Existence of global strong solutions to the Navier-Stokes equations with large input data

B. T. Kien* and N. H. Son†

Abstract. This paper deals with the existence of global strong solutions to the three-dimentional Navier-Stokes equations with large initial data. We show that if $\Omega \subset \mathbb{R}^3$ is the interior of a torus and the input data are axially symmetric vector fields then the Navier-Stokes equations have a unique global strong solution on $(0, \infty)$. Here, we do not require that the swirls of the data are zero. The obtained result is proved without any requirement on size of the data.
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1 Introduction

In this paper we are going to study the Navier-Stokes equations, where the incompressible fluid fills the domain Ω:

$$\text{(NSE)} \qquad \begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla p = f(t,x) & (t,x) \in (0,\infty) \times \Omega, \\ \text{div } v = 0 & (t,x) \in [0,\infty) \times \Omega, \\ v = 0 & (t,x) \in [0,\infty) \times \partial\Omega, \\ v(0,x) = v_0(x) & x \in \Omega, \end{cases}$$

where $v_0 : \Omega \to \mathbb{R}^3$ is a divergence-free vector field, that is, $\text{div} v_0 = 0$ and Ω is the interior of a torus which is defined by

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | (R_2 - \sqrt{x_1^2 + x_2^2})^2 + x_3^2 < R_1^2\} \tag{1}$$

and its boundary is the torus

$$\partial\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | (R_2 - \sqrt{x_1^2 + x_2^2})^2 + x_3^2 = R_1^2\}. \tag{2}$$

Here R_1 and R_2 are positive radiuses which satisfy the condition

$$R_2 > 3R_1. \tag{3}$$

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So far there have been a lot of books and articles on the mathematical theory of the Navier-Stokes equations (NSEs for short). The classical results could be found in the books of the authors R. Temam [31], [32], O.A. Ladyzhenskaya [21], P. Constantin and C. Foias [7]. The modern references can be found in the recent books of H. Bahouri et al. [2], M. Cannone [4] and P. G. L. Rieusset [29].

It is known that, under smallness condition of v_0 and f, the Navier-Stokes equations have a unique global strong/smooth solution (see, for instance [1], [13], [14], [17], [18], [26]). However, proving the existence of smooth/strong solutions to the Navier-Stokes equations without smallness conditions on the initial data has been a challenge for mathematicians so far (see [6]).

Although it was shown that in the case $\Omega = \mathbb{R}^3$, there are some models for which finite time blow-up of solutions can be proved for some classes of large data (see [12], [27] and [33]), we believe that there are some classes of large input data under which the Navier-Stokes equations have global strong/smooth solutions. In fact, in [16] and [28] the authors showed that when the initial data have a large two-dimension part and a small three-dimension part, then the equations have a unique global in time solution. In recent paper of J. Y. Chemin et al [8], the authors considered the Navier-Stokes equations for the case of initial data of the form

$$v_0(x_h, x_3) := (v^1(x_h, \epsilon x_3), v^2(x_h, \epsilon x_3), \frac{1}{\epsilon} v^3(x_h, \epsilon x_3)),$$

where x_h belongs to torus $\mathbb{T}^2$ and $x_3 \in \mathbb{R}$. They showed that under a smalless condition of v_0 the Navier-Stokes equations have a global smooth solution in this case. Note that such initial data may be arbitrarily large in the norm of $\dot{B}^{-1}_{\infty,\infty}$.

In general the existence of global strong solutions of the NSEs with large input data is guaranteed whenever the data have good structures. In 1968, Ukhovskii and Iudovich [35] and Ladyzhenskaya [22] (see also [23]) showed that if v_0 and f are axially symmetric with zero swirls, then the Navier-Stokes equations have a unique global strong solution on $(0, T)$ with $T < +\infty$. The obtained results in [35] and [22] were proved without any size requirement on the data. In the same direction, Mahalov et al. [25] studied the existence of global solution for the Navier-Stokes equations under requirement that the input data v_0 and f are helical symmetric. Unfortunately, the obtained results in [25] were based on key lemma but its proof is incorrect (see Lemma 3.1 in [25] and its proof).

From the above one may ask whether there is a class of large input data and some bounded domains Ω under which the Navier-Stokes equations have a unique global strong solution on $(0, \infty)$. The aim of this paper is to address this question. Note that the existence of global strong solutions to the NSEs plays an important role not only in the theory of partial differential equations but also in optimal control problems. Based on the existence of global strong solutions of NSEs, we can establish the Pontryagin maximum principle for optimal control problems governed by NSEs (see for instance [19]).

Recall that any vector field $v(x)$ in $\mathbb{R}^3$ can represent in the cylindrical coordinate under the form

$$v = v_r(r, x_3, \theta)e_r + v_\theta(r, x_3, \theta)e_\theta + v_z(r, x_3, \theta)e_z,$$

where $r = \sqrt{x_1^2 + x_2^2}$ and

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \ e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \ e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The component v_θ is called the swirl of v. When v_r, v_θ, v_z do not depend on θ, we say that v is *axially symmetric*.

In this paper we continue to develop the obtained results in [35], [22] and [23] by considering the Cauchy problem for the Navier-Stokes equations (NSE) in the bounded domain Ω which is the interior of a torus and assume that the input data are axially symmetric and *their swirls are not necessary to equal zero*. Under this assumptions, we show that the NSEs have a unique global strong solution on $(0, \infty)$ without any requirement on size of the input data. It is worth pointing out that the technique for the proof of our result is different from mentioned papers. In [35], [22] and [23], the authors used curl operator in order to transform the NSEs into vorticity equations. This method bases on regularity of solution of NSEs heavily. Here, we give a direct proof by using the energy method as in [7] and exploiting the structures of the data. In some sense, our obtained result is an extension of preceding results for the case where the swirls of data are nonzero.

Let us define

$$\begin{aligned}
&V = \{u \in H_0^1(\Omega)^3 \mid \text{div } u = 0\},\\
&V_0 = \{u \in V \mid u \text{ is axially symmetric}\},\\
&H = \{u \in L^2(\Omega)^3 \mid \text{div } u = 0, u \cdot \overrightarrow{n} = 0 \text{ on } \partial\Omega\},\\
&\overrightarrow{n} \text{ is the unit outward normal vector on } \partial\Omega,\\
&V_{\text{as}} = \text{the closure of } H^2(\Omega)^3 \cap V_0 \text{ in } V,\\
&H_{\text{as}} = \text{the closure of } H^2(\Omega)^3 \cap V_0 \text{ in } H.
\end{aligned}$$

It is know that H is a Hilbert space with the scalar product $(\cdot, \cdot)$ and norm $|\cdot|$ which are induced by the scalar product and norm in $L^2(\Omega)^3$. Also, V is a Hilbert space with the scalar product $((\cdot, \cdot))$ and norm $\|\cdot\|$ which are induced by the scalar product and norm in $H_0^1(\Omega)^3$, where

$$((v, w)) = \sum_{i=1}^{3}(D_i v, D_i w), \forall v, w \in V.$$

We are ready to state our main result.

Theorem 1.1 *Suppose that Ω satisfies condition (3), $v_0 \in V_{\rm as}$ and $f \in L^2((0,\infty);H_{\rm as}) \cap L^1((0,\infty);H_{\rm as})$. Then the Navier-Stokes equations (NSE) have a unique global strong solution v which is axially symmetric and*

$$v \in L^\infty((0,\infty);V_{\rm as}) \cap L^2((0,\infty);H^2(\Omega)^3),\ \frac{dv}{dt} \in L^2((0,\infty);H_{\rm as}).$$

Moreover, the following energy inequalities are valid:

$$|v(t)|^2 \le (|v_0|^2 + \int_0^\infty |f(s)|ds) \exp\big(\int_0^\infty |f(s)|ds\big)\ \forall t \ge 0 \tag{4}$$

and

$$\int_0^\infty |\nabla v(s)|^2 ds \le \frac{1}{2}|v_0|^2 + \big(|v_0|^2 + \int_0^\infty |f(s)|ds\big)^{1/2} \exp\big(\frac{1}{2}\int_0^\infty |f(s)|ds\big)\int_0^\infty |f(s)|ds. \tag{5}$$

Remark 1.1 *In the case of finite interval $[0,T_0]$ with $0 < T_0 < +\infty$, if we require that $v_0 \in V_{\rm as}$ and $f \in L^2((0,T_0),H_{\rm as})$, then the NSEs have a unique global strong solution v on $(0,T_0)$ with*

$$v \in L^\infty((0,T_0);V_{\rm as}) \cap L^2((0,T_0);H^2(\Omega)^3),\ \frac{dv}{dt} \in L^2((0,T_0);H_{\rm as}).$$

Let us give an illustrative example where the initial datum satisfies assumptions of Theorem 1.1.

Example 1.1 We consider the vector field $v_0 = (v_{01}, v_{02}, v_{03})$, where

$$\begin{aligned} v_{01} &= \big[(R_2 - \sqrt{x_1^2+x_2^2})^2 + x_3^2 - R_1^2\big]\frac{x_3(x_1-x_2)}{x_1^2+x_2^2},\\ v_{02} &= \big[(R_2 - \sqrt{x_1^2+x_2^2})^2 + x_3^2 - R_1^2\big]\frac{x_3(x_1+x_2)}{x_1^2+x_2^2},\\ v_{03} &= \frac{1}{2}\big[(R_2 - \sqrt{x_1^2+x_2^2})^2 + x_3^2 - R_1^2\big]\frac{R_2 - \sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2}}. \end{aligned}$$

It is obvious that $v_0|_{\partial\Omega} = 0$, ${\rm div} v_0 = 0$ and v_0 is of class $C^\infty(\Omega)^3$. In the cylindrical coordinate: $x_1 = r\cos\theta, x_2 = r\sin\theta$ and $x_3 = z$, we have

$$\begin{aligned} v_{01} &= \big[(R_2 - r)^2 + z^2 - R_1^2\big]\frac{z}{r}(\cos\theta - \sin\theta),\\ v_{02} &= \big[(R_2 - r)^2 + z^2 - R_1^2\big]\frac{z}{r}(\sin\theta + \cos\theta),\\ v_{03} &= \frac{1}{2}\big[(R_2 - r)^2 + z^2 - R_1^2\big]\frac{R_2 - r}{r}. \end{aligned}$$

Hence, $v_0 = v_r e_r + v_\theta e_\theta + v_z e_z$ with

$$\begin{aligned} v_r = v_\theta &= \big[(R_2 - r)^2 + z^2 - R_1^2\big]\frac{z}{r},\\ v_z &= \frac{1}{2}\big[(R_2 - r)^2 + z^2 - R_1^2\big]\frac{R_2 - r}{r}. \end{aligned}$$

Since v_r, v_θ and v_z do not depend on θ, v_0 is axially symmetric and so $v_0 \in C^\infty(\Omega)^3 \cap V_0$.

The proof of Theorem 1.1 is provided in Section 3. In order to prove the main result we need to establish some auxiliary results which are given in Section 2 bellow.

2 Some auxiliary results

Hereafter we shall use the following function spaces:

$$
\begin{aligned}
&\mathbf{L}^2(\Omega) := (L^2(\Omega))^3 = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega),\\
&\mathbf{H}^1_0(\Omega) := (H^1_0(\Omega))^3,\ \mathbf{H}^m(\Omega) := H^m(\Omega))^3 \text{ with norm } \|\cdot\|_m,\\
&\mathcal{V} = \{y \in \mathcal{D}(\Omega)^3 \mid \operatorname{div} y = 0\},\\
&V = \text{the closure of } \mathcal{V} \text{ in } \mathbf{H}^1_0(\Omega) = \{y \in \mathbf{H}^1_0(\Omega) \mid \operatorname{div} y = 0\},\\
&H = \text{the closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega) = \{y \in \mathbf{L}^2(\Omega) \mid \operatorname{div} y = 0, y \cdot \overrightarrow{n} = 0 \text{ on } \partial\Omega\},\\
&H^\perp = \{\phi \in \mathbf{L}^2(\Omega) \mid \phi = \nabla p, p \in H^1(\Omega)\},\\
&W = V \cap \mathbf{H}^2(\Omega),\\
&W^{1,2}(0,T;E_1,E_2) = \{v \in L^2(0,T;E_1) \mid \frac{dv}{dt} \in L^2(0,T;E_2)\},
\end{aligned}
$$

where E_1, E_2 are Banach spaces.

For convenience, we shall denote by $\langle \cdot,\cdot \rangle_e$ and $|\cdot|_e$ the scalar product and the Euclid norm in $\mathbb{R}^n$ with $n = 2,3$, respectively. It is well known that the imbeddings

$$W \hookrightarrow V \hookrightarrow H$$

are compact and each space is dense in the following one.

Let us denote by $\mathbb{P} : \mathbf{L}^2(\Omega) \to H$ the Leray projection on H. We then define the Stokes operator $A : W \to H$ by setting $A = -\mathbb{P}\Delta$ and mappings B, $\mathbf{b}$ which are given by

$$
\begin{aligned}
B(u,u) &= \mathbb{P}(u \cdot \nabla)u,\\
\mathbf{b}(u,v,w) &= (B(u,v), w)
\end{aligned}
$$

for $u, v \in \mathbf{H}^1_0(\Omega)$ and $w \in \mathbf{L}^2(\Omega)$.

Note that via transformation $x_1 = r\cos\theta, x_2 = r\sin\theta, x_3 = z$, the domain Ω is transformed into

$$G \times (-\pi,\pi] = \{(r,z) \in \mathbb{R}^2 \mid (R_2 - r)^2 + z^2 < R_1^2\} \times (-\pi,\pi] \tag{6}$$

with

$$\partial G = \{(r,z) \in \mathbb{R}^2 \mid (R_2 - r)^2 + z^2 = R_1^2\}.$$

Proposition 2.1 *If $f \in H_{\text{as}}$ then f is axially symmetric.*

Proof. By definition of H_{as}, there exists a sequence $f_n \in \mathbf{H}^2(\Omega) \cap V_0$ such that f_n converges to f strongly in $\mathbf{L}^2(\Omega)$. In the cylindrical coordinate, we can can present f in the form:

$$\begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix} = f^r \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + f^\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} + f^z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $f^r = f^r(r,z,\theta)$, $f^\theta = f^\theta(r,z,\theta)$ and $f^z = f^z(r,z,\theta)$.

Since $f_n = (f_n^1, f_n^2, f_n^3)$ is axially symmetric, it has the presentation:

$$\begin{pmatrix} f_n^1 \\ f_n^2 \\ f_n^3 \end{pmatrix} = f_n^r \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + f_n^\theta \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} + f_n^z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $f_n^r = f_n^r(r,z)$, $f_n^\theta = f_n^\theta(r,z)$ and $f^z = f_n^z(r,z)$. Since

$$|f_n - f|_e^2 = |f_n^1 - f^1|^2 + |f_n^2 - f^1|^2 + |f_n^3 - f^3|^2 = |f_n^r - f^r|^2 + |f_n^\theta - f^\theta|^2 + |f_n^z - f^z|^2,$$

we see that

$$\|f_n - f\|_{\mathbf{L}^2(\Omega)}^2 = \sum_{j\in\{r,\theta,z\}} \int_\Omega |f_n^j - f^j|^2 dx = \sum_{j\in\{r,\theta,z\}} \int_{G\times(-\pi,\pi)} |f_n^j(r,z) - f^j(r,z,\theta)|^2 r dr dz d\theta. \tag{7}$$

Hence for each $j \in \{r,\theta,z\}$, we have

$$\int_{G\times(-\pi,\pi)} |f_n^j(r,z) - f^j(r,z,\theta)|^2 r dr dz d\theta \to 0 \text{ as } n \to \infty.$$

Since $r > R_2 - R_1 > 0$, we obtain

$$\int_{G\times(-\pi,\pi)} |f_n^j(r,z) - f^j(r,z,\theta)|^2 dr dz d\theta \to 0 \text{ as } n \to \infty. \tag{8}$$

This implies that for a.e. $\theta \in (-\pi,\pi)$, we have

$$\int_G |f_n^j(r,z) - f^j(r,z,\theta)|^2 dr dz \to 0 \text{ as } n \to \infty. \tag{9}$$

From (8), we have

$$\int_{G\times(-\pi,\pi)} |f_n^j(r,z)|^2 dr dz d\theta$$

is bounded and so is

$$\int_G |f_n^j(r,z)|^2 dr dz.$$

Hence we can assume that $f_n^j(r,z)$ converges weakly to a function $f_0^j(r,z)$ in $L^2(G)$. On the other hand form (9), we have $f_n^j(r,z)$ converges strongly to a function $f^j(r,z,\theta)$ in $L^2(G)$. Consequently, we must have $f_0^j(r,z) = f^j(r,z,\theta)$ for a.e. $\theta \in (-\pi,\pi)$. Hence f is axially symmetric. The proof is complete. □

Proposition 2.2 *Let $h \in H_{\text{as}}$. Then there exists a unique $u \in \mathbf{H}^2(\Omega) \cap V_0$ and $p \in H^1(\Omega)$ which solve the Stokes system:*

$$\begin{cases} -\Delta u + \nabla p = h & \text{in } \Omega, \\ \text{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Proof. By Theorem 3.11 in [7], the Stokes system has a unique solution $u \in H^2(\Omega) \cap V$ and $p \in H^1(\Omega)$ such that

$$\|u\|_{H^2(\Omega)^3} + \|p\|_{H^1(\Omega)} \le C|h| \tag{11}$$

for some absolute constant $C > 0$. By applying the Leray projection on the first equation of (10), we obtain

$$Au = h, \ \ \|u\|_{H^2(\Omega)^3} \le C|Au|. \tag{12}$$

It remains to show that u is axially symmetric.

Since h is axially symmetric, $h = h_r e_r + h_\theta e_\theta + h_z e_z$, where $h_r = h_r(r, z)$, $h_\theta = h_\theta(r, z)$ and $h_z = h_z(r, z)$. We want to find u in the form $u = u_r e_r + u_\theta e_\theta + u_z e_z$, where u_r, u_θ and u_z depend only on (r, z) and $p = p(r, z)$. Then in the cylindrical coordinates the system (10) is transformed into the system

$$-\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_r}{\partial r}\Big) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\Big) + \frac{\partial p}{\partial r} = h_r \text{ in } G, \tag{13}$$

$$-\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_z}{\partial r}\Big) + \frac{\partial^2 u_z}{\partial z^2}\Big) + \frac{\partial p}{\partial z} = h_z \text{ in } G, \tag{14}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \text{ in } G \tag{15}$$

$$-\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_\theta}{\partial r}\Big) + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2}\Big) = h_\theta \text{ in } G, \tag{16}$$

$$(u_r, u_\theta, u_z) = (0, 0, 0) \text{ on } \partial G. \tag{17}$$

Recall that

$$G = \{(r, z) \in \mathbb{R}^2 \mid (R_2 - r)^2 + z^2 < R_1^2\},$$
$$\partial G = \{(r, z) \in \mathbb{R}^2 \mid (R_2 - r)^2 + z^2 = R_1^2\}, \ R_2 > 3R_1.$$

Define $\tilde{v} = (v_r, v_z) = r(u_r, u_z)$. Then the equations (13), (14), (15) and (17) become

$$\begin{cases} -\frac{1}{r}\Delta_{(r,z)} v_r + \frac{1}{r^2}\partial_r v_r + \partial_r p = h_r \\ -\frac{1}{r}\Delta_{(r,z)} v_z + \frac{1}{r^2}\partial_r v_z - \frac{v_z}{r^3} + \partial_z p = h_z \\ \text{div}\tilde{v} = 0, \\ \tilde{v}|_{\partial G} = 0. \end{cases} \tag{18}$$

Meanwhile, equations (16) and (17) become

$$\begin{cases} -\Delta_{(r,z)} v_\theta + \frac{1}{r}\partial_r v_\theta = r h_\theta, \\ v_\theta = 0 \quad \text{on} \quad \partial G. \end{cases} \tag{19}$$

Hereafter, $\Delta_{(r,z)}$ and $\nabla_{(r,z)}$ are defined by

$$\Delta_{(r,z)} w = D_r^2 w + D_z^2 w, \ \nabla_{(r,z)} w = (D_r w, D_z w) \ \text{ for } w = w(r, z).$$

Let us denote by $V_2 = \{v \in H^1_0(G)^2 \mid \mathrm{div} v = 0\}$. It is known that V_2 is the closure of $\mathcal{V}_2$ in $H^1_0(G)^2$ with

$$\mathcal{V}_2 = \{w \in C^\infty_0(G)^2 \mid \mathrm{div} w = 0\}.$$

Recall that a vector $\tilde{v} = (v_r, v_z) \in V_2$ is said to be weak solution of (18) if

$$\begin{aligned}&(-\frac{1}{r}\Delta_{(r,z)}v_r + \frac{1}{r^2}\partial_r v_r + \partial_r p, w_r) + (-\frac{1}{r}\Delta_{(r,z)}v_z + \frac{1}{r^2}\partial_r v_z - \frac{v_z}{r^3} + \partial_z p, w_z)\\ &= (w_r, h_r) + (w_z, h_z)\ \forall \tilde{w} = (w_r, w_z) \in V_2.\end{aligned}$$

This variational formulation is equivalent to

$$(-\frac{1}{r}\Delta_{(r,z)}v_r + \frac{1}{r^2}\partial_r v_r, w_r) + (-\frac{1}{r}\Delta_{(r,z)}v_z + \frac{1}{r^2}\partial_r v_z - \frac{v_z}{r^3}, w_z) = (w_r, h_r) + (w_z, h_z)$$

for all $\tilde{w} = (w_r, w_z) \in V_2$ or equivalently,

$$\begin{aligned}&\int_G \frac{1}{r}\langle\nabla_{(r,z)}v_r, \nabla_{(r,z)}w_r\rangle_e drdz + \int_G \frac{1}{r}\langle\nabla_{(r,z)}v_z, \nabla_{(r,z)}w_z\rangle_e drdz - \int_G \frac{1}{r^3}v_z w_z drdz\\ &= \int_G (h_r w_r + h_z w_z) drdz\ \forall (w_r, w_z) \in V_2.\end{aligned}$$

Let us define a bilinear mapping $T : V_2 \times V_2 \to \mathbb{R}$ by setting

$$T(\tilde{v}, \tilde{w}) = \int_G \frac{1}{r}\langle\nabla_{(r,z)}v_r, \nabla_{(r,z)}w_r\rangle_e drdz + \int_G \frac{1}{r}\langle\nabla_{(r,z)}v_z, \nabla_{(r,z)}w_z\rangle_e drdz - \int_G \frac{1}{r^3}v_z w_z drdz$$

for $\tilde{v} = (v_r, v_z), \tilde{w} = (w_r, w_z) \in V_2$. We now show that T is continuous and coercive. In fact, using the fact $R_2 - R_1 \le r \le R_2 + R_1$, we have

$$\begin{aligned}|T(\tilde{v}, \tilde{w})| &\le \frac{1}{R_2 - R_1}(|\nabla_{(r,z)}\tilde{v}||\nabla_{(r,z)}\tilde{w}|) + \frac{1}{(R_2 - R_1)^3}|\tilde{v}||\tilde{w}|\\ &\le C\|\tilde{v}\|\|\tilde{w}\|\ \forall \tilde{v}, \tilde{w} \in V_2.\end{aligned}$$

Taking $\tilde{w} = \tilde{v}$, we have

$$\begin{aligned}T(\tilde{v}, \tilde{v}) &= \int_G \frac{1}{r}|\nabla_{(r,z)}\tilde{v}|^2 drdz - \int_G \frac{1}{r^3}|v_z|^2 drdz\\ &\ge \frac{1}{R_2 + R_1}|\nabla_{(r,z)}\tilde{v}|^2 - \frac{1}{(R_2 - R_1)^3}|\tilde{v}|^2.\end{aligned} \quad (20)$$

Note that

$$G \subset \{(r, z) \in \mathbb{R}^2 : |\langle(r, z), (0, 1)\rangle_e| \le R_1\}.$$

Therefore, from the Poincaré inequality (see [9, Theorem 2.8]), we have

$$|\tilde{v}|^2 \le 2R_1^2|\nabla_{(r,z)}\tilde{v}|^2. \quad (21)$$

From this and (20), we get

$$\begin{aligned}T(\tilde{v}, \tilde{v}) &\ge \frac{1}{R_2 + R_1}|\nabla_{(r,z)}\tilde{v}|^2 - \frac{2R_1^2}{(R_2 - R_1)^3}|\nabla_{(r,z)}\tilde{v}|^2\\ &\ge (\frac{1}{R_2 + R_1} - \frac{2R_1^2}{(R_2 - R_1)^3})|\nabla_{(r,z)}\tilde{v}|^2.\end{aligned} \quad (22)$$

Since $R_2 > 3R_1$, we have $\alpha := \dfrac{1}{R_2+R_1} - \dfrac{2R_1^2}{(R_2-R_1)^3} > 0$. Hence

$$T(\tilde{v},\tilde{v}) \geq \alpha|\nabla_{(r,z)}\tilde{v}|^2 \ \forall \tilde{v} \in V_2.$$

By the Stampacchia theorem (see [3, Theorem 5.6]), there exists a unique $\tilde{v} = (v_r, v_z) \in V_2$ such that

$$T(\tilde{v},\tilde{w}) = (h,\tilde{w}) \ \forall \tilde{w} \in V_2.$$

Hence there exists $p \in L^2(G)$ such that $(\tilde{v}, p)$ is a weak solution of (18). Note that system (18) is equivalent to

$$\begin{cases} -\Delta_{(r,z)}v_r + \frac{1}{r}\partial_r v_r + r\partial_r p = rh_r, \\ -\Delta_{(r,z)}v_z + \frac{1}{r}\partial_r v_z - \frac{v_z}{r^2} + r\partial_z p = rh_z, \\ \mathrm{div}\tilde{v} = 0, \\ \tilde{v}|_{\partial G} = 0. \end{cases}$$

Since the function $\phi(r) = r$ is of $C^\infty(G)$, we have $r\partial_r p = \partial_r(rp) - p$ in the sense of distribution. Hence the above system can be written in the form

$$\begin{cases} -\Delta_{(r,z)}v_r + \partial_r(rp) = rh_r + p - \frac{1}{r}\partial_r v_r, \\ -\Delta_{(r,z)}v_z + \partial_z(rp) = rh_z - \frac{1}{r}\partial_r v_z + \frac{v_z}{r^2}, \\ \mathrm{div}\tilde{v} = 0, \\ \tilde{v}|_{\partial G} = 0, \end{cases}$$

where the terms on the right hand sides of the first equation and the second equation are of $L^2(G)$. By results on regularity of solutions to the Navier-Stokes equations, we see that $(v_r, v_z) \in H^2(G)^2 \cap V_2$ and $rp \in H^1(G)$ and so $p \in H^1(G)$. Also, by simple arguments, wee see that the elliptic equation (19) has a unique solution $v_\theta \in H^2(G) \cap H_0^1(G)$. We now define $\hat{u} = \frac{1}{r}(v_r, v_\theta, v_z)$. Then $(\hat{u}, p)$ satisfies equations (13)–(17). Define

$$\vartheta = \hat{u}_r e_r + \hat{u}_\theta e_\theta + \hat{u}_z e_z, \ \tilde{p}(x_1,x_2,x_3) = p(r, x_3),$$

where $\hat{u}_r = \frac{1}{r}v_r$, $\hat{u}_\theta = \frac{1}{r}v_\theta$ and $\hat{u}_z = \frac{1}{r}v_z$. Then $(\vartheta, \tilde{p})$ is a solution of (10) with $\vartheta \in H^2(\Omega)^3 \cap V_0$, $\tilde{p} \in H^1(\Omega)$. By applying the Leray projection, we get $A\vartheta = h$. By the uniqueness, we obtain $u = \vartheta$. The proof is complete. $\square$

In the sequel we shall denote $W_{\mathrm{as}} = \mathbf{H}^2(\Omega) \cap V_0$ and define a mapping

$$\widetilde{A} : \mathcal{D}(\widetilde{A}) \subset H_{\mathrm{as}} \to H_{\mathrm{as}} \text{ with } \mathcal{D}(\widetilde{A}) = W_{\mathrm{as}}$$

by setting

$$\widetilde{A}u = Au \text{ for } u \in \mathcal{D}(\widetilde{A}).$$

Thus $\widetilde{A}$ is a restriction of the Stokes operator A on W_{as}. By the Proposition 2.2, we see that $\widetilde{A}$ is bijective and densely defined on H_{as}. The following proposition gives some properties of $\widetilde{A}$.

Proposition 2.3 *The following assertions are valid:*
(*a*) *The operator* $\widetilde{A} : \mathcal{D}(\widetilde{A}) \subset H_{\rm as} \to H_{\rm as}$ *is symmetric, i.e.,*

$$(\widetilde{A}u, v) = (u, \widetilde{A}v) \ \forall u, v \in \mathcal{D}(\widetilde{A}).$$

(*b*) *The inverse operator* $(\widetilde{A})^{-1}$ *is compact in* $H_{\rm as}$.
(*c*) *The operator* $\widetilde{A} : \mathcal{D}(\widetilde{A}) \subset H_{\rm as} \to H_{\rm as}$ *is self-adjoint.*
(*d*) *The inverse operator* $(\widetilde{A})^{-1}$ *is also self-adjoint.*
(*e*) *For all* $u \in \mathcal{D}(\widetilde{A})$ *and* $v \in V_{\rm as}$, *one has*

$$(\widetilde{A}u, v) = ((u, v)). \tag{23}$$

Proof. (*a*) From [7, Proposition 4.2], A is symmetric. Hence

$$(\widetilde{A}u, v) = (Au, v) = (u, Av) = (u, \widetilde{A}v) \ \forall u, v \in \mathcal{D}(\widetilde{A}).$$

(*b*) Since $\mathcal{D}(\widetilde{A}) \hookrightarrow H_{\rm as}$ is a compact embedding, the operator $(\widetilde{A})^{-1} : H_{\rm as} \to \mathcal{D}(\widetilde{A}) \subset H_{\rm as}$ is a compact operator in $H_{\rm as}$.
(*c*) Let us show that $\mathcal{D}(\widetilde{A}^*) \subset \mathcal{D}(\widetilde{A})$. Indeed, take any $u \in \mathcal{D}(\widetilde{A}^*)$. By definition, there exists $h \in H_{\rm as}$ such that $(\widetilde{A}v, u) = (v, h)$ for all $v \in \mathcal{D}(\widetilde{A})$. Since $h \in H_{\rm as}$ and by Proposition 2.2, we can find a vector $\tilde{u} \in \mathcal{D}(\widetilde{A})$ such that $\widetilde{A}\tilde{u} = h$. Taking any $g \in H_{\rm as}$ and using Proposition 2.2 again, we see that there exists $v \in \mathcal{D}(\widetilde{A})$ such that $\widetilde{A}v = g$. Therefore from (*a*), we have

$$(g, u - \tilde{u}) = (\widetilde{A}v, u) - (\widetilde{A}v, \tilde{u}) = (v, h) - (v, A\tilde{u}) = (v, h) - (v, h) = 0.$$

Since g is arbitrary in $H_{\rm as}$, we get $u = \tilde{u} \in \mathcal{D}(\widetilde{A})$. Consequently, $\mathcal{D}(\widetilde{A}) = \mathcal{D}(\widetilde{A}^*)$. By (*a*), for all $u, v \in \mathcal{D}(\widetilde{A})$, we have

$$(\widetilde{A}^*u, v) = (u, \widetilde{A}v) = (\widetilde{A}u, v).$$

Since $\mathcal{D}(\widetilde{A})$ is dense in $H_{\rm as}$, we obtain $\widetilde{A}u = \widetilde{A}^*u$.
(*d*) By Theorem 10.2.2 in [20], we have

$$(\widetilde{A}^{-1})^* = ((\widetilde{A})^*)^{-1} = \widetilde{A}^{-1}.$$

We obtain the conclusion.
(*e*) Since

$$(\mathbb{P}\phi, \psi) = (\phi, \mathbb{P}\psi) \ \forall \phi, \psi \in H,$$

we have

$$(\widetilde{A}u, v) = -(\Delta u, \mathbb{P}v) = -(\Delta u, v) = (\nabla u, \nabla v) = ((u, v)) \tag{24}$$

for all $u \in \mathcal{D}(\widetilde{A})$ and $v \in V_{\rm as}$. □

From Proposition 2.3, $(\widetilde{A})^{-1}$ is self-adjoint and compact. By a well known theorem of Hilbert (see [3, Theorem 6.11]), there exists a sequence of positive number μ_j with $\mu_{j+1} \leq \mu_j$ and an orthogonal basis $\{w_j\}$ of H_{as} such that $(\widetilde{A})^{-1} w_j = \mu_j w_j$. We denote $\lambda_j = \dfrac{1}{\mu_j}$. Since $(\widetilde{A})^{-1}$ has range $\mathcal{D}(\widetilde{A})$, we get

$$\begin{aligned}
& A w_j = \widetilde{A} w_j = \lambda_j w_j, \ w_j \in W_{\text{as}}, \\
& 0 < \lambda_1 < \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots, \\
& \lim_{j \to \infty} \lambda_j = +\infty, \\
& (w_j)_{j=1,\ldots} \text{are an orthogonal basis of } H_{\text{as}}.
\end{aligned}$$

Proposition 2.4 *For all* $j \geq 1$, $w_j \in W_{\text{as}} \cap C^\infty(\Omega)$.

Proof. By [31, Proposition 2.2, Chapter 1] and the fact that Ω is of class C^∞, we have $w_j \in C^\infty(\Omega)$. □

Let us define the fractional power $\widetilde{A}^\alpha$ of $\widetilde{A}$ by setting

$$\widetilde{A}^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha \mu_j w_j \quad \text{for} \quad u = \sum_{j=1}^{\infty} \mu_j w_j, \ u \in \mathcal{D}(\widetilde{A}^\alpha), \tag{25}$$

$$\mathcal{D}(\widetilde{A}^\alpha) = \{u \in H_{\text{as}} \mid u = \sum_{j=1}^{\infty} \mu_j w_j, \ \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |\mu_j|^2 < +\infty, \ \mu_j \in \mathbb{R}\}. \tag{26}$$

The spaces $\mathcal{D}(\widetilde{A}^\alpha)$ carry a natural scalar product $\langle \cdot, \cdot \rangle_\alpha$ which is defined by setting

$$\langle u, v \rangle_\alpha = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \mu_j \eta_j \quad \text{whenever} \quad u = \sum_{j=1}^{\infty} \mu_j w_j, \ v = \sum_{j=1}^{\infty} \eta_j w_j. \tag{27}$$

For this scalar product, the sequence $\{\lambda_j^{-\alpha} w_j\}$ form an orthogonal system which is complete in $\mathcal{D}(\widetilde{A}^\alpha)$. Based on this fact, we have the following proposition.

Proposition 2.5 *The system* $\{\lambda_j^{-1/2} w_j\}$ *form an orthogonal basis of* V_{as}. *Moreover,* $\mathcal{D}((\widetilde{A})^{1/2}) = V_{\text{as}}$ *and* V_{as} *is dense in* H_{as}.

Proof. Since $\{\lambda_j^{-1/2} w_j\}$ form an orthogonal basis in $\mathcal{D}((\widetilde{A})^{1/2})$, it is sufficient to show that $\mathcal{D}((\widetilde{A})^{1/2}) = V_{\text{as}}$. In fact, the vectors $\{\lambda_j^{-1/2} w_j\}$ are in V_{as} and

$$\langle \lambda_j^{-1/2} w_j, \lambda_k^{-1/2} w_k \rangle_{1/2} = \delta_{jk} = (A(\lambda_j^{-1/2} w_j), \lambda_k^{-1/2} w_k) = ((\lambda_j^{-1/2} w_j, \lambda_k^{-1/2} w_k)). \tag{28}$$

Hence $\mathcal{D}((\widetilde{A})^{1/2}) \subset V_{\text{as}}$. Conversely, if V_{as} is not contained in $\mathcal{D}((\widetilde{A})^{1/2})$ then there exists a nonzero vector $\overline{v} \in V_{\text{as}}$ such that $\overline{v}$ is orthogonal with $\mathcal{D}((\widetilde{A})^{1/2})$ in V_{as}. Hence

$$0 = ((\overline{v}, \lambda_j^{-1/2} w_j)) = (\overline{v}, \widetilde{A}(\lambda_j^{-1/2} w_j)) = \lambda_j^{1/2} (\overline{v}, w_j)$$

for all j. Since $\overline{v} \in V_{\rm as} \subset H_{\rm as}$, we obtain $\overline{v} = 0$. The proof is complete. $\square$

Let us denote by X_m the finite dimensional space which is spanned by $\{w_1, w_2, ..., w_m\}$. For each $h \in H_{\rm as}$ and $g \in V_{\rm as}$, we shall denote by $P_m h$ and $P_m^{V_{\rm as}} g$ the projections of h and g on X_m in $H_{\rm as}$ and $V_{\rm as}$, respectively.

Proposition 2.6 *The following formulae are valid*

$$P_m h = \sum_{j=1}^{m} (h, w_j) w_j,$$
$$P_m^{V_{\rm as}} g = \sum_{j=1}^{m} \lambda_j^{-1} ((g, w_j)) w_j.$$

Moreover, if $\varphi \in V_{\rm as}$ *then* $P_m \varphi = P_m^{V_{\rm as}} \varphi$ *and*

$$|P_m \varphi| \leq |\varphi|, \ \ \|P_m \varphi\| \leq \|\varphi\|. \tag{29}$$

Proof. The first conclusion is obvious. For the second formula we note that $\{\lambda_j^{-1/2} w_j\}$ is an orthogonal basis of $V_{\rm as}$. When $\varphi \in V_{\rm as}$, we have

$$\begin{aligned} P_m \varphi = \sum_{j=1}^{m} (w_j, \varphi) w_j &= \sum_{j=1}^{m} (\lambda_j^{-1} \widetilde{A}(w_j), \varphi) w_j \\ &= \sum_{j=1}^{m} \lambda_j^{-1} ((w_j, \varphi)) w_j = P_m^{V_{\rm as}} \varphi. \end{aligned}$$

Estimation (29) follows from property of orthogonal projections. The proof is complete. $\square$

Let $u \in C^\infty(\Omega) \cap V_0$. Then $u = u_r e_r + u_\theta e_\theta + u_z e_z$. This is equivalent to

$$\begin{cases} u_1 &= u_r(t, r, z) \cos\theta - u_\theta(t, r, z) \sin\theta, \\ u_2 &= u_r(t, r, z) \sin\theta + u_r\theta(t, r, z) \cos\theta, \\ u_3 &= u_z(t, r, z). \end{cases} \tag{30}$$

It is easy to see that

$$|u|_e^2 = (u_r)^2 + (u_\theta)^2 + (u_z)^2. \tag{31}$$

Since $u|_{\partial\Omega} = 0$, we have $u_r|_{\partial G} = u_\theta|_{\partial G} = u_z|_{\partial G} = 0$. By applying formulae $D_1 =$

$\cos\theta D_r - \frac{1}{r}\sin\theta D_\theta$, $D_2 = \sin\theta D_r + \frac{1}{r}\cos\theta D_\theta$ and $D_3 = D_z$ for u_1, u_2 and u_3, we get

$$\begin{aligned}
D_1u_1 &= \cos^2\theta D_r u_r - \frac{1}{2}\sin 2\theta D_r u_\theta + \sin^2\theta\frac{u_r}{r} + \frac{1}{2}\sin 2\theta\frac{u_\theta}{r},\\
D_2u_1 &= \frac{1}{2}\sin 2\theta D_r u_r - \sin^2\theta D_r u_\theta - \frac{1}{2}\sin 2\theta\frac{u_r}{r} - \cos^2\theta\frac{u_\theta}{r},\\
D_3u_1 &= D_z u_r\cos\theta - D_z u_\theta\sin\theta,\\
D_1u_2 &= \frac{1}{2}\sin 2\theta D_r u_r + \cos^2\theta D_r u_\theta - \frac{1}{2}\sin 2\theta\frac{u_r}{r} + \sin^2\theta\frac{u_\theta}{r},\\
D_2u_2 &= \sin^2\theta D_r u_r + \frac{1}{2}\sin 2\theta D_r u_\theta + \cos^2\theta\frac{u_r}{r} - \frac{1}{2}\sin 2\theta\frac{u_\theta}{r},\\
D_3u_2 &= \sin\theta D_z u_r + \cos\theta D_z u_\theta,\\
D_1u_3 &= D_r u_z\cos\theta,\\
D_2u_3 &= D_r u_z\sin\theta,\\
D_3u_3 &= D_z u_z.
\end{aligned}$$

It follows that

$$\sum_{i,j=1}^{3}(D_iu_j)^2 = (D_ru_r)^2 + (D_ru_\theta)^2 + (D_ru_z)^2 + (D_zu_r)^2 + (D_zu_\theta)^2 + (D_zu_z)^2 + \frac{1}{r^2}(u_r^2 + u_\theta^2). \tag{32}$$

Lemma 2.1 *There exist absolute constants $C_1, C_2, C_3 > 0$ such that*

$$\|\tilde{u}\|_{L^2(G)^3} \le C_1\|u\|_{L^2(\Omega)^3}, \tag{33}$$

$$\|\nabla_{(r,z)}\tilde{u}\|_{L^2(G)^3} \le C_2\|\nabla u\|_{L^2(\Omega)^3}, \tag{34}$$

$$\|\tilde{u}\|_{H^2(G)^3} \le C_3\|u\|_{H^2(\Omega)^3} \tag{35}$$

for all $u \in C^\infty(\Omega)\cap V_0$, where $\tilde{u} = (u_r, u_\theta, u_z)$.

Proof. By (31), we have

$$\begin{aligned}
\|\tilde{u}\|^2_{L^2(G)^3} &= \int_G((u_r)^2 + (u_\theta)^2 + (u_z)^2)drdz = \int_G\frac{1}{r}((u_r)^2 + (u_\theta)^2 + (u_z)^2)rdrdz\\
&\le \frac{1}{R_2 - R_1}\int_G((u_r)^2 + (u_\theta)^2 + (u_z)^2)rdrdz\\
&= \frac{1}{2\pi(R_2 - R_1)}\int_{-\pi}^{\pi}\int_G((u_r)^2 + (u_\theta)^2 + (u_z)^2)rdrdzd\theta\\
&= \frac{1}{2\pi(R_2 - R_1)}\|u\|^2_{L^2(\Omega)^3}.
\end{aligned}$$

Therefore, inequality (33) is proved. Inequality (34) is established similarly. It remains to prove inequality (35).

By some computations, we have the following formulae:

$$D_1^2u_1 = \cos^3\theta D_r^2 u_r - \cos^2\theta\sin\theta D_r^2 u_\theta + 3\cos\theta\sin^2\theta(\frac{D_r u_r}{r} - \frac{u_r}{r^2}) +$$
$$+ (\cos^2\theta\sin\theta + \sin\theta\cos 2\theta)(\frac{D_r u_\theta}{r} - \frac{u_\theta}{r^2}), \tag{36}$$
$$D_1D_3u_1 = \cos^2\theta D_rD_z u_r - \sin\theta\cos\theta D_rD_z u_\theta + \sin^2\theta\frac{D_z u_r}{r} + \sin\theta\cos\theta\frac{D_z u_\theta}{r}, \tag{37}$$
$$D_3^2u_1 = \cos\theta D_z^2 u_r - \sin\theta D_z^2 u_\theta, \tag{38}$$
$$D_1^2u_3 = \cos^2\theta D_r^2 u_z + \sin^2\theta\frac{D_r u_z}{r}, \tag{39}$$
$$D_3D_1u_3 = \cos\theta D_rD_z u_z, \tag{40}$$
$$D_3^2u_3 = D_z^2 u_z. \tag{41}$$

By putting

$$T_1 = 3\cos\theta\sin^2\theta(\frac{D_r u_r}{r} - \frac{u_r}{r^2}) + (\cos^2\theta\sin\theta + \sin\theta\cos 2\theta)(\frac{D_r u_\theta}{r} - \frac{u_\theta}{r^2}),$$

we have from (36) that

$$(D_1^2u_1)^2 = \big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)^2 + 2\big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)T_1 + T_1^2.$$

Hence

$$\int_\Omega (D_1^2u_1)^2dx = \int_{-\pi}^{\pi}\int_G (D_1^2u_1)^2 r dr dz d\theta \geq (R_2 - R_1)\int_{-\pi}^{\pi}\int_G (D_1^2u_1)^2 dr dz d\theta$$
$$= (R_2 - R_1)\int_{-\pi}^{\pi}\int_G \big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)^2 dr dz d\theta$$
$$+ (R_2 - R_1)\int_{-\pi}^{\pi}\int_G \big[2\big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)T_1 + T_1^2\big] dr dz d\theta$$
$$\geq (R_2 - R_1)\int_{-\pi}^{\pi}\int_G \big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)^2 dr dz d\theta$$
$$+ (R_2 - R_1)\int_{-\pi}^{\pi}\int_G \big[2\big(\cos^3\theta D_r^2u_r - \cos^2\theta\sin\theta D_r^2u_\theta\big)T_1\big] dr dz d\theta$$
$$= (R_2 - R_1)\int_G \big(\frac{5\pi}{8}(D_r^2u_r)^2 + \frac{\pi}{8}(D_r^2u_\theta)^2\big) dr dz$$
$$+ 2(R_2 - R_1)\int_G \big(\frac{3\pi}{8}(D_r^2u_r)(\frac{D_r u_r}{r} - \frac{u_r}{r^2}) - \frac{\pi}{8}(D_r^2u_\theta)(\frac{D_r u_\theta}{r} - \frac{u_\theta}{r^2})\big) dr dz$$
$$\geq (R_2 - R_1)\int_G \big(\frac{5\pi}{8}(D_r^2u_r)^2 + \frac{\pi}{8}(D_r^2u_\theta)^2\big) dr dz$$
$$+ (R_2 - R_1)\int_G \big(- \frac{\pi}{8}(D_r^2u_r)^2 - \frac{9\pi}{8}(\frac{D_r u_r}{r} - \frac{u_r}{r^2})^2 - \frac{\pi}{16}(D_r^2u_\theta)^2 - \frac{\pi}{4}(\frac{D_r u_\theta}{r} - \frac{u_\theta}{r^2})^2\big) dr dz$$
$$= (R_2 - R_1)\int_G \big(\frac{\pi}{2}(D_r^2u_r)^2 + \frac{\pi}{16}(D_r^2u_\theta)^2 - \frac{9\pi}{8}(\frac{D_r u_r}{r} - \frac{u_r}{r^2})^2 - \frac{\pi}{4}(\frac{D_r u_\theta}{r} - \frac{u_\theta}{r^2})^2\big) dr dz$$
$$\geq (R_2 - R_1)\int_G \Big(\frac{\pi}{2}(D_r^2u_r)^2 + \frac{\pi}{16}(D_r^2u_\theta)^2 - \frac{9\pi}{4}((\frac{D_r u_r}{r})^2 + (\frac{u_r}{r^2})^2) - \frac{\pi}{2}((\frac{D_r u_\theta}{r})^2 + (\frac{u_\theta}{r^2})^2)\Big) dr dz.$$

This implies that

$$\int_\Omega (D_1^2 u_1)^2 dx + (R_2 - R_1)\int_G \Big(\frac{9\pi}{4}((\frac{D_r u_r}{r})^2 + (\frac{u_r}{r^2})^2) + \frac{\pi}{2}((\frac{D_r u_\theta}{r})^2 + (\frac{u_\theta}{r^2})^2)\Big) drdz \geq$$
$$\geq (R_2 - R_1)\int_G \Big(\frac{\pi}{2}(D_r^2 u_r)^2 + \frac{\pi}{16}(D_r^2 u_\theta)^2\Big) drdz.$$

Since $r \geq R_2 - R_1$, we obtain

$$\int_\Omega (D_1^2 u_1)^2 dx + \frac{1}{(R_2 - R_1)}\int_G \Big(\frac{9\pi}{4}(D_r u_r)^2 + \frac{\pi}{2}(D_r u_\theta)^2 + \frac{u_r^2 + u_\theta^2}{(R_2 - R_1)^2}\Big) drdz \geq$$
$$\geq (R_2 - R_1)\int_G \Big(\frac{\pi}{2}(D_r^2 u_r)^2 + \frac{\pi}{16}(D_r^2 u_\theta)^2\Big) drdz. \tag{42}$$

By the same procedure, we obtain from (37) to (41) the following estimations:

$$\int_\Omega (D_1 D_3 u_1)^2 dx + \frac{1}{R_2 - R_1}\int_G \big(\frac{\pi}{4}(D_z u_r)^2 + \frac{\pi}{2}(D_z u_\theta)^2\big) drdz \geq$$
$$\geq (R_2 - R_1)\int_G \big(\frac{\pi}{2}(D_r D_z u_r)^2 + \frac{\pi}{8}(D_r D_z u_\theta)^2\big) drdz, \tag{43}$$

$$\int_\Omega (D_3^2 u_1)^2 dx \geq (R_2 - R_1)\int_G \big(\pi(D_z^2 u_r)^2 + \pi(D_z^2 u_\theta)^2\big) drdz, \tag{44}$$

$$\int_\Omega (D_1^2 u_3)^2 dx + \frac{1}{R_2 - R_1}\int_G \frac{\pi}{4}(D_r u_z)^2 drdz \geq (R_2 - R_1)\int_G \frac{\pi}{2}(D_r^2 u_z)^2 drdz, \tag{45}$$

$$\int_\Omega (D_3 D_1 u_3)^2 dx \geq (R_2 - R_1)\int_G \pi(D_r D_z u_z)^2 drdz, \tag{46}$$

$$\int_\Omega (D_3^2 u_3)^2 dx \geq (R_2 - R_1)\int_G 2\pi(D_z^2 u_z)^2 drdz. \tag{47}$$

By adding inequalities (42)–(47), we see that there exists positive constants α, β such that

$$\|u\|^2_{H^2(\Omega)^3} + \alpha\int_G \big((D_r u_r)^2 + (D_r u_\theta)^2 + (D_z u_r)^2 + (D_z u_\theta)^2\big) drdz + \int_G \frac{u_r^2 + u_\theta^2}{(R_2 - R_1)^3} drdz \geq$$
$$\geq \beta\int_G \big(\sum_{i,j,k\in\{r,\theta,z\}} (D_i D_j u_k)^2\big) drdz.$$

By adding both sides with $\beta\|\nabla_{(r,z)}\tilde{u}\|^2_{L^2(G)^3}$ and $\beta\|\tilde{u}\|^2_{L^2(G)^3}$, we obtain

$$\|u\|^2_{H^2(\Omega)^3} + \alpha'\|\nabla_{(r,z)}\tilde{u}\|^2_{L^2(G)^3} + \gamma'\|\tilde{u}\|^2_{L^2(G)^3} \geq \beta'\|\tilde{u}\|^2_{H^2(G)^3}$$

for some positive constants α', β' and γ'. Using (33) and (34), we get

$$(1 + \alpha' C_1 + \gamma' C_2)\|u\|^2_{H^2(\Omega)^3} \geq \beta'\|\tilde{u}\|^2_{H^2(G)^3}.$$

The proof of the lemma is complete. □

In the sequel we shall need some estimations of $|\mathbf{b}(u,u,\widetilde{A}u)|$. For this we consider the mapping $\widetilde{\mathbf{b}}$ which is defined by setting

$$\widetilde{\mathbf{b}}(\tilde{u},\tilde{v},\tilde{w}) = \int_G \Big|\big((u_r D_r v_r + u_z D_z v_r - \frac{1}{r}u_\theta v_\theta)w_r + (u_r D_r v_\theta + u_z D_z v_\theta + \frac{1}{r}u_\theta v_r)w_\theta \\ + (u_r D_r v_z + u_z D_z v_z)w_z\big)\Big| drdz \tag{48}$$

with $\tilde{u} = (u_r, u_\theta, u_z), \tilde{v} = (v_r, v_\theta, v_z)$ and $\tilde{w} = (w_r, w_\theta, w_z)$ belong to $C^\infty(G)^3$.

We now have the following key lemma.

Lemma 2.2 *Let $0 \le s_i < 1$ and $s_1 + s_2 + s_3 \ge 1$. Then there exists an absolute constant $C_4 > 0$ depending on G and s_i such that*

$$\widetilde{\mathbf{b}}(\tilde{u},\tilde{v},\tilde{w}) \le C_4 \|\tilde{u}\|_0^{1-s_1}\|\tilde{u}\|_1^{s_1]}\|\tilde{v}\|_1^{1-s_2}\|\tilde{v}\|_2^{s_2}\|\tilde{w}\|_0^{1-s_3}\|\tilde{w}\|_1^{s_3}$$

for all $\tilde{u},\tilde{v},\tilde{w} \in C^\infty(G)^3$.

Proof. Let us set $\hat{u} = E^1\tilde{u}$, $\hat{v} = E^2\tilde{v}$ and $\hat{w} = E^1\tilde{w}$, where $E^l : H^l(G) \to H^l(\mathbb{R}^2)$ is a linear extension operator. Recall that for all $\phi \in H^{1+[s]}(\mathbb{R}^n)$ with $[s]$ is the integer part of s, we have the following interpolation inequalities:

$$\|\phi\|_{H^s(\mathbb{R}^n)} \le \|\phi\|_{H^{[s]}(\mathbb{R}^n)}^{1-(s-[s])}\|\phi\|_{H^{[s]+1}(\mathbb{R}^n)}^{s-[s]}. \tag{49}$$

By assumption, we have $n = 2$ and $0 \le s_i < n/2$. Define q_i by $\dfrac{1}{q_i} = \dfrac{1}{2} - \dfrac{s_i}{n}$ wit $i = 1,2,3$ and q_4 by $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} = 1$. This is possible because $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \le 1$. We now have

$$\begin{aligned}\widetilde{\mathbf{b}}(\tilde{u},\tilde{v},\tilde{w}) &\le \int_G \Big|(u_r D_r v_r + u_z D_z v_r - \frac{1}{r}u_\theta v_\theta)w_r\Big| drdz \\ &+ \int_G \Big|(u_r D_r v_\theta + u_z D_z v_\theta + \frac{1}{r}u_\theta v_r)w_\theta\Big| drdz + \int_G \Big|(u_r D_r v_z + u_z D_z v_z)w_z\Big| drdz \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{50}$$

Using the Hölder inequality, the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ with $s < \dfrac{n}{2}$, $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{s}{n}$

and interpolation inequality (49), we have

$$\begin{aligned}
\Sigma_1 &\le \int_G \big(|u_r D_r v_r w_r| + |u_z D_z v_r w_r| + |\frac{1}{r} u_\theta v_\theta w_r|\big) drdz \\
&\le \|u_r\|_{L^{q_1}} \|D_r v_r\|_{L^{q_2}} \|w_r\|_{L^{q_3}} \|1\|_{L^{q_4}} + \|u_z\|_{L^{q_1}} \|D_z v_r\|_{L^{q_2}} \|w_r\|_{L^{q_3}} \|1\|_{L^{q_4}} + \\
&+ \frac{1}{R_2 - R_1} \|u_\theta\|_{L^{q_1}} \|v_\theta\|_{L^{q_2}} \|w_r\|_{L^{q_3}} \|1\|_{L^{q_4}} \\
&\le |G|^{\frac{1}{q_4}} \|\hat{u}_r\|_{L^{q_1}} \|D_r \hat{v}_r\|_{L^{q_2}} \|\hat{w}_r\|_{L^{q_3}} + |G|^{\frac{1}{q_4}} \|\hat{u}_z\|_{L^{q_1}} \|D_z \hat{v}_r\|_{L^{q_2}} \|\hat{w}_r\|_{L^{q_3}} + \\
&+ \frac{|G|^{\frac{1}{q_4}}}{R_2 - R_1} \|\hat{u}_\theta\|_{L^{q_1}} \|\hat{v}_\theta\|_{L^{q_2}} \|\hat{w}_r\|_{L^{q_3}} \\
&\le |G|^{\frac{1}{q_4}} \|\hat{u}_r\|_{s_1} \|D_r \hat{v}_r\|_{s_2} \|\hat{w}_r\|_{s_3} + |G|^{\frac{1}{q_4}} \|\hat{u}_z\|_{s_1} \|D_z \hat{v}_r\|_{s_2} \|\hat{w}_r\|_{s_3} + \\
&+ \frac{|G|^{\frac{1}{q_4}}}{R_2 - R_1} \|\hat{u}_\theta\|_{s_1} \|\hat{v}_\theta\|_{s_2} \|\hat{w}_r\|_{s_3} \\
&\le |G|^{\frac{1}{q_4}} \|\hat{u}_r\|_{[s_1]}^{1-s_1+[s_1]} \|\hat{u}_r\|_{[s_1]+1}^{s_1-[s_1]} \|D_r \hat{v}_r\|_{[s_2]}^{1-s_2+[s_2]} \|D_r \hat{v}_r\|_{[s_2]+1}^{s_2-[s_2]} \|\hat{w}_r\|_{[s_3]}^{1-s_3+[s_3]} \|\hat{w}_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&+ |G|^{\frac{1}{q_4}} \|\hat{u}_z\|_{[s_1]}^{1-s_1+[s_1]} \|\hat{u}_z\|_{[s_1]+1}^{s_1-[s_1]} \|D_z \hat{v}_r\|_{[s_2]}^{1-s_2+[s_2]} \|D_z \hat{v}_r\|_{[s_2]+1}^{s_2-[s_2]} \|\hat{w}_r\|_{[s_3]}^{1-s_3+[s_3]} \|\hat{w}_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&+ \frac{|G|^{\frac{1}{q_4}}}{R_2 - R_1} \|\hat{u}_\theta\|_{[s_1]}^{1-s_1+[s_1]} \|\hat{u}_\theta\|_{[s_1]+1}^{s_1-[s_1]} \|\hat{v}_\theta\|_{[s_2]}^{1-s_2+[s_2]} \|\hat{v}_\theta\|_{[s_2]+1}^{s_2-[s_2]} \|\hat{w}_r\|_{[s_3]}^{1-s_3+[s_3]} \|\hat{w}_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&\le \gamma_1 \|u_r\|_{[s_1]}^{1-s_1+[s_1]} \|u_r\|_{[s_1]+1}^{s_1-[s_1]} \|D_r v_r\|_{[s_2]}^{1-s_2+[s_2]} \|D_r v_r\|_{[s_2]+1}^{s_2-[s_2]} \|w_r\|_{[s_3]}^{1-s_3+[s_3]} \|w_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&+ \gamma_1 \|u_z\|_{[s_1]}^{1-s_1+[s_1]} \|u_z\|_{[s_1]+1}^{s_1-[s_1]} \|D_z v_r\|_{[s_2]}^{1-s_2+[s_2]} \|D_z v_r\|_{[s_2]+1}^{s_2-[s_2]} \|w_r\|_{[s_3]}^{1-s_3+[s_3]} \|w_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&+ \gamma_1 \|u_\theta\|_{[s_1]}^{1-s_1+[s_1]} \|u_\theta\|_{[s_1]+1}^{s_1-[s_1]} \|v_\theta\|_{[s_2]}^{1-s_2+[s_2]} \|v_\theta\|_{[s_2]+1}^{s_2-[s_2]} \|w_r\|_{[s_3]}^{1-s_3+[s_3]} \|w_r\|_{1+[s_3]}^{s_3-[s_3]} \\
&\le 3\gamma_1 \|\tilde{u}\|_{[s_1]}^{1-s_1+[s_1]} \|\tilde{u}\|_{[s_1]+1}^{s_1-[s_1]} \|\tilde{v}\|_{[s_2]+1}^{1-s_2+[s_2]} \|\tilde{v}\|_{[s_2]+2}^{s_2-[s_2]} \|\tilde{w}\|_{[s_3]}^{1-s_3+[s_3]} \|\tilde{w}\|_{1+[s_3]}^{s_3-[s_3]}
\end{aligned}$$

for some absolute constant $\gamma_1 > 0$. Hence

$$\Sigma_1 \le 3\gamma_1 \|\tilde{u}\|_{[s_1]}^{1-s_1+[s_1]} \|\tilde{u}\|_{[s_1]+1}^{s_1-[s_1]} \|\tilde{v}\|_{[s_2]+1}^{1-s_2+[s_2]} \|\tilde{v}\|_{[s_2]+2}^{s_2-[s_2]} \|\tilde{w}\|_{[s_3]}^{1-s_3+[s_3]} \|\tilde{w}\|_{1+[s_3]}^{s_3-[s_3]}. \tag{51}$$

By similar arguments, we can show that

$$\Sigma_2 \le 3\gamma_2 \|\tilde{u}\|_{[s_1]}^{1-s_1+[s_1]} \|\tilde{u}\|_{[s_1]+1}^{s_1-[s_1]} \|\tilde{v}\|_{[s_2]+1}^{1-s_2+[s_2]} \|\tilde{v}\|_{[s_2]+2}^{s_2-[s_2]} \|\tilde{w}\|_{[s_3]}^{1-s_3+[s_3]} \|\tilde{w}\|_{1+[s_3]}^{s_3-[s_3]}. \tag{52}$$

and

$$\Sigma_3 \le 2\gamma_3 \|\tilde{u}\|_{[s_1]}^{1-s_1+[s_1]} \|\tilde{u}\|_{[s_1]+1}^{s_1-[s_1]} \|\tilde{v}\|_{[s_2]+1}^{1-s_2+[s_2]} \|\tilde{v}\|_{[s_2]+2}^{s_2-[s_2]} \|\tilde{w}\|_{[s_3]}^{1-s_3+[s_3]} \|\tilde{w}\|_{1+[s_3]}^{s_3-[s_3]} \tag{53}$$

for some absolute constants $\gamma_2, \gamma_3 > 0$. Combining (51)–(53) and (50), we obtain the conclusion of the lemma with $C_4 = 3\gamma_1 + 3\gamma_2 + 2\gamma_3$ and $[s_1] = [s_2] = [s_3] = 0$. □

When $s_1 = s_2 = 1/2$ and $s_3 = 0$, we have the following estimation.

