A Hausdorff-type distance, a directional derivative of a set-valued map and applications in set optimization

Truong Xuan Duc Ha*

April 24, 2017

Dedicated to Professor Johannes Jahn in honor of his 65th birthday

Abstract

In this paper, we follow Kuroiwa's set approach in set optimization, which proposes to compare values of a set-valued objective map F respect to various set order relations. We introduce a Hausdorff-type distance relative to an ordering cone between two sets in a Banach space and use it to define a directional derivative for F. We show that the distance has nice properties regarding set order relations and the directional derivative enjoys most properties of the one of a scalar- single-valued function. These properties allow us to derive necessary and/or sufficient conditions for various types of maximizers and minimizers of F.

Key Words: Set-valued map, directional derivative, coderivative, set optimization, optimality condition

Mathematics subject classifications (MSC 2010): 49J53, 90C46

1 Introduction

Optimization problems with set-valued data arose originally inside of the theory of vector optimization and have recently been attracted more attention due to their important real-world applications in socio-economics, see [4, 18], and a survey given in [27]. An objective map in a set optimization problem (SP) is a set-valued map $F : \Omega \subseteq X \rightrightarrows Y$, where Ω is a nonempty set and X, Y are vector spaces. There are several approaches to define a solution of (SP) (or a minimizer and a maximizer of F over Ω) but we restrict ourselves here mainly to the classical vector approach and the Kuroiwa's set approach. Let a convex cone $K \subset Y$ be given. Then K induces a partial order in Y and various set order relations in 2^Y. In the first approach, one

*Institute of Mathematics, Vietnam Academy of Sciences and Technology, 18 Hoang Quoc Viet Road, 10307 Hanoi, Viet Nam, email: txdha@math.ac.vn

compares elements of the image $F(\Omega) := \cup_{x\in\Omega} F(x)$ w.r.t. the partial order in Y (see e.g. [25]) while in the second approach, one compares the sets $F(x)$ w.r.t. set order relations in 2^Y (see e.g. [22]). We refer an interested reader to [14, 20] for surveys on set order relations and set-valued optimization problems whose solