Computation of the Łojasiewicz exponent for a germ of a smooth function in two variables

Ha Huy Vui *
Institute of Mathematics
18 Hoang Quoc Viet, Cau Giay District
10307, Ha noi, Viet Nam
Email: hhvui@math.ac.vn

Abstract

Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ be a germ of a smooth function. We give a sufficient condition such that the Łojasiewicz inequality holds, i.e. there exist a neighborhood Ω of the origin and constants $c, \alpha > 0$ such that

$$|f(x)| \geq cd(x, f^{-1}(0))^{\alpha}$$

for all $x \in \Omega$. Then, under this condition, we compute the Łojasiewicz exponent of f. As a by-product we obtain a formula for the Łojasiewicz exponent of a germ of an analytic function, which is different from that of T.C.Kuo (Comment. Math. Helv. 49 (1974), pp.201-213.

1 Introduction

Let f be a germ of an analytic function at the origin of $\mathbb{R}^n$. Let $V := \{x \in \mathbb{R}^n | f = 0\}$. The Łojasiewicz inequality says that there exist $c > 0$ and $\alpha > 0$ and a neighbourhood Ω of the origin such that

$$|f(x)| \geq cd(x, V)^{\alpha} \tag{1.1}$$

for all $x \in \Omega$.

The Łojasiewicz inequality was born to solve a problem in analysis [L1].

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Later, it finds many applications and generates many results in other fields [Bi-M], [Br], [H], [Kur-M-P], [T].

Let $L(f) := inf\{\alpha | \exists c > 0$ and Ω s.t. (1) holds $\}$ be the *Łojasiewicz exponent* of f.
If $f(x,y)$ is an analytic function, $L(f)$ is computed via Puiseux expansions of f [Kuo].

In general, Inequality (1) is false for smooth functions. In this paper we assume f to be a germ of a smooth function in two variables. Firstly, we give a sufficient condition such that inequality (1) holds for f. Then, under this condition, the Łojasiewicz exponent $L(f)$ is computed explicitly.
Our method is based on the smooth Puiseux theorem of V. Rychkov [R] and the following observation: let $f(x_1, x_2, ..., x_n) = c_d x_n^d + c_{d-1}(x_1, ..., x_{n-1}) x_n^{d-1} + ... + c_0(x_1, ..., x_{n-1})$ be the Malgrange-Weierstrass form of f, where d is the multiplicity of f at the origin, then the set $\{x \in \mathbb{R}^n | \dfrac{\partial f}{\partial x_n} = 0\}$ can be considered as a testing set for the existence of the Łojasiewicz inequality for f. As a by-product we obtain a formula for the Łojasiewicz exponent of a germ of an analytic function, which is different from that of [Kuo].

Recently, a version of the Łojasiewicz inequality for smooth functions was given in [H-N-S].

2 Statements of results

Lemma 2.1. *Let $f(x', x_n) = x_n^d + a_{d-1}(x') x_n^{d-1} + ... + a_0(x')$ be a germ of a smooth function at $0 \in \mathbb{R}^{n-1} \times \mathbb{R}$. Assume that, for some constants $c_0 > 0, \epsilon > 0$ and $\beta > 0$ we have*

$$|f(x)| \geq c_0 dist(x, V)^\beta \tag{2.1}$$

for all $x \in V_1 \cap B_\epsilon$, where $V_1 := \{x \in \mathbb{R}^n | \dfrac{\partial f}{\partial x_n} = 0\}$ and $B_\epsilon = \{x \in \mathbb{R}^n | \|x\| < \epsilon\}$. Then there exist a neighbourhood Ω of $0 \in \mathbb{R}^n$ and an $c > 0$ such that

$$|f(x)| \geq c dist(x, V)^\alpha$$

for all $x \in \Omega$, where $\alpha = max\{d, \beta\}$.

Let $L(f, V_1) :=$ inf $\{\beta > 0 | \exists c, \epsilon > 0$ such that (2.1) holds $\}$. We call $L(f, V_1)$ the *Łojasiewicz exponent of f with respect to V_1*.
Our problems now become:

i) To find conditions under which $L(f, V_1) < \infty$.

ii) To compute $L(f, V_1)$ and then to compute $L(f)$.

We recall now the smooth Puiseux theorem of V. Rychkov.
Let $\mathbb{C}[[x]]$ be the ring of formal series with coefficients in $\mathbb{C}$. We use notation of following rings of germs of $\mathbb{C}-$ valued functions.

- $C^\infty((x))$ and $C^\infty((x,y))$ are rings of C^∞ functions at the origin of $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

- $C_+((x))$ and $C_+^\infty((x))$ are rings of one-sided germs, consisting of (the equivalent classes of) functions $g(x)$ defined in a left half -neighbourhood of zero of the form $[0, \epsilon)$, where $\epsilon > 0$ can depend on $g(x)$, which are continuous, respectively C^∞, up to zero.

- $A_+((x^\gamma)), \gamma > 0$, the subring of $C_+((x))$ consisting of germs $g(x)$, for which there exists a series $\widetilde{g}(x) \in \mathbb{C}[[x^\gamma]]$, $\widetilde{g}(x) = \sum_{n=0}^\infty c_n x^{n\gamma}$, such that $g(x) \sim \widetilde{g}(x)$ in the sense that for any N

$$g(x) - \sum_{n=0}^{N} c_n x^{n\gamma} = O(x^{(N+1)\gamma}), x \longrightarrow 0,$$

such an $\widetilde{g}(x)$ is uniquely determined and is called the asymptotic expansion of $g(x)$.

The ring of germs of $\mathbb{R}$ - valued functions will be denoted by adding an $\mathbb{R}$ to the above notation, e.g. $\mathbb{R}A_+((x))$, $\mathbb{R}C_+((x))$.
Let $F(x,y) \in \mathbb{R}C^\infty((x,y))$ and $\overline{F}(x,y) = \sum c_{ij}x^i y^j$ be the formal Taylor series of F at the origin. Put

$$supp(F) = \{(i,j) \in \mathbb{R}^2 | c_{ij} \neq 0\}$$

Let $\Gamma(F)$ be the convex hull of the set

$$\{(i,j) + \mathbb{R}_+^2 | (i,j) \in supp(F)\}$$

Then $\Gamma(F)$ is the Newton polygon of F. Let E be a compact edge of the boundary of $\Gamma(F)$ joining points (A_E, B_E) and (A'_E, B'_E), where $B'_E > B_E$, we put

$$n_E = B'_E - B_E, \gamma_E = \frac{A_E - A'_E}{B'_E - B_E}$$

Let A be the x-coordinate of the vertical infinite edge and B be the y-coordinate of the horizontal infinite edge of $\Gamma(F)$.

Let $F(x,y) \in \mathbb{R}C^\infty((x,y))$ such that its formal Taylor series $\overline{F}$ is different from 0. Assume that the formal Taylor $\overline{F}$ of $F(x,y)$ at the origin is not identically equal to zero.

Theorem 2.2. *[R] The germ $F(x,y)$ admits in region $x, y > 0$ a factorization of the form*

$$F(x,y) = u(x,y)\prod_{i=1}^{A}(x - X_i(y))\prod_{i=1}^{B}(y - Y_i(x))\prod_{E}\prod_{i=1}^{n_E}(y - Y_{E_i}(x))$$

where

i) $u(x,y) \in \mathbb{R}C^\infty((x,y)), u(0,0) \neq 0$

ii) *all* $X_i(y) \in C_+((y))$, *(resp.* $Y_i(x) \in C_+((x))$*) and* $X_i(y) = O(y^N)$, *as* $y \longrightarrow 0$ *(resp.* $Y_i(x) = O(x^N)$, *as* $x \longrightarrow 0$*) for any* $N > 0$

iii) *all* $Y_{E_i}(x) \in A_+((x^{\frac{1}{n!}}))$ *for* $n = B + \sum_E n_E$ *with asymptotic expansions of the form* $Y_{E_i}(x) = c_{E_i}x^{\gamma_E} + ...$, *as* $x \longrightarrow 0$, *where* $c_{E_i} \neq 0$.

Remark 2.3. The asymptotic series $\widetilde{Y}_{E_i}(x) = c_{E_i}x^{\gamma_i} + ... \in \mathbb{C}[[x^{\frac{1}{n!}}]]$ of $Y_{E_i}(x)$ is computed from the formal Taylor series $\overline{F}$ of F by the Newton-Puiseux algorithm which is described in [W].

Let $f(x,y) \in \mathbb{R}C^\infty((x,y))$ and $d := min\{i + j | \frac{\partial^{i+j}f}{\partial x^i \partial y^j}(0,0) \neq 0\}$ be the multiplicity of f at the origin. By the Malgrange preparation theorem [M], we can assume that f is of the form

$$f(x,y) = y^d + a_{d-1}(x)y^{d-1} + ... + a_0(x)$$

where $a_i(x) \in C^\infty((x)), a_i(0) = 0, i = 0, 1, ..., d-1$.
It follows from Theorem 2.2 that

i) $f(x,y)$ admits in the region $x > 0$ a factorization

$$f(x,y) = \prod_{i=1}^{B}(y - Y_i(x))\prod_{E}\prod_{i=1}^{n_E}(x - Y_{E_i}(x)) \tag{2.2}$$

where $Y_i(x) \in C_+((x))$ and $Y_i(x) = O(x^N)$, as $x \longrightarrow 0$, for any $N > 0, Y_{E_i}(x) \in A_+((x^{\frac{1}{d!}})), B$ is the y-coordinate of the infinite horizontal edge of $\Gamma(f)$ and E runs through all compact edges of $\Gamma(f)$.

ii) $\dfrac{\partial f}{\partial y}(x,y)$ admits in the region $x>0$ a factorization

$$(2.3) \qquad \frac{\partial f}{\partial y}(x,y) = \prod_{i=1}^{B'}(y - Y'_i(x)) \prod_{E'} \prod_{i=1}^{n_{E'}} (y - Y_{E'_i}(x))$$

where $Y'_i(x) \in C_+((x))$ and $Y'_i(x) = O(x^N)$, as $x \longrightarrow 0$, for any $N>0, Y_{E'_i}(x) \in A_+((x^{\frac{1}{(d-1)!}})), B'$ is the y-coordinate of the infinite horizontal edge of $\Gamma(\dfrac{\partial f}{\partial y})$ and E' runs through all compact edges of $\Gamma(\dfrac{\partial f}{\partial y})$.

Notation 2.4. *Let $\mathcal{P}_{\mathbb{C}}(f)$ denote the sets of all functions $Y_i(x)$ and $Y_{E_i}(x)$ in (2.2) and $\mathcal{P}_{\mathbb{C}}(\dfrac{\partial f}{\partial y})$ denote the set of all functions $Y'_i(x), Y'_{E'_i}(x)$ in (2.3).*

Put

$$\mathcal{P}_{\mathbb{R}}(f) := (\mathcal{P}_{\mathbb{C}}(f) \cap \mathbb{R}C_+((x))) \bigcup (\mathcal{P}_{\mathbb{C}}(f) \cap \mathbb{R}A_+((x^{\frac{1}{d!}})))$$

and

$$\mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) := (\mathcal{P}_{\mathbb{C}}(\frac{\partial f}{\partial y}) \cap \mathbb{R}C_+((x))) \bigcup (\mathcal{P}_{\mathbb{C}}(\frac{\partial f}{\partial y}) \cap \mathbb{R}A_+((x^{\frac{1}{(d-1)!}})))$$

Then, as germs of sets at the origin, we have

$$V^+ := V \cap \{(x,y) \in \mathbb{R}^2 | x \geq 0\} = \bigcup_{Y \in \mathcal{P}_{\mathbb{R}}(f)} \{(x,y) \in \mathbb{R}^2 | x \geq 0, y = Y(x)\}$$

and

$$V_1^+ := V_1 \cap \{(x,y) \in \mathbb{R}^2 | x \geq 0\} = \bigcup_{Y' \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})} \{(x,y) \in \mathbb{R}^2 | x \geq 0, y = Y'(x)\}.$$

For two real value functions $\varphi(x) > 0$ and $\psi(x) > 0$ we write $\varphi \asymp \psi$ if and only if $\dfrac{\varphi}{\psi}$ lies between two positive constants, as x is near $0 \in \mathbb{R}$.
If $\varphi(x) \asymp x^s$ for some s, we put $v(\varphi) := s$. If $\varphi(x) = o(x^N)$ as $x \longrightarrow 0$ for any $N > 0$, we put $v(\varphi) := \infty$. If $\varphi(x)$ and $\psi(x)$ are two functions, we put $\theta(\varphi, \psi) := v(|\varphi - \psi|)$ and call it the *contact order* of φ and ψ.
For each $Y' \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ we denote

$$\theta_{Y'} = \sum_{Y \in \mathcal{P}_{\mathbb{C}}(f)} \theta(Y', Y).$$

Clearly $\theta_{Y'} \in \mathbb{Q} \cup \{\infty\}$.

We also put

$$(2.4) \qquad \delta_{Y'} := \begin{cases} 1 & \text{if } \mathcal{P}_{\mathbb{R}}(\mathrm{f}) = \emptyset \\ max\{\theta(Y', Y) | Y \in \mathcal{P}_{\mathbb{R}}(f)\} & \text{if otherwise} \end{cases} .$$

Definition 2.5. We say that f satisfies the *finite contact condition* if for every $Y' \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) \setminus \mathcal{P}_{\mathbb{C}}(f)$, we have $\theta_{Y'} < \infty$.

Theorem 2.6. *Let $f(x,y) = y^d + a_{d-1}(x)y^{d-1} + ... + a_0(x) \in \mathbb{R}C^{\infty}((x,y))$, where d is the multiplicity of f at $0 \in \mathbb{R}^2$. Assume that f satisfies the finite contact condition, then the number*

$$L_+(f, V_1) = max\{\frac{\theta_{Y'}}{\delta_{Y'}}\}$$

where Y' runs through all elements of $\mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) \setminus \mathcal{P}_{\mathbb{R}}(f)$, has the property that there exist $c > 0$ and $\epsilon > 0$ such that

$$|f(x,y)| \geq cdist((x,y), V)^{L_+(f;V_1)}$$

for all $(x,y) \in V_1 \cap \{(x,y) \in \mathbb{R}^2 | x \geq 0\} \cap B_\epsilon$, where $B_\epsilon = \{(x,y) \in \mathbb{R}^2 | \|(x,y)\| < \epsilon\}$. Moreover, $L_+(f;V_1)$ is the smallest number with this property.

Corollary 2.7. *Assume that both $f(x,y)$ and $\widetilde{f}(x,y) = f(-x,y)$ satisfy the finite contact condition then we have*

$$L(f;V_1) = max\{L_+(f;V_1), L_+(\widetilde{f};V_1)\}$$

Theorem 2.8. *Let $f(x,y) = y^d + a_{d-1}(x)y^{d-1} + ... + a_0(x) \in \mathbb{R}C^{\infty}((x,y))$, where d is the multiplicity of f at $0 \in \mathbb{R}^2$. Assume that both $f(x,y)$ and $f(-x,y)$ satisfy the finite contact condition, then we have*

$$L(f) = max\{L(f;V_1), d\}.$$

Remark 2.9. The condition of the existence of the Łojasiewcz inequality given in Theorem 2.8 is not sharp, as it is shown by the following example. Let $f(x,y) = y^2 - e^{-1/x^2}$. Then the ideal $f\mathcal{C}^{\infty}(\overline{B_\epsilon})$ is closed, hence there exist $c > 0$ and $\alpha > 0$ such that

$$|f(x,y)| \geq cd((x,y), f^{-1}(0))^{\alpha}$$

for all $(x,y) \in \overline{B_\epsilon} = \{(x,y) \in \mathbb{R}^2 | \ ||(x,y)|| \leq \epsilon\}$ (see [M, §2, Chapter VI].)

Nevertheless, f does not satisfy the finite contact condition.

Now we consider the two following cases, where the finite contact condition holds automatically

(i) f is a germ of a smooth function, which is non–degenerate with respect to its Newton polygon;

(ii) f is a germ of an analytic function.

Case 1:

Let denote the formal Taylor series of f at the origin by

$$\overline{f}(x,y) = \sum_{i+j=0}^{\infty} c_{ij}x^i y^j$$

For a compact edge E of $\Gamma(f)$, we put

$$f_E = \sum_{(i,j)\in E} c_{ij}x^i y^j$$

Definition 2.10. [Ko] We say that f is *nondegenerate* if for any compact edge E, the system

$$\frac{\partial f_E}{\partial x} = \frac{\partial f_E}{\partial y} = 0$$

has no solutions in $(\mathbb{R}\setminus 0)^2$.
We say that f is *convenient* if $\Gamma(f)$ intersects both coordinate axes.

For each compact edge of $\Gamma(f)$, joining the points (A_E, B_E) and (A'_E, B'_E), where $B'_E > B_E$, let as before $\gamma_E = \dfrac{A_E - A'_E}{B'_E - B_E}$. We put

$$\gamma(f) := \{\gamma_E | E \text{ runs through all compact edges of } \Gamma(f)\}.$$

Theorem 2.11. *Let $f(x,y) = y^d + a_{d-1}(x)y^{d-1} + ... + a_0(x) \in \mathbb{R}C^\infty((x,y))$, where d is the multiplicity of f at $0 \in \mathbb{R}^2$. Assume that f is convenient and nondegenerate. Then the Łojasiewicz exponent of f is finite and can be expressed in terms of the set $\gamma(f)\cup\{d\}$.*

Case 2:

Let $f(x,y)$ be a germ of an analytic function at $0 \in \mathbb{R}^2$. Then the sets $\mathcal{P}_{\mathbb{C}}(f)$, $\mathcal{P}_{\mathbb{R}}(f)$ (resp., $\mathcal{P}_{\mathbb{C}}(\dfrac{\partial f}{\partial y})$, $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$) are respectively the sets of complex and real Puiseux series of $f = 0$ (resp., of $\dfrac{\partial f}{\partial y} = 0$). Clearly, the finite contact condition holds automatically. Hence, Theorem 2.8 gives the following method of computation of the Łojasiewicz exponent of an analytic germ.

Firstly we find all the Puiseux series of $f = 0$ and $\dfrac{\partial f}{\partial y} = 0$. This gives the sets $\mathcal{P}_{\mathbb{C}}(f)$ and $\mathcal{P}_{\mathbb{C}}(\dfrac{\partial f}{\partial y})$. Then, in assuming that we are in the domain $\mathbb{R}^2_{\geq 0}$, we define the sets $\mathcal{P}_{\mathbb{R}}(f)$ and $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ of real Puiseux series. We have

$$L_+(f; V_1) = \max\{\frac{\theta_{Y'}}{\delta_{Y'}}\},$$

where Y' runs through the set $Y' \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \setminus \mathcal{P}_{\mathbb{C}}(f)$. Then, we compute $L_+(\tilde{f}; V_1)$, $\tilde{f} = f(-x, y)$, by similar way and finally we use Theorem 2.8 to get the Łojasiewicz exponent of an analytic germ. This method of computation of $L(f)$ for analytic germs is different from that of [Kuo] in the following aspect: instead of using the so called *real approximations* of Puiseux series of $f = 0$ to express $L(f)$ as in [Kuo], we use real Puiseux series of $\dfrac{\partial f}{\partial y}$ and Puiseux series of $f = 0$.

Let us use this method to compute the exponent $L_+(f, V_1)$ for some examples taken from [Kuo].

Example 2.12. (Example (3.6) of [Kuo]).

$$f(x, y) = (y - x^2)(y^4 + x^{10})$$

$$\frac{\partial f}{\partial y} = 0 \Leftrightarrow 5y^4 - 4y^3x^2 + x^{10} = 0$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives 4 Puiseux series}$$

$$Y_1' := Y_1(x) = \frac{4}{5}x^2 + ...$$

$$Y_2' := Y_2(x) = \left(\frac{1}{4}\right)^{1/3} x^{8/3} + ...$$

$$Y_3' := Y_3(x) = \left(\frac{1}{4}\right)^{1/3} e^{i\pi/3} x^{8/3} + ...$$

$$Y_4' := Y_4(x) = \left(\frac{1}{4}\right)^{1/3} e^{i2\pi/3} x^{8/3} + ...$$

Clearly $Y_3', Y_4' \notin \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$. We show that $Y_1', Y_2' \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$.
Let $Y_1(x) = x^2\left(\dfrac{4}{5} + \varphi(x)\right)$ with $v(\varphi(x)) > 0$. Since $\dfrac{\partial f}{\partial y}(x, Y_1(x)) = 0$ we have

$$0 = 6\left(\frac{4}{5}\right)^2 \varphi + x^2 + \text{ terms of degree } > 1 \text{ in } \varphi.$$

Hence, it follows from Implicit function theorem that $\varphi(x)$ is an analytic function and therefore $Y_1(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$. Since the number of non-real Puiseux series is even, Y_2' also belongs to $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$. We find that $\theta_{Y_1'} = 10$, $\theta_{Y_2'} = 12$.
Since $\mathbb{R}^2 \cap \{f = 0\} \Leftrightarrow \{y - x^2 = 0\}$, we have

$$\delta_{Y_1'} = \delta_{Y_2'} = 2.$$

Hence $L(f) = \max\{5, 6\} = 6$.

Example 2.13. (Example (3.8) of [Kuo]). This example is due to Łojasiewicz [L2]. It shows that the Łojasiewicz exponent can be larger than the degree of a polynomial.

$$f(x,y) = y^{2n} + (y - x^n)^2$$
$$\frac{\partial f}{\partial y} = 2ny^{2n-1} + 2(y - x^n)$$

Hence $\dfrac{\partial f}{\partial y} = 0$ gives a single Puiseux series

$$Y' := y(x) = x^n + ...$$

which belongs to $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ by Implicit function theorem. More careful computation gives

$$y(x) = x^n - nx^{2n^2-n} + \text{ terms of degree } > 2\mathrm{n}^2 - \mathrm{n}.$$

Hence $\theta_{Y'} = v(f(x, y(x))) = 2n^2$.
Since $\mathcal{P}_{\mathbb{R}}(f) = \emptyset$, $\delta_{Y'} = 1$. Hence $L(f) = 2n^2$.

3 Proofs

3.1 Proof of Lemma 2.1

Proof. Let us begin with determining the neighbourhood Ω. We fix two points $(0, a)$ and $(0, -a) \in (\mathbb{R}^{n-1} \times \mathbb{R}) \cap B_\epsilon$. Since $c_i(0) = 0, i = 0, 1, ..., d-1$, we have $|f(0,-a)| = |f(0,a)| = |a|^d$. Hence, we can find an $b > 0$ such that if $\|x'\| < b$ then

$$min\{|f(x',a)|, |f(x',-a)|\} \geq \frac{|a|^d}{2}.$$

We put $\Omega := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | \|x'\| < b, |x_n| < |a|\}$.
Let $\Omega_1 := \{(x', x_n) \in \Omega | f \geq \frac{|a|^d}{2}\}$ and $\Omega_2 := \Omega \setminus \Omega_1$.
Clearly, one can find $c > 0$ such that $|f(x)| \geq c dist(x, V)^\alpha$, for all $x \in \Omega_1$ where $\alpha = max\{\beta, d\}$.
Let $x_0 \in \Omega_2$, we denote $r = |f(x_0)|$ and

$$W(x_0) := \{x \in \mathbb{R}^n | |f(x)| \leq r\} \cap V_1.$$

Claim 3.1.

$$|f(x_0)| \geq (\frac{1}{d!2d})^d [min\{dist(x_0, V), dist(x_0, W(x_0)\}]^d.$$

Proof. Put $\sum(x'_0) = \{x_n \in [-a, +a] | |f(x'_0, x_n)| \leq r\}$. Then $\sum(x'_0)$ is a non-empty closed semi-algebraic set in $\mathbb{R}^1$, hence it is the finite union of closed intervals and isolated points. Since $x_0 \in \Omega_2$, we see that $a \notin \sum(x'_0)$ and $-a \notin \sum(x'_0)$. Hence, if $[a_i, b_i] \subset \sum(x'_0)$, we have $|f(x'_0, a_i)| = |f(x'_0, b_i)| = r_0$ and $|f(x'_0, t)| < r$ for any $t \in (a_i, b_i)$. Clearly, if c_j is an isolated point of $\sum(x'_0)$, we also have $|f(x'_0, c_j)| = r$.
Since $\frac{\partial^d f(x'_0, x_n)}{\partial x_n^d} = d!$, it follows from the Van der Corput lemma [C-C-W] that the following estimate for the Lebesgue's measure of $\sum(x'_0)$ holds:

$$mes \sum(x'_0) \leq d!(2d) r^{\frac{1}{d}}$$

Consequently,

$$|b_i - a_i| \leq d!(2d) r^{\frac{1}{d}} \tag{3.1}$$

for each $[a_i, b_i] \subset \sum(x'_0)$.
Since $|f(x_0)| = r$, either the point x_{n_0} coincides with one of a_i and b_i or it is an isolated point of $\sum(x'_0)$. Assume that $x_{n_0} = a_i$.

i) If $f(x'_0, x_{n_0}).f(x'_0, b_i) < 0$, then there exists $t_0 \in (x_{n_0}, b_i)$ such that $f(x'_0, t_0) = 0$, i.e. $(x'_0, t_0) \in V$. It follows from (3.1) that

$$dist(x_0, V) \leq |x_{n_0} - t_0| \leq |x_{n_0} - b_i| \leq d!(2d) r^{\frac{1}{d}}$$

or

$$|f(x_0)| \geq (\frac{1}{d!2d})^d dist(x_0, V)^d$$

and claim 3.1 holds.

ii) Assume that $f(x'_0, x_{n_0}).f(x'_0, b_i) > 0$, then, by Rolle's theorem there exists $t_1 \in (x_{n_0}, b_i)$ such that $\frac{\partial f}{\partial x_n}(x'_0, t_1) = 0$, i.e., $(x'_0, t_1) \in W(x'_0)$ and the claim holds again by (3.1). Now assume that x_{n_0} is an isolated point of $\sum(x'_0)$. It is easy to see that x_{n_0} is a local minimum of a function $g(x_n) := f(x'_0, x_n)$, therefore $\frac{\partial f}{\partial x_n}(x'_0, x_{n_0}) = 0$ and the claim holds automatically.

□

Now we derive the proof of Lemma 2.1 from Claim 3.1. Let $\varphi(x_0)$ and $\psi(x_0)$ be the points of $W(x_0)$ and V, respectively, such that $dist(x_0, W(x_0)) = dist(x_0, \varphi(x_0))$ and $dist(\varphi(x_0), V) = dist(\varphi(x_0), \psi(x_0))$.
We have

$$dist(x_0, V) \leq dist(x_0, \psi(x_0)) \leq dist(x_0, \varphi(x_0)) + dist(\varphi(x_0), \psi(x_0))$$

If $dist(x_0, \varphi(x_0)) \geq dist(\varphi(x_0), \psi(x_0))$ then $dist(x_0, V) \leq 2dist(x_0, \varphi(x_0)) = 2dist(x_0, W(x_0))$. Hence, by Claim 3.1, we have
(3.2)

$$|f(x_0)| \geq (\frac{1}{d!2d})^d min\{dist(x_0, V), dist(x_0, W(x_0))\}^d \geq (\frac{1}{2^d d!2d})^d dist(x_0, V)^d$$

Assume that $dist(x_0, \varphi(x_0)) \leq dist(\varphi(x_0), \psi(x_0))$, then $dist(x_0, V) \leq 2dist(\varphi(x_0), \psi(x_0))$.
Since $\varphi(x_0) \in W(x_0)$, $|f(x_0)| \geq |f(\varphi(x_0))|$, by the hypothesis, we have
(3.3)

$$|f(x_0)| \geq |f(\varphi(x_0))| \geq c_0 dist(\varphi(x_0), V)^\beta \geq c_0(\frac{dist(x_0, V)}{2})^\beta = \frac{c_0}{2^\beta} dist(x_0, V)^\beta$$

Lemma 2.1 follows from (3.2) and (3.3). □

Lemma 3.2. *Let $Y(x) \in \mathcal{P}_{\mathbb{R}}(f)$ and X_Y be the germ at the origin of the set $\{(x, y) \in \mathbb{R}^2 | x \geq 0, y = Y(x)\}$. Let $Y'(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$. Then we have*

$$dist((x, Y'(x)), X_Y) \asymp |Y'(x) - Y(x)|.$$

Proof. We recall that $f(x, y) \in \mathbb{R}C^\infty((x, y))$ and is of the form

$$f(x, y) = y^d + a_{d-1}(x)y^{d-1} + ... + a_0(x),$$

where d is the multiplicity of f at the origin. Hence, $f(x, y)$ admits in the region $x > 0$ a functorization of the form (3), i.e.

$$f(x, y) = \prod_{i=1}^{B}(y - Y_i(x)) \prod_{E} \prod_{i=1}^{n_E}(x - Y_{E_i}(x)),$$

where B is the $y-$coordinate of the infinity horizontal edge of $\Gamma(f)$ and E runs through all compact edges of $\Gamma(f)$. Each function $Y_i(x) \in C_+((x))$, in particular, $Y_i(x) = O(x^N)$ as $x \to 0$, for any $N > 0$. Each $Y_{E_i}(x) \in A_+((x^{\frac{1}{d!}}))$ and has asymptotic expansion of the form $Y_{E_i}(x) = c_{E_i} x^{\gamma_E} + ...$, as $x \to 0$, with $c_{E_i} \neq 0$.

Claim 1. For each compact edge E of $\Gamma(f)$, we have $\gamma_E \geq 1$.

Proof. Since the multiplicity of f at the origin is equal to d, it is easy to see that the line $i + j = d$ is a support line of $\Gamma(f)$. Hence, all the compact edges of $\Gamma(f)$, except possibly the edge containing the vertex $(0, d)$, are situated strictly higher the line $i + j = d$. Therefore, the inequality $\gamma_E \geq 1$ follows trivially from the definition of γ_E.

Let W be a neighbourhood of $(0, 0)$ and $\widetilde{W} := W \cap \{(x, y) \in \mathbb{R}^2 | x > 0\}$. Let

$$\phi : \widetilde{W} \longrightarrow \phi(\widetilde{W}),$$

defined by $\phi(x, y) = (u, v)$, with $u = x, v = y - Y(x)$.

Claim 2. If $Y(x) \in P_{\mathbb{R}}(f)$, then there exist constants $c_1 > 0$ and $c_2 > 0$ such that for any two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ from $\widetilde{W}$, we have

$$c_1 ||z_1 - z_2|| \leq ||\phi(z_1) - \phi(z_2)|| \leq c_2 ||z_1 - z_2||. \tag{3.4}$$

Proof. It is convenient to use l^1 norm in $\mathbb{R}^2$, i.e.

$$||(x, y)||_{l^1} := |x| + |y|.$$

Let $Y(x) \in C_+((x))$, we have

$$||z_1 - z_2||_{l^1} = ||(x_1, y_1) - (x_2, y_2)||_{l^1} = |x_1 - x_2| + |y_1 - y_2| =$$

$$= |x_1 - x_2| + |y_1 - Y(x_1) - (y_2 - Y(x_2)) + (Y(x_1) - Y(x_2))|$$

$$\geq |x_1 - x_2| + |y_1 - Y(x_1) - (y_2 - Y(x_2))| - |(Y(x_1) - Y(x_2))|.$$

Since $Y(x) \in C_+((x))$, if W is sufficiently small, we have

$$|Y(x_1) - Y(x_2)| \leq \frac{1}{2}|x_1 - x_2|.$$

We obtain that

$$||z_1 - z_2||_{l^1} \geq |x_1 - x_2| + |(y_1 - Y(x_1)) - (y_2 - Y(x_2))| - \frac{1}{2}|x_1 - x_2|$$

$$\geq \frac{1}{2}[|x_1 - x_2| + |(y_1 - Y(x_1)) - (y_2 - Y(x_2))|] = \frac{1}{2}||\phi(z_1) - \phi(z_2)||_{l^1}.$$

Conversely,

$$||\phi(z_1)-\phi(z_2)||_{l^1} = |x_1-x_2| + |(y_1-Y(x_1))-(y_2-Y(x_2))|$$

$$\geq |x_1-x_2|+|y_1-y_2|-|Y(x_1)-Y(x_2)| \geq |x_1-x_2|+|y_1-y_2|-\frac{1}{2}|x_1-x_2|$$

$$\geq \frac{1}{2}||z_1-z_2||_{l^1}.$$

Thus, Claim 2 holds if $Y(x)\in C_+((x))$.

Assume that $Y(x)=Y_{E_i}(x)\in A_+((x^{\frac{1}{d!}}))$. Then, $Y(x)=c_{E_i}x^{\gamma_E}+o(x^{\gamma_E})$, as $x\to 0$.

By Claim 1, $\gamma_E \geq 1$. We distinguish two cases. Firstly, assume that $\gamma_E>1$, then it is easy to see that

$$|Y(x_1)-Y(x_2)| \leq \frac{1}{2}|x_1-x_2|,$$

for any couple of points $x_1>0$ and $x_2>0$, sufficiently closed to 0. Then, analogously as above, we can see that Claim 2 holds. Now, assume that $\gamma_E=1$, then

$$Y(x)=c_{E_i}x+cx^{\gamma}+o(x^{\gamma}),\ \text{as x}\to 0,\ \text{where } \gamma>1.$$

Let

$$\phi_0:\widetilde{W}\longrightarrow \phi_0(\widetilde{W})$$
$$(x,y)\mapsto (x,y-c_{E_i}x)$$

and

$$\phi_1:\phi_0(\widetilde{W})\longrightarrow \phi(\widetilde{W})$$
$$(x,y)\mapsto (x,y-(Y(x)-c_{E_i}x)).$$

Then we have $\phi=\phi_1\circ\phi_0$. Clearly,

$$c_1'||z_1-z_2|| \leq ||\phi_0(z_1)-\phi_0(z_2)|| \leq c_2'||z_1-z_2|| \tag{3.5}$$

holds for some $c_1'>0$ and $c_2'>0$.

Moreover, since $Y-c_Ex=cx^{\gamma}+o(x^{\gamma})$, where $\gamma>1$, we can see that the inequalities

$$c_1''||z_1-z_2|| \leq ||\phi_1(z_1)-\phi_1(z_2)|| \leq c_2''||z_1-z_2|| \tag{3.6}$$

hold for some $c_1''>0$ and $c_2''>0$. The Claim 2 follows then from (3.5) and (3.6).

Now we finish the proof of Lemma 3.2.

We see that

$$\phi(X_Y) = \{v = 0\} \subset \mathbb{R}^2_{u,v}$$

and $\phi((x, Y'(x))) = (u, v)$ with $u = x$, $v = Y'(x) - Y(x)$, hence

$$dist(\phi((x, Y'(x)), \phi(X_Y)) = |Y'(x) - Y(x)|.$$

The proof of Lemma 3.2 follows from this equality and Claim 2.

□

3.2 Proof of Theorem 2.6

Theorem 2.6 follows directly from factorizations of the germs f and $\dfrac{\partial f}{\partial y}$, Lemma 3.2 and the definition of the number $\theta_{Y'}$ and $\delta_{Y'}$.

3.3 Proof of Theorem 2.8

Proof. Clearly $L(f; V_1) \leq L(f)$. By Lemma 2.1, $L(f) \leq max\{L(f; V_1), d\}$
Hence, it is enough to prove that $L(f) \geq d$.
We will use the max–norm in $\mathbb{R}^2$, i.e. $||(x, y)|| = \max\{|x|, |y|\}$.

Claim 3.3. $dist((0, y), V) \asymp |y|$

Proof. Since $(0, 0) \in V$, $dist((0, y), V) \leq |y|$.
We will prove that there exists $c > 0$ such that $|y| \leq dist((0, y), V)$ for y sufficiently small. Let $dist((0, y), V) = dist((0, y), A(y))$, with $A(y) \in V$. We may assume that $A(y)$ is of the form $A(y) = (\tau(y), Y(\tau(y)))$, where $Y(x) \in \mathcal{P}_{\mathbb{R}}(f)$.
Then we have
(3.7)
$$|y| \leq dist((0, y), (\tau(y), Y(\tau(y)))) + |Y(\tau(y))| = max\{|\tau(y)|, |y - Y(\tau(y))|\} + |Y(\tau(y))|$$

Since $Y(x) \in \mathcal{P}_{\mathbb{R}}(f)$, either $Y(x) = O(x^N)$ for any $N > 0$ or $Y(x) = c_E x^{\gamma_E} + o(x^{\gamma_E})$, with $\gamma_E \geq 1$, there exists $\delta > 0$ such that $|Y(\tau(y))| \leq \delta|\tau(y)|$, for all y sufficiently small. Since we may assume that $\delta > 1$, continuing (3.7), we have

$$|y| \leq max\{\delta|\tau(y)|, \delta|y - Y(\tau(y))|\} + \delta|\tau(y)| \leq (\delta + 1)dist((0, y), V),$$

and the claim is proved. □

Now, by the claim $|f(0, y)| = |y|^d \asymp dist((0, y), V)^d$, which implies that $L(f) \geq d$. □

3.4 Proof of Theorem 2.11

Proof. Let $f := (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$, $f(x,y) = y^d + c_{d-1}(x)y^{d-1} + ... + c_0(x)$, where d is the multiplicity of f at $(0,0) \in \mathbb{R}^2$.

