpha}) - \frac{1}{\alpha}\exp\left(\lambda^{\frac{1}{\alpha}} t\right) \right| \leq \frac{m(\alpha,\lambda)}{t^{\alpha}},
$$

$$
\left| t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}\exp\left(\lambda^{\frac{1}{\alpha}} t\right) \right| \leq \frac{m(\alpha,\lambda)}{t^{\alpha+1}};
$$

(ii) if $\lambda \in \Lambda_\alpha^s$ then

$$\left|t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right| \leq \frac{m(\alpha,\lambda)}{t^{\alpha+1}}.$$

For a proof of this theorem one uses integral representations of Mittag-Leffler functions and the method of estimation of the integrals similar to that of the proofs of Theorem 1.3 and Theorem 1.4 in [Pod99, pp. 32–34].

Lemma 3. Let $\lambda \in \mathbb{C} \setminus \{0\}$. There exists a positive constant $K(\alpha,\lambda)$ such that for all $t \geq 0$ the following estimates hold:

(i) if $\lambda \in \Lambda_\alpha^u$ then

$$\int_t^\infty \left|\lambda^{\frac{1}{\alpha}-1}E_\alpha(\lambda t^\alpha)\exp(-\lambda^{\frac{1}{\alpha}}\tau)\right|\,d\tau \leq K(\alpha,\lambda),$$

$$\int_0^t \left|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) - \lambda^{\frac{1}{\alpha}-1}E_\alpha(\lambda t^\alpha)\exp(-\lambda^{\frac{1}{\alpha}}\tau)\right|\,d\tau \\ \leq K(\alpha,\lambda);$$

(ii) if $\lambda \in \Lambda_\alpha^s$ then

$$\int_0^t \left|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)\right|\,d\tau \leq K(\alpha,\lambda).$$

Proof. The proof of this lemma follows easily by using Lemma 2 and repeating arguments used in the proof of [CDST14, Lemma 5]. □

Lemma 4. For any function $g \in C_b(\mathbb{R}_{\geq 0};\mathbb{R})$ and $\lambda \in \Lambda_\alpha^u$, we have

$$\lim_{t\to\infty}\int_0^t (t-\tau)^{\alpha-1}\frac{E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)}{E_\alpha(\lambda t^\alpha)}g(\tau)\,d\tau \\ = \lambda^{\frac{1}{\alpha}-1}\int_0^\infty \exp(-\lambda^{\frac{1}{\alpha}}\tau)g(\tau)\,d\tau. \tag{6}$$

Proof. Use Lemma 2, Lemma 3 and arguments analogous to those used in the proof of [CDST14, Lemma 8]. □

3 A Perron type theorem for fractional differential equations

This section is devoted to a Perron-type theorem for fractional systems. The main result of this section is stated as follows.

Theorem 5 (Perron-type theorem for fractional differential equations)**.** Let $A \in \mathbb{R}^{d\times d}$ and $\alpha \in (0,1)$. The inhomogeneous system

$$^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(t)$$

has at least one bounded solution for any $f \in C_b(\mathbb{R}_{\geq 0};\mathbb{R}^d)$ if only if the matrix A satisfies the following condition:

$$\sigma(A) \cap \left\{\lambda \in \mathbb{C} : \lambda = 0 \ \text{ or } \ |\arg(\lambda)| = \frac{\alpha\pi}{2}\right\} = \emptyset.$$

The proof of Theorem 5 is divided into the sufficient part (Proposition 6) and the necessary part (Proposition 11). Firstly, we show the sufficient part.

Proposition 6 (Sufficient part of Theorem 5)**.** Let $A \in \mathbb{R}^{d\times d}$ satisfy the hyperbolic condition (2):

$$\sigma(A) \cap \left\{\lambda \in \mathbb{C} : \lambda = 0 \ \text{ or } \ |\arg(\lambda)| = \frac{\alpha\pi}{2}\right\} = \emptyset.$$

Then, for any $f \in C_b(\mathbb{R}_{\geq 0};\mathbb{R}^d)$, the corresponding inhomogeneous system

$$^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(t), \tag{7}$$

has at least one bounded solution.