Corollary 2.1 *There exists an absolute constant $C_5 > 0$ depending on G such that*

$$\begin{aligned}
\widetilde{\mathbf{b}}(\tilde{u}, \tilde{v}, \tilde{w}) = \int_G \Big| &(u_r D_r v_r + u_z D_z v_r - \frac{1}{r} u_\theta v_\theta) w_r + (u_r D_r v_\theta + u_z D_z v_\theta + \frac{1}{r} u_\theta v_r) w_\theta \\
&+ (u_r D_r v_z + u_z D_z v_z) w_z \Big| drdz \le C_5 \|\tilde{u}\|_0^{1/2} \|\tilde{u}\|_1^{1/2} \|\tilde{v}\|_1^{1/2} \|\tilde{v}\|_2^{1/2} \|\tilde{w}\|_0
\end{aligned} \tag{54}$$

for all $\tilde{u}, \tilde{v}, \tilde{w} \in C^\infty(G)^3$.

3 Proof of the main result

By applying the Leray projection, the system (NSE) becomes

$$\begin{cases} \dfrac{du}{dt} + Au + B(u,u) = f(t) \\ v(0) = v_0. \end{cases} \tag{55}$$

In the sequel, we shall show that the system (55) has a unique solution v satisfying

$$v \in L^\infty((0,\infty); V_{\text{as}}) \cap L^2((0,\infty); H^2(\Omega)^3).$$

For each $m \geq 1$, we consider the Galerkin system of finding $u_m(t) \in X_m$ such that

$$\frac{du_m}{dt} + \widetilde{A}u_m + P_m B(u_m, u_m) = g_m, \tag{56}$$

$$u_m(0) = u_{0,m}, \tag{57}$$

where $u_{0,m} = P_m v_0$ and $g_m = P_m f$. Let $u_m = \sum_{j=1}^m \xi_j(t) w_j$ and $g_m = \sum_{j=1}^m \eta_j(t) w_j$. Since $\widetilde{A}w_j = \lambda_j w_j$, the above system is equivalent to the system of ordinary differential equations for $\xi_j(t)$:

$$\frac{d\xi_j}{dt} + \lambda_j \xi_j + \sum_{k,l=1}^m \mathbf{b}(w_k, w_l, w_j)\xi_k \xi_l = \eta_j, \ \ j = 1, 2, ..., m,$$

$$\xi_j(0) = \xi_j^0, \ \ j = 1, 2, ..., m,$$

or

$$\begin{cases} \frac{d\xi_j}{dt} = F_j(t,\xi), \\ \xi_j(0) = \xi_j^0, \ \ j = 1, 2, ..., m, \end{cases} \tag{58}$$

where $\xi_j^0 = (u_{0,m}, w_j)$, $\eta_j(t) = (f(t,\cdot), w_j)$ and

$$F_j(t,\xi) = \eta_j(t) - \lambda_j \xi_j - \sum_{k,l=1}^m \mathbf{b}(w_k, w_l, w_j)\xi_k \xi_l.$$

It is clear that $F_j(t,\xi)$ is locally Lipschitz in ξ. Therefore, system (58) has a maximal solution defined on some interval $[0, t_m)$. If $t_m < \infty$ then $|\xi(t)|_e = |u_m(t)|$ must tend to $+\infty$ as $t \to t_m$ (see for instance [15, Corollary 3.2, p.14]). However, the a priori estimate we shall prove later show that this does not happen and therefore $t_m = +\infty$. Indeed, taking the scalar product of (56) with u_m and using property of $\mathbf{b}$, we get

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + |\nabla u_m(t)|^2 = (f, u_m).$$

Since $|(f, u_m)| \leq |f||u_m| \leq \frac{1}{2}(|f||u_m|^2 + |f|)$, we obtain

$$\frac{d}{dt}|u_m(t)|^2 + 2|\nabla u_m(t)|^2 \leq 2|f(t)||u_m(t)| \leq |f(t)||u_m(t)|^2 + |f(t)|. \tag{59}$$

The Gronwal inequality implies that

$$|u_m(t)|^2 \le (|u_m(0)|^2 + \int_0^t |f(s)|ds)\exp\,(\int_0^t |f(s)|ds)$$
$$\le (|v_0|^2 + \int_0^\infty |f(s)|ds)\exp\,(\int_0^\infty |f(s)|ds). \tag{60}$$

Hence $\lim_{t\to t_m}|u_m(t)| < +\infty$ and so $t_m = +\infty$. For convenience, we put

$$M_1^2 = (|v_0|^2 + \int_0^\infty |f(s)|ds)\exp\,(\int_0^\infty |f(s)|ds).$$

Then we have

$$|u_m(t)| \le M_1 \ \forall t \ge 0. \tag{61}$$

By integrating two sides of (59), we get

$$2\int_0^t |\nabla u_m|^2 ds \le |u_{0,m}|^2 + 2\int_0^t |f(s)||u_m(s)|ds$$
$$\le |v_0|^2 + 2M_1\int_0^\infty |f(s)|ds.$$

Hence

$$\int_0^\infty |\nabla u_m(s)|^2 ds \le M_2^2 \tag{62}$$

with

$$M_2^2 := \frac{1}{2}|v_0|^2 + M_1\int_0^\infty |f(s)|ds.$$

Taking the scalar product both sides of (56) with $\widetilde{A}u_m$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + |\widetilde{A}u_m(t)|^2 = -\mathbf{b}(u_m, u_m, \widetilde{A}u_m) + (g_m, \widetilde{A}u_m). \tag{63}$$

By (12), we have the estimate $C\|u_m\|_{\mathbf{H}^2(\Omega)} \le |\widetilde{A}u_m|$. From this estimation and (63), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + C\|u_m(t)\|^2_{\mathbf{H}^2(\Omega)} \le |\mathbf{b}(u_m, u_m, \widetilde{A}u_m)| + |g_m||\widetilde{A}u_m|$$
$$\le |\mathbf{b}(u_m, u_m, \widetilde{A}u_m)| + |f(t)|\|u_m\|_{\mathbf{H}^2(\Omega)}. \tag{64}$$

It is clear that u_m and $\widetilde{A}u_m$ are axially symmetric. Hence $u_m = u_{mr}e_r + u_{m\theta}e_\theta + u_{mz}e_z$ and $\widetilde{A}u_m = w_{mr}e_r + w_{m\theta}e_\theta + w_{mz}e_z$. For convenience we remove index m and write u

instead of u_m. Put $\tilde{u} = (u_r, u_\theta, u_z)$ and $\tilde{w} = (w_r, w_\theta, w_z)$. Then we have

$$\begin{aligned}
&\big|\mathbf{b}(u,u,\widetilde{A}u)\big| = \Big|\int_\Omega \langle (u\cdot\nabla)u, \widetilde{A}u\rangle_e dx\Big| \\
&= \Big|\int_\Omega \Big((u_r D_r u_r + u_z D_z u_r - \frac{1}{r}u_\theta u_\theta)w_r + (u_r D_r u_\theta + u_z D_z u_\theta + \frac{1}{r}u_\theta u_r)w_\theta \\
&+ (u_r D_r u_z + u_z D_z u_z)w_z\Big)dx\Big| \\
&\le \int_\Omega \Big|(u_r D_r u_r + u_z D_z u_r - \frac{1}{r}u_\theta u_\theta)w_r + (u_r D_r u_\theta + u_z D_z u_\theta + \frac{1}{r}u_\theta u_r)w_\theta \\
&+ (u_r D_r u_z + u_z D_z u_z)w_z\Big|dx \\
&\le \int_{-\pi}^{\pi}\int_G \Big|(u_r D_r u_r + u_z D_z u_r - \frac{1}{r}u_\theta u_\theta)w_r + (u_r D_r u_\theta + u_z D_z u_\theta + \frac{1}{r}u_\theta u_r)w_\theta \\
&+ (u_r D_r u_z + u_z D_z u_z)w_z\Big| r dr dz d\theta \\
&\le 2\pi(R_2+R_1)\int_G \Big|(u_r D_r u_r + u_z D_z u_r - \frac{1}{r}u_\theta u_\theta)w_r + (u_r D_r u_\theta + u_z D_z u_\theta + \frac{1}{r}u_\theta u_r)w_\theta \\
&+ (u_r D_r u_z + u_z D_z u_z)w_z\Big| dr dz \\
&= 2\pi(R_2+R_1)\widetilde{\mathbf{b}}(\tilde{u},\tilde{u},\tilde{w}).
\end{aligned}$$