Let us number all the vertices $(A_0, B_0), ..., (A_k, B_k)$ of $\Gamma(f)$ such that $B_0 > B_1 > ... > B_k$. ($B_0 = d, B_k = 0$, since f is convenient and the multiplicity of f at the origin is d). Let E_i be the compact edge joining the points (A_{i-1}, B_{i-1}) and $(A_i, B_i), i = 1, ..., k$ and $\gamma_{E_i} = \dfrac{A_i - A_{i-1}}{B_{i-1} - B_i}$. Then, by the convexity of $\Gamma(f)$, we have $\gamma_{E_1} < \gamma_{E_2} < ... < \gamma_{E_k}$.

Now, we number the edges E'_i of $\Gamma(\dfrac{\partial f}{\partial y})$ such that if $i < j$ then $\gamma_{E'_i} < \gamma_{E'_j}$.

We see that if $i = 1, ..., k-1$, then E'_i is the edge of $\Gamma(\dfrac{\partial f}{\partial y})$, joining the points $(A_{i-1}, B_{i-1} - 1)$ and $(A_i, B_i - 1)$. We have then

$$\gamma_{E'_i} = \frac{A_i - A_{i-1}}{(B_{i-1} - 1) - (B_i - 1)} = \frac{A_i - A_{i-1}}{B_{i-1} - B_i} = \gamma_{E_i}$$

Hence $\gamma_{E'_i} = \gamma_{E_i}$ for $i = 1, ..., k-1$.

Further, let us to look at how the edge E_k of $\Gamma(f)$ "produces" the corresponding edges of $\Gamma(\dfrac{\partial f}{\partial y})$.

Firstly, assume that E_k does not contain integer points different from (A_{k-1}, B_{k-1}) and (A_k, B_k). Then E_k "produces" the infinite horizontal edge E'_∞ of $\Gamma(\dfrac{\partial f}{\partial y})$ and the set of all adges of $\Gamma(\dfrac{\partial f}{\partial y})$ are $E'_1, ..., E'_{k-1}, E'_\infty$.

Assume that E_k contains integer points, different from (A_{k-1}, B_{k-1}) and (A_k, B_k). Then, beside of the edge E'_k of $\Gamma(\dfrac{\partial f}{\partial y})$, with $\gamma_{E'_k} = \gamma_{E_k}$, possibly there are also edges $E'_{k+1}, ..., E'_{k'}$, and E'_∞ of $\Gamma(\dfrac{\partial f}{\partial y})$, with $\gamma_{E'_i} > \gamma_{E_k}$ for $i = k+1, ..., k'$ (if $k' > k$). Now we compute $L(f, V_1)$ via the set $\gamma(f) \cup \{d\}$.

Case 1. Let either $Y' \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \cap C_+((x))$ or $Y' \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \cap A_+((x^{\frac{1}{(d-1)!}}))$ such that $Y'(x) = c_{E'}x^{\gamma_{E'}} + o(x^{\gamma_{E'}})$, with $\gamma'_{E'} > \gamma_{E_k}$. We have then $v(|Y'(x) - Y(x)|) = v(|Y(x)|)$, for all $Y \in \mathcal{P}_{\mathbb{C}}(f)$. Hence

$$|f(x, Y'(x))| = \prod_{Y \in \mathcal{P}_{\mathbb{C}}(f)} |Y'(x) - Y(x)| \asymp |x|^{\theta_{Y'}}$$

where

$$\theta_{Y'} = \sum_{i=1}^{k} n_{E_i}\gamma_{E_i} \tag{3.8}$$

Further

$$dist((x,Y'(x)),V) = min\{dist((x,Y'(x)),X_Y) | Y \in \mathcal{P}_{\mathbb{R}}(f)\}$$

where $X_Y := \{(x,y)|y = Y(x)\}$
By Lemma 3.2, we have

$$dist((x,Y'(x)),V) \asymp |x|^{\delta_{Y'}}$$

where

$$\delta_{Y'} = \begin{cases} 1 & if \quad \mathcal{P}_{\mathbb{R}}(f) = \emptyset \\ max\{v(|Y|) | Y \in \mathcal{P}_{\mathbb{R}}(f)\} \end{cases} \tag{3.9}$$

Hence, on the branch $(x,Y'(x))$ of V_1, we have

$$|f(x,Y'(x))| \geq c\,dist((x,Y'(x)),V)^{\frac{\theta_{Y'}}{\delta_{Y'}}}$$

where $\theta(Y')$ and $\delta(Y')$ are given by (3.8) and (3.9) respectively.
Case 2. $Y' \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$ and $Y'(x) = c_{E'_s}x^{\gamma_{E'_s}} + o(x^{\gamma_{E'_s}})$ with $\gamma_{E'_s} = \gamma_{E_s}$, (i.e. either $s \leq k-1$ or $s = k$ if $\gamma_{E'_k} = \gamma_{E_k}$).

Claim 3.4. For $Y \in \mathcal{P}_{\mathbb{C}}(f)$, we have

$$v(|Y'(x) - Y(x)|) = min\{\gamma_{E_s}, v(|Y(x)|\}.$$

Proof. Since $\gamma_{E'_s} = \gamma_{E_s}$, the claim is trivial if $v(|Y(x)|) \neq \gamma_{E_s}$.
Now, assume $Y(x) = c_{E_s}x^{\gamma_{E_s}} + o(x^{\gamma_{E_s}})$. To prove the claim, we will show that $c_{E'_s} \neq c_{E_s}$. Let $\overline{f}(x,y) = \sum_{i+j=0}^{\infty} c_{ij}x^iy^j$ and $\overline{\frac{\partial f}{\partial y}}(x,y) = \sum_{i+j=0}^{\infty} jc_{ij}x^iy^{j-1}$ be respectively the formal Taylor series of f and $\frac{\partial f}{\partial y}$. It follows from the Newton- Puiseux algorithm [W] that $f_{E_s}(1,c_{E_s}) = 0$ and $\widehat{f}_{E'_s}(1,c'_{E'_s}) = 0$, where $f_{E_s}(x,y) = \sum_{(i,j)\in E_s} c_{ij}x^iy^j$ and $\widehat{f}_{E'_s}(x,y) = \sum_{(i,j-1)\in E'_s} jc_{ij}x^iy^{j-1}$.
It is clear that, $\widehat{f}_{E'_s}(x,y) = \frac{\partial f_{E_s}}{\partial y}$. Thus, we have