Before proving Proposition 6, we transform the matrix A of the system (7) into its Jordan normal form to obtain a simpler system. Let $T \in \mathbb{R}^{d\times d}$ be a nonsingular matrix transforming A into its Jordan normal form, i.e.,

$$T^{-1}AT = \operatorname{diag}(A_1, \dots, A_n),$$

where for $j = 1, \dots, n$, the block A_j is of the following form

$$A_j := \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix}_{d_j\times d_j},$$

with $\lambda_j \in \sigma(A) \cap \mathbb{R}$, or

$$A_j = \begin{pmatrix} D_j & I & 0 & \cdots & 0 \\ 0 & D_j & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & D_j & I \\ 0 & 0 & \cdots & 0 & D_j \end{pmatrix}_{d_j \times d_j},$$

here

$$D_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a_j, b_j \in \mathbb{R}, \quad b_j \neq 0,$$

and $\lambda_j = a_j + ib_j \in \sigma(A)$. By the change of variable $x = Ty$, the system (7) is transformed into the equation

$$^{C}D^{\alpha}_{0+} y(t) = By(t) + g(t), \tag{8}$$

where B is the real Jordan normal form of A, i.e.,

$$B = T^{-1}AT = \operatorname{diag}(A_1, \ldots, A_n), \quad \text{and} \quad g(t) = T^{-1}f(t).$$

On the other hand, without loss of generality, we may rewrite (8) into the form

$$^{C}D^{\alpha}_{0+} y(t) = \operatorname{diag}(B^s, B^u)y(t) + (g^s(t), g^u(t))^{\mathrm{T}}, \tag{9}$$

where $B^{s/u}$ is the part of B corresponding to the collection of all blocks with the eigenvalues belonging to $\Lambda^{s/u}_{\alpha}$. Note that the system (7) has at least one bounded solution for any bounded continuous function f if and only if the system (9) has at least one bounded solution for any bounded continuous function g. Thus, we only focus on the system (9). We need the following preparatory lemmas for the proof of Proposition 6.

Lemma 7. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the inhomogeneous equation

$$^{C}D^{\alpha}_{0+} x(t) = \lambda x(t) + g(t), \tag{10}$$

where $g \in C_b(\mathbb{R}_{\geq 0}; \mathbb{C})$. Then, the following statements hold:

(i) if $\lambda \in \Lambda^s_{\alpha}$, then all solutions of (10) are bounded on $\mathbb{R}_{\geq 0}$;

(ii) if $\lambda \in \Lambda^u_{\alpha}$, then the equation (10) has exactly one bounded solution.

Proof. (i) Using the variation of constants formula provided by Theorem 1, for any $\xi \in \mathbb{C}$, the solution $\varphi(\cdot, \xi)$ of (10) has the representation

$$\varphi(t,\xi) = E_\alpha(\lambda t^\alpha)\xi + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) g(\tau)\, d\tau, \quad \forall t \geq 0.$$

From [Pod99, Theorem 1.4, p. 33], we see that the quantity $E_\alpha(\lambda t^\alpha)\xi$ is bounded on $[0,\infty)$. On the other hand, due to Lemma 3(ii), there exists a positive constant C such that

$$\int_0^t |(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) g(\tau)|\, d\tau \leq C \sup_{\tau \geq 0} |g(\tau)|, \quad \forall t \geq 0.$$

Thus $\varphi(\cdot,\xi)$ is bounded for any $\xi \in \mathbb{C}$.

(ii) Let

$$\xi^* := -\lambda^{\frac{1}{\alpha}-1} \int_0^\infty \exp(-\lambda^{\frac{1}{\alpha}} \tau) g(\tau)\, d\tau.$$

By the variation of constants formula provided by Theorem 1, we see that the function

$$\begin{aligned}\varphi(t,\xi^*) = E_\alpha(\lambda t^\alpha)\Big(-\lambda^{\frac{1}{\alpha}-1} \int_0^\infty \exp(-\lambda^{\frac{1}{\alpha}} \tau) g(\tau)\, d\tau \Big) \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) g(\tau)\, d\tau, \quad \forall t \geq 0,\end{aligned}$$

is a solution of (10). We will prove that this function is the only bounded solution. Indeed, for any $t \geq 0$, we have

$$\begin{aligned}|\varphi(t,\xi^*)| \leq & \int_t^\infty \Big|\lambda^{\frac{1}{\alpha}-1} E_\alpha(\lambda t^\alpha) \exp(-\lambda^{\frac{1}{\alpha}}\tau)\Big|\, |g(\tau)|\, d\tau \\ & + \int_0^t \Big|(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) - \lambda^{\frac{1}{\alpha}-1} E_\alpha(\lambda t^\alpha)\exp(-\lambda^{\frac{1}{\alpha}}\tau)\Big|\, |g(\tau)|\, d\tau,\end{aligned}$$

which together with Lemma 3(i) imply that

$$|\varphi(t,\xi^*)| \leq 2K(\alpha,\lambda) \sup_{\tau \geq 0} |g(\tau)|, \quad \forall t \geq 0.$$

Thus, $\varphi(\cdot,\xi^*)$ is bounded on $[0,\infty)$. Now, assume that $\varphi(\cdot,\xi)$ is another bounded solution of (10) for some $\xi \in \mathbb{C}$. Then,

$$\varphi(t,\xi^*) - \varphi(t,\xi) = E_\alpha(\lambda t^\alpha)(\xi^* - \xi), \quad \forall t \geq 0.$$

Because $\lim_{t\to\infty} E_\alpha(\lambda t^\alpha) = \infty$, we have $\xi^* = \xi$. This implies that

$$\varphi(t, \xi^*) = \varphi(t, \xi), \quad \forall t \geq 0.$$

Hence, the equation (10) has exactly one bounded solution. The proof is complete. $\square$

Remark 8. Consider the system

$$ {}^C D^\alpha_{0+} x_1(t) = a x_1(t) - b x_2(t) + g_1(t), \tag{11}$$

$$ {}^C D^\alpha_{0+} x_2(t) = b x_1(t) + a x_2(t) + g_2(t), \tag{12}$$

where $a, b \in \mathbb{R}$ and $g_1, g_2 \in C_b(\mathbb{R}_{\geq 0}; \mathbb{R})$. In the light of Proposition 7, we obtain the following result:

(i) if $a + ib \in \Lambda^s_\alpha$, then all solutions of the system (11)-(12) are bounded;

(ii) if $\lambda := a + ib \in \Lambda^u_\alpha$, then the system (11)-(12) has exactly one bounded solution as

$$(x_1(t), x_2(t))^{\mathrm{T}} = (\Re u(t), \Im u(t))^{\mathrm{T}}, \quad \forall t \geq 0,$$

where

$$\begin{aligned} u(t) &= E_\alpha(\lambda t^\alpha) \left(-\lambda^{\frac{1}{\alpha}-1} \int_0^\infty \exp\left(-\lambda^{1/\alpha}\tau\right)(g_1(\tau) + i g_2(\tau))\, d\tau \right) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)(g_1(\tau) + i g_2(\tau))\, d\tau, \quad \forall t \geq 0. \end{aligned}$$

Indeed, set $u(t) = x_1(t) + i x_2(t)$ and $g(t) = g_1(t) + i g_2(t)$, then the system (11)-(12) is equivalent to the equation

$$ {}^C D^\alpha_{0+} u(t) = \lambda u(t) + g(t)$$

and Lemma 7 is applied.

Next, we prove an analogous result of Lemma 7 for inhomogeneous systems whose linear parts are of the form $J_{k,\lambda} := \lambda\, \mathrm{id}_k + N_k$, where $k \in \mathbb{N}$, id_k denotes the unit matrix in $\mathbb{R}^{k\times k}$ and

$$N_k := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{k\times k}.$$

Lemma 9. Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the inhomogeneous system

$$ {}^{C}D^{\alpha}_{0+}x(t) = J_{k,\lambda}x(t) + g(t), \quad t \geq 0, \tag{13} $$

where $g \in C_b(\mathbb{R}_{\geq 0}; \mathbb{C}^k)$. Then, the following statements hold:

(i) if $\lambda \in \Lambda^s_\alpha$, then all solutions of (13) are bounded;

(ii) if $\lambda \in \Lambda^u_\alpha$, then the equation (13) has exactly one bounded solution.

Proof. By the definition of $J_{k,\lambda}$, the system (13) is rewritten in the form

$$ {}^{C}D^{\alpha}_{0}x_i(t) = \lambda\, x_i(t) + x_{i+1}(t) + g_i(t), \quad i = 1, \dots, k-1 \tag{14} $$

and

$$ {}^{C}D^{\alpha}_{0}x_k(t) = \lambda\, x_k(t) + g_k(t). \tag{15} $$

(i) Assume that $g \in C_b(\mathbb{R}_{\geq 0}; \mathbb{C}^k)$. Let

$$ x^0 = (x^0_1, \cdots, x^0_k)^{\mathrm{T}} \in \mathbb{C}^k $$

be an arbitrary vector and $\varphi(t, x^0) = (\varphi_1(t), \cdots, \varphi_k(t))^{\mathrm{T}}$ denote the solution of (13) satisfying the initial condition $\varphi(t, x^0) = x^0$. From (15), we have

$$ \varphi_k(t) = E_\alpha(\lambda t^\alpha)x^0_k + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)g_k(s)\, ds. \tag{16} $$

It follows from Lemma 7(i) that φ_k is bounded in $\mathbb{R}_{\geq 0}$. Substitute φ_k into (14) and applying Lemma 7(i) again we get that φ_{k-1} is also bounded. Continue this process we will get that $\varphi_{k-2}, \dots, \varphi_1$ are all bounded.

(ii) Using arguments similar to that of the part (i) above and with the application of Lemma 7(ii), we obtain the proof of Lemma 9(ii). □

Corollary 10. For $\lambda := a + ib \in \mathbb{C} \setminus \{0\}$, we consider the equation

$$ {}^{C}D^{\alpha}_{0+}x(t) = J_{2k,\lambda}x(t) + f(t), \tag{17} $$

where $f \in C_b(\mathbb{R}_{\geq 0}; \mathbb{R}^{2k})$ and $J_{2k,\lambda}$ is a real Jordan block in the form

$$ J_{2k,\lambda} := \begin{pmatrix} D & I & 0 & \cdots & 0 \\ 0 & D & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & D & I \\ 0 & 0 & \cdots & 0 & D \end{pmatrix}_{2k \times 2k}, $$

with

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, the following statements hold:

(i) if $\lambda \in \Lambda_\alpha^s$, then all solutions of (17) are bounded;

(ii) ff $\lambda \in \Lambda_\alpha^u$, then all solutions of (17) has exactly one bounded solution.

Proof. By using the change $u_j(t) = x_{2j-1}(t) + ix_{2j}(t)$, for $j = 1, \ldots, k$, from the system (17), we obtain the system

$$\begin{aligned} {}^C D_{0+}^\alpha u_1(t) &= \lambda u_1(t) + u_2(t) + g_1(t) & (18) \\ {}^C D_{0+}^\alpha u_2(t) &= \lambda u_2(t) + u_3(t) + g_2(t) & (19) \\ &\cdots \\ {}^C D_{0+}^\alpha u_{k-1}(t) &= \lambda u_{k-1}(t) + u_k(t) + g_{k-1}(t) & (20) \\ {}^C D_{0+}^\alpha u_k(t) &= \lambda u_k(t) + g_k(t), & (21) \end{aligned}$$