From this, Corollary 2.1 and Lemma 2.1, we obtain

$$\begin{aligned}
\big|\mathbf{b}(u,u,\widetilde{A}u)\big| &\le 2\pi(R_2+R_1)\widetilde{\mathbf{b}}(\tilde{u},\tilde{u},\tilde{w}) \\
&\le 2\pi(R_2+R_1)C_5\|\tilde{u}\|^{1/2}_{L^2(G)^3}\|\tilde{u}\|_{H^1(G)^3}\|\tilde{u}\|^{1/2}_{H^2(G)^3}\|\tilde{w}\|_{L^2(G)} \\
&\le 2\pi(R_2+R_1)C_5(C_1)^{1/2}|u|^{1/2}C_2\|u\|C_3^{1/2}\|u\|_2^{1/2}C_1|\widetilde{A}u| \\
&\le C_6|u|^{1/2}\|u\|\|u\|_2^{1/2}|\widetilde{A}u| \qquad (65)
\end{aligned}$$

for some constant $C_6 > 0$. Since $|\widetilde{A}u| \le \|u\|_2$, we get

$$\big|\mathbf{b}(u_m,u_m,\widetilde{A}u_m)\big| \le C_6|u_m|^{1/2}\|u_m\|\|u_m\|_2^{3/2}. \qquad (66)$$

Combining this with (64) yields

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + C\|u_m(t)\|_2^2 \le C_6|u_m|^{1/2}\|u_m\|\|u_m\|_2^{3/2} + |f(t)|\|u_m\|_2. \qquad (67)$$

Using Young's inequality

$$ab \le \epsilon a^p + C(\epsilon)b^q,\ (a,b>0,\epsilon>0,p,q>0,1/p+1/q=1,C(\epsilon)=\frac{1}{q(\epsilon p)^{q/p}}),$$

we have

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + C\|u_m(t)\|_2^2 \le C_7|u_m|^2\|u_m\|^4 + \frac{C}{4}\|u_m\|_2^2 + \frac{|f(t)|^2}{C} + \frac{C\|u_m\|_2^2}{4}.$$

This implies that

$$\frac{d}{dt}\|u_m(t)\|^2 + C\|u_m(t)\|_2^2 \le 2C_7|u_m|^2\|u_m\|^4 + \frac{2|f(t)|^2}{C}. \tag{68}$$

By Gronwall's inequality and (62), we get

$$\begin{aligned}\|u_m(t)\|^2 &\le \Big[\|u_{0,m}\|^2 + \frac{2}{C}\int_0^t |f(s)|^2 ds\Big] \exp\big(2C_7\int_0^t |u_m(s)|^2\|u_m(s)\|^2 ds\big)\\ &\le \Big[\|v_0\|^2 + \frac{2}{C}\int_0^\infty |f(s)|^2 ds\Big] \exp\big(2C_7 M_1^2 M_2^2\big).\end{aligned}$$

Putting

$$M_3^2 = \Big[\|v_0\|^2 + \frac{2}{C}\int_0^\infty |f(s)|^2 ds\Big] \exp\big(2C_7 M_1^2 M_2^2\big),$$

we have

$$\|u_m(t)\|^2 \le M_3^2 \;\forall t \ge 0. \tag{69}$$

Integrating both sides of (68) and using (61), (62) and (69), we get

$$\|u_m(t)\|^2 + C\int_0^t \|u_m(s)\|_2^2 ds \le \|v_0\|^2 + 2C_7(M_1M_2M_3)^2 + \frac{2}{C}\int_0^\infty |f(s)|^2 ds.$$

This implies that

$$\int_0^\infty \|u_m(s)\|_2^2 ds \le M_4^2, \tag{70}$$

where

$$M_4^2 := \frac{1}{C}\big(\|v_0\|^2 + 2C_7(M_1M_2M_3)^2 + \frac{2}{C}\int_0^\infty |f(s)|^2 ds\big).$$

Let us give a bound for $\frac{du_m}{dt}$. From (56), we have

$$\begin{aligned}|\frac{du_m}{dt}| &\le |\widetilde{A}u_m| + |B(u_m,u_m)| + |g_m|\\ &\le \|u_m\|_2 + C_5|u_m|^{1/2}\|u_m\|\|u_m\|_2^{1/2} + |f|,\end{aligned}$$

where the estimate $|B(u_m,u_m)| \le C_5|u_m|^{1/2}\|u_m\|\|u_m\|_2^{1/2}$ follows from (65). It follows that

$$|\frac{du_m}{dt}|^2 \le C_8(\|u_m\|_2^2 + |u_m|\|u_m\|^2\|u_m\|_2 + |f|^2)$$

for some constant $C_8 > 0$. Hence

$$\begin{aligned}\int_0^\infty |\frac{du_m}{dt}|^2 dt &\le C_8\big(\int_0^\infty \|u_m(t)\|_2^2 dt + M_1M_3\int_0^\infty \|u_m\|\|u_m\|_2 dt + \int_0^\infty |f(t)|^2 dt\big)\\ &\le C_8\big(M_4^2 + M_1M_3(\int_0^\infty \|u_m\|^2 dt)^{1/2}(\int_0^\infty \|u_m\|_2^2 dt)^{1/2} + \int_0^\infty |f(t)|^2 dt\big)\\ &\le C_8\big(M_4^2 + M_1M_3M_2M_4 + \int_0^\infty |f(t)|^2 dt\big).\end{aligned}$$

Consequently,

$$\int_0^\infty |\frac{du_m}{dt}|^2 dt \le M_5^2 \quad \text{with} \quad M_5^2 := C_8\big(M_4^2 + M_1M_3M_2M_4 + \int_0^\infty |f(t)|^2 dt\big). \tag{71}$$

In summary, we conclude from (62), (69)–(71) that

$$\begin{aligned} &\{u_m\} \quad \text{is bounded in} \quad L^\infty((0,\infty), V_{\text{as}}),\\ &\{u_m\} \quad \text{is bounded in} \quad L^2((0,\infty), \mathbf{H}^2(\Omega)),\\ &\{\frac{du_m}{dt}\} \quad \text{is bounded in} \quad L^2((0,\infty), H_{\text{as}}). \end{aligned}$$

Passing subsequences, we can assume that

$$\begin{aligned} &u_n \to v \quad \text{weakly star in} \quad L^\infty((0,\infty), V_{\text{as}}),\\ &u_n \to v \quad \text{weakly in} \quad L^2((0,\infty), \mathbf{H}^2(\Omega)),\\ &\frac{du_m}{dt} \to \phi \quad \text{weakly in} \quad L^2((0,\infty), H_{\text{as}}). \end{aligned}$$

Let $\psi(t) \in C_0^\infty(0,\infty)$. Then there exists $T' > 0$ such that $\text{supp}(\psi) \subseteq [0, T']$. Taking the scalar product of (56) with $\psi(t)w_j$ and integrating, we obtain

$$\begin{aligned} &\int_0^\infty (\frac{d}{dt}u_m, \psi(t)w_j)dt - \int_0^\infty (\Delta u_m, \psi(t)w_j)dt + \int_0^\infty \mathbf{b}(u_m, u_m, \psi(t)w_j)dt\\ &= \int_0^\infty (g_m, \psi(t)w_j)dt \end{aligned} \tag{72}$$

or equivalently,

$$\begin{aligned} &-\int_0^\infty (u_m, \psi(t)'w_j)dt - \int_0^\infty (\Delta u_m, \psi(t)w_j)dt + \int_0^\infty \mathbf{b}(u_m, u_m, \psi(t)w_j)dt\\ &= \int_0^\infty (g_m(t), \psi(t)w_j)dt. \end{aligned} \tag{73}$$

Note that $\{u_m\}$ is bounded in $W^{1,2}((0,T'); V_{\text{as}}, H_{\text{as}})$ and u_m converges weakly to v in $W^{1,2}((0,T); V_{\text{as}}, H_{\text{as}})$. Since V_{as} is dense in H_{as} and the embedding $V_{\text{as}} \hookrightarrow H_{\text{as}}$ is compact, the Aubin theorem implies that the embedding

$$W^{1,2}((0,T'); V_{\text{as}}, H_{\text{as}}) \hookrightarrow L^2((0,T'); H_{\text{as}})$$

is compact. Hence u_m converges strongly to v in $L^2((0,T'); H_{\text{as}})$. By [31, Lemma 3.2, p. 289], we have

$$\int_0^{T'} \mathbf{b}(u_m, u_m, \psi(t)w_j)dt \to \int_0^{T'} \mathbf{b}(v, v, \psi(t)w_j)dt \quad \text{when} \quad m \to \infty.$$

This means that

$$\int_0^\infty \mathbf{b}(u_m, u_m, \psi(t)w_j)dt \to \int_0^\infty \mathbf{b}(v, v, \psi(t)w_j)dt \quad \text{as} \quad m \to \infty.$$

Taking the limit both sides of (72) and (73), we obtain

$$\int_0^\infty (\phi, \psi(t)w_j)dt - \int_0^\infty (\Delta v, \psi(t)w_j)dt + \int_0^\infty \mathbf{b}(v, v, \psi(t)w_j)dt = \int_0^\infty (f(t), \psi(t)w_j)dt$$

and

$$-\int_0^\infty (v, \psi(t)'w_j)dt - \int_0^\infty (\Delta v, \psi(t)w_j)dt + \int_0^\infty \mathbf{b}(v, v, \psi(t)w_j)dt = \int_0^\infty (f(t), \psi(t)w_j)dt.$$

From the above, we see that $(\phi, w_j) = (\frac{dv}{dt}, w_j)$ and so

$$(\frac{d}{dt}v, w_j) + (\widetilde{A}v, w_j) + (B(v, v), w_j) = (f(t), w_j) \ \forall j \geq 1.$$

Hence

$$(\frac{dv}{dt}, w) + (\widetilde{A}v, w) + (B(v, v), w) = (f(t), w) \ \forall w \in H_{\text{as}}$$

or equivalently,

$$(\frac{dv}{dt} + \widetilde{A}v + B(v, v) - f(t), w) = 0 \ \forall w \in H_{\text{as}}.$$

Since $\frac{dv}{dt} + \widetilde{A}v + B(v, v) - f(t) \in H_{\text{as}}$, we must have

$$\frac{dv}{dt} + \widetilde{A}v + B(v, v) - f(t) = 0.$$

Let us choose a continuously differentiable function ψ on $[0, \infty)$ such that $\psi(0) = 1$ and $\psi(t) = 0$ for all $t \geq T'$ for some $0 < T' < \infty$. Taking the scalar product of (56) with $\psi(t)w_j$ again and using similar arguments as in the proof of [31, Theorem 3.1, p. 289], we can show that $v(0) = v_0$. The energy inequalities (4) and (5) follows from (61) and (62). The proof of Theorem 1.1 is complete. □

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