$$f_{E_s}(1,c_{E_s}) = 0 \tag{3.10}$$

and

$$\frac{\partial f_{E_s}}{\partial y}(1,c_{E'_s}) = 0 \tag{3.11}$$

Since E_s belongs to the line joining the points (A_{s-1}, B_{s-1}) and (A_s, B_s) with $\dfrac{A_s - A_{s-1}}{B_{s-1} - B_s} = \gamma_{E_s}$, $f_{E_s}(x, y)$ can be written in the form

$$f_{E_s}(x,y) = x^{A_{s-1}} y^{B_s} \prod_{k=1}^{l} (y - c_k x^{\gamma_{E_s}})^{\alpha_k}$$

where $c_1, ..., c_l$ are non-zero roots of $f_{E_s}(1, y) = 0$.
By contradiction, assume that $c_{E_s} = c_{E'_s}$, then it follows from (3.10) and (3.11) that c_{E_s} is a nonzero root of $f_{E_s}(1, y) = 0$ of multiplicity greater than 1. As consequence, the system $\dfrac{\partial f_{E_s}}{\partial x}$ and $\dfrac{\partial f_{E_s}}{\partial y}$ has $(1, c_{E_s})$ as a solution in $(\mathbb{R}\backslash 0)^2$, which is possible, since f is non-degenerate. The claim is proved. $\square$

Now,

$$|f(x, Y'(x))| = \prod_{Y \in \mathcal{P}_{\mathbb{C}}(f)} (Y' - Y(x)) \asymp |x|^{\sum v(|Y' - Y|)},$$

By Claim 3.4

$$v(|Y'(x) - Y(x)|) = \begin{cases} v(|Y(x)|) & if \quad v(|Y(x)|) < \gamma_{E_s} \\ \gamma_{E_s} & if \quad v(|Y(x)|) \geq \gamma_{E_s} \end{cases} \tag{3.12}$$

Putting

$$\theta_{Y'} := \sum_{Y \in \mathcal{P}_{\mathbb{C}}(f)} v(|Y' - Y)|)$$

we get

$$\theta_{Y'} = \sum_{i=1}^{s-1} n_{E_i} \gamma_{E_i} + (\sum_{i=s}^{k} n_{E_i}) \gamma_{E_s} \tag{3.13}$$

It follows from Lemma 3.2 and Claim 3.4 that $dist((x, Y'(x)), V) \asymp x^{\delta_{Y'}}$, where

$$\delta_{Y'} = \begin{cases} 1 & if \quad \mathcal{P}_{\mathbb{R}}(f) \neq \emptyset \\ min\{\gamma_{E_s}, \gamma^{\mathbb{R}}(f)\} \end{cases} \tag{3.14}$$

with

$$\gamma^{\mathbb{R}}(f) := max\{v(|Y|) | Y \in \mathcal{P}_{\mathbb{R}}(f)\} \tag{3.15}$$

Therefore, on the branch $(x, Y(x))$ we have

$$|f(x, Y'(x))| \geq c.dist((x, Y'(x)), V)^{\frac{\theta_{Y'}}{\delta_{Y'}}}$$

for some $c > 0$, where the number $\theta(Y')$ and $\delta(Y')$ are determined by (3.8) - (3.9) or (3.14) -(3.15). Thus, the exponent $L_+(f; V_1)$, as well as $L_+(\hat{f}; V_1)$, can be computed via the set $\{\gamma_{E_1}, ..., \gamma_{E_k}, d\}$. The proof of Theorem 2.11 follows now from Theorem 2.8. $\square$

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References

[Bi-M] E. Bierstone, P. D. Milman, *Semianalytic and Subanalytic sets*, Publ. Math Inst. Hautes Etudes Sci. 67(1988),5-42.

[Br] W. Dale Brownawell, *Bounds for the degrees in the Nullstellensatz,* Ann. of Math. 126 (1987), no.3, 577-591.

[C-C-W] A. Carberry, M.Christ, J. Wright, *Multidimensional Vander Corput lemma and sublevel set estimates.* J. Amer. Math. Soc. iz(1999), no 41 1981-1015.

[H] H.V.Ha, *Nombres de Łojasiewicz et singularites l' infini des polynmes de deux variables complexes.* C. R. Acad. Sci- Paris, Serie 1, 311 (1990), 429-432.

[H-N-S] H.V.Ha, H.V.Ngai, T.S.Pham, *A global smooth version of the classical Łojasiewicz inequality.* J. Math. Anal. Appl. 421(2015), 1559-1572.

[Ko] A. G. Kouchnirenko, *Polyhedres de Newton et nombres de Milnor,* Invent.Math, 32 (1976), 1-31.

[Kuo] T.C.Kuo, *Computation of Lojasiewicz exponent of* $f(x, y)$*,* Comment. Math. Helv., 49 (1974), pp.201-213.

[Kur] K. Kurdyka, *On gradient of function definable in o-minimal structures.* Ann. de l'institut Fourier, t.48, no 3(1998), 769-783.

[Kur-M-P] K. Kurdyka, T. Mostowski, A. Parusinski, *Proof of the gradient conjecture of R. Thom.* Ann of Math., 152 (2000), 763-792.

[L1] S. Lojasiewicz, *Sur le probleme de la division*, Studia Math. 18 (1959) 87136.

[L2] S. Lojasiewicz, *Emsembles Semi-Analytiques*, Memeo. I.H.E.S 18 (1965) 87136.

[M] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press and Tata Institute of Fundamental Research, 1996.

[R] V. S. Rychkov, *Sharp L^2 bounds for oscillatory integral operators with C^∞ phase,* Math. Z. 236 (2001), 461 -489.

[T] B. Teissier, *Variétés polaires 1*, Inventiones Math. 40 (1977), 267-292.

[W] R. J. Walker, *Algebraic Curves,* Princeton Univ. Press., 1950.