here $g_j(t) = f_{2j-1}(t) + if_{2j}(t)$, for $j = 1, \ldots, k$. If $\lambda \in \Lambda_\alpha^s$, according to Lemma 9(i), we see that all solutions of this system are bounded which implies that all solutions of the system (17) are bounded, too. In the case $\lambda \in \Lambda_\alpha^u$, from Lemma 9(ii), this system has exactly one bounded solution $u(t) = (u_1(t), \ldots, u_k)^{\mathrm{T}}$. Hence, the system (17) has exactly one bounded solution $\varphi(t) = (\varphi_1(t), \ldots, \varphi_{2k}(t))^{\mathrm{T}}$, where for all $t \geq 0$,

$$\varphi_{2j-1}(t) = \Re u_j(t), \quad j = 1, \ldots, k,$$

and

$$\varphi_{2j}(t) = \Im u_j(t), \quad j = 1, \ldots, k.$$

□

We now prove the sufficient part of Theorem 5.

Proof of Proposition 6. Due to the fact that the system (7) has at least one bounded solution for any bounded continuous function f if and only if the system (9) has at least one bounded solution for any bounded continuous function g. We now focus on the system (9). Applying Lemma 9(ii) and Corollary 10(ii) to each Jordan block from B^u, we can find exactly one bounded solution $u(t)$ of the equation (the unstable part of the equation (9))

$$^{C}D^{\alpha}_{0+}y^u(t) = B^u y^u(t) + g^u(t);$$

and applying Lemma 9(i) and Corollary 10(i) to each Jordan block from B^s, we see that all solutions of the equation (the stable part of the equation (9))

$$^{C}D^{\alpha}_{0+}y^s(t) = B^s y^s(t) + g^s(t), \qquad y^s(0) = y^s_0,$$

are bounded. Put $\varphi(t) := (u(t), v(t))^{\mathrm{T}}$ for any $t \geq 0$, where $v(t)$ is an arbitrary solution of (9). Then, this function is a bounded solution of the system (9). The proof is complete. $\square$

Finally, we prove the necessary condition in the statement of Theorem 5.

Proposition 11 (Necessary part of Theorem 5)**.** Consider the system

$$^{C}D^{\alpha}_{0+}x(t) = Ax(t) + f(t), \tag{22}$$

where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$. Assume that for any bounded continuous function f, the system (22) has at least one bounded solution. Then, the matrix A has to satisfy the hyperbolic condition (2):

$$\sigma(A) \cap \left\{\lambda \in \mathbb{C} : \lambda = 0 \text{ or } |\arg(\lambda)| = \frac{\alpha\pi}{2}\right\} = \emptyset.$$

Before going to the proof of this theorem, we need the following technical proposition.

Lemma 12. Consider the equation

$$^{C}D^{\alpha}_{0+}x(t) = \lambda x(t) + f(t), \tag{23}$$

where $\lambda = 0$ or $\arg(\lambda) = \frac{\alpha\pi}{2}$ and $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$. Then, there exists a bounded continuous function f such that every solutions of this equation are unbounded.

Proof. First, we consider the case $\lambda = 0$. In this case, we choose $f(t) = \Gamma(1+\alpha)$. It is obvious that for any $x_0 \in \mathbb{R}$, the equation (23) with the initial condition $x(0) = x_0$, has a unique solution as

$$\varphi(t, x_0) = x_0 + t^\alpha, \quad \forall t \geq 0.$$

This solution is unbounded for any $x_0 \in R$.

In the case $\arg(\lambda) = \frac{\alpha\pi}{2}$, we write λ in the form $\lambda = r(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2})$ and choose $f(t) = \exp(ir^{\frac{1}{\alpha}}t)$. By Theorem 1, the solution $\varphi(\cdot, x_0)$ of (23) starting from x_0 satisfies

$$\varphi(t, x_0) = E_\alpha(\lambda t^\alpha)\, x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) \exp(ir^{\frac{1}{\alpha}}\tau)\, d\tau.$$

We claim that this solution is unbounded. Indeed, the quantity $x_0 E_\alpha(\lambda t^\alpha)$ is bounded due to [Pod99, Theorem 1.1, p. 30], while the quantity

$$\int_{t-1}^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)\, d\tau$$

is bounded by the following estimate

$$\Big|\int_{t-1}^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)\, d\tau\Big| \leq \int_0^1 s^{\alpha-1} |E_{\alpha,\alpha}(\lambda s^\alpha)|\, ds.$$

Furthermore, using [Pod99, Theorem 1.1, p. 30], we show that the quantity

$$\int_0^{t-1} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) \exp(ir^{\frac{1}{\alpha}}\tau)\, d\tau$$

is unbounded. Indeed, for $\theta \in (\frac{\alpha\pi}{2}, \alpha\pi)$ is arbitrary but fixed and $\varepsilon \in (0, \frac{|\lambda|}{2})$ satisfies

$$|\lambda| t^\alpha - \varepsilon \geq |\lambda| t^\alpha \sin(\theta - \frac{\alpha\pi}{2}) \tag{24}$$

for all $t \geq 1$, we denote by $\gamma(\varepsilon, \theta)$ the contour consisting of the following three parts

(i) $\arg(z) = -\theta$, $|z| \geq \varepsilon$;

(ii) $-\theta \leq \arg(z) \leq \theta$, $|z| = \varepsilon$;

(iii) $\arg(z) = \theta$, $|z| \geq \varepsilon$.

The contour $\gamma(\varepsilon, \theta)$ divides the complex plane (z) into two domains, which we denote by $G^-(\varepsilon, \theta)$ and $G^+(\varepsilon, \theta)$. These domains lie correspondingly on the left and on the right side of the contour $\gamma(\varepsilon,\theta)$. According to [Pod99, Theorem 1.1, p. 30], we have

$$\begin{aligned}
&\int_0^{t-1} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) \exp\left(ir^{\frac{1}{\alpha}}\tau\right) d\tau \\
&= \int_0^{t-1} (t-\tau)^{\alpha-1} \frac{1}{\alpha} \lambda^{\frac{1-\alpha}{\alpha}} (t-\tau)^{1-\alpha} \exp(\lambda^{\frac{1}{\alpha}}(t-\tau)) \exp(ir^{\frac{1}{\alpha}}\tau)\, d\tau \\
&\qquad\qquad + \int_0^{t-1} (t-\tau)^{\alpha-1} \frac{1}{2\alpha\pi i} \int_{\gamma(\varepsilon,\theta)} \frac{\exp(\xi^{\frac{1}{\alpha}})\xi^{\frac{1-\alpha}{\alpha}}}{\xi-\lambda(t-\tau)^\alpha}\, d\xi \exp(ir^{\frac{1}{\alpha}}\tau)\, d\tau \\
&= I_4(t) + I_5(t).
\end{aligned}$$

Clearly, we see that

$$\begin{aligned}
I_4(t) &= \int_0^{t-1} (t-\tau)^{\alpha-1} \frac{1}{\alpha} \lambda^{\frac{1-\alpha}{\alpha}} (t-\tau)^{1-\alpha} \exp(\lambda^{\frac{1}{\alpha}}(t-\tau)) \exp(ir^{\frac{1}{\alpha}}\tau)\, d\tau \\
&= \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}(t-1)\exp(\lambda^{\frac{1}{\alpha}} t). \qquad (25)
\end{aligned}$$

On the other hand, due to (24), we obtain

$$\begin{aligned}
|I_5(t)| &\le \frac{\int_{\gamma(\varepsilon,\theta)} |\exp(\xi^{\frac{1}{\alpha}})\xi^{\frac{1-\alpha}{\alpha}}|\, d\xi}{2\alpha\pi \sin(\theta-\frac{\alpha\pi}{2})} \int_0^{t-1} \frac{(t-\tau)^{\alpha-1}}{|\lambda|(t-\tau)^\alpha}\, d\tau \\
&\le \frac{\int_{\gamma(\varepsilon,\theta)} |\exp(\xi^{\frac{1}{\alpha}})\xi^{\frac{1-\alpha}{\alpha}}|\, d\xi}{2\alpha\pi|\lambda| \sin(\theta-\frac{\alpha\pi}{2})} \log t. \qquad (26)
\end{aligned}$$

From (25) and (26), this implies that the quantity

$$\int_0^{t-1} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) \exp\left(ir^{\frac{1}{\alpha}}\tau\right) d\tau$$

is unbounded. So, the solution $\varphi(\cdot, x_0)$ is unbounded for any $x_0 \in \mathbb{C}$. The proof is complete. □

Corollary 13. Assume that $\lambda = a + ib$ is a complex number satisfying $\arg(\lambda) = \frac{\alpha\pi}{2}$. Consider the fractional differential equation

$$^{C}D^{\alpha}_{0+}x_1(t) = ax_1(t) - bx_2(t) + f_1(t), \qquad (27)$$

$$^{C}D^{\alpha}_{0+}x_2(t) = bx_1(t) + ax_2(t) + f_2(t). \qquad (28)$$

Then, we can find $f_1, f_2 \in C_b(\mathbb{R}_{\ge 0}; \mathbb{R})$ such that all solutions of this system are unbounded.

Proof. Consider the equation

$$^{C}D^{\alpha}_{0+}u(t) = \lambda u(t) + f(t), \quad t > 0,$$

where $f(t) = f_1(t) + if_2(t)$. From the proof of Lemma 12, choosing the function f as $f(t) = \exp(ir^{\frac{1}{\alpha}}t)$ (where $r = \sqrt{a^2+b^2}$), we see that all solutions of this equation are unbounded. Hence, by choosing $f_1(t) = \cos(r^{\frac{1}{\alpha}}t)$ and $f_2(t) = \sin(r^{\frac{1}{\alpha}}t)$, all solutions of the system (27)-(28) are unbounded. The proof is complete. □

We are now in a position to prove Proposition 11.

Proof of Proposition 11. First, we consider the case $0 \in \sigma(A)$. Without loss of generality (transform A to the Jordan form and change the order of coordinates if necessary), we can write the matrix A in the equation (22) in the form

$$A = \begin{pmatrix} \hat{A} & A_1 \\ 0 & 0 \end{pmatrix}.$$

Choosing $f = (\hat{f}(t), \Gamma(1+\alpha))^{\mathrm{T}}$ with $\hat{f} \in C_b(\mathbb{R}_{\geq 0}; \mathbb{R}^{d-1})$. Due to Lemma 12, the last coordinate of any solution of (22) is unbounded. Hence, all solutions of (22) are unbounded.

Next, in the case the spectrum $\sigma(A)$ has at least one eigenvalue λ such that $\arg(\lambda) = \frac{\alpha\pi}{2}$, we can assume that

$$A = \begin{pmatrix} \hat{A} & A_2 \\ 0 & D \end{pmatrix}$$

where

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

with $a = \Re\lambda$, $b = \Im\lambda$. Due to Corollary 13, by choosing the function $f = (\hat{f}, f_1, f_2)^{\mathrm{T}}$, where $\hat{f} \in C_b(\mathbb{R}_{\geq 0}; \mathbb{R}^{d-2})$, and

$$f_1(t) = \cos(r^{\frac{1}{\alpha}}t), \quad f_2(t) = \sin(r^{\frac{1}{\alpha}}t),$$

with $r = |\lambda|$, we see that at least one of the last two coordinates of any solution of (22) is unbounded. Hence, all solutions of (22) are unbounded. The proof is complete. □

Proof of Theorem 5. The proof of Theorem 5 follows directly from Proposition 6 and Proposition 11. □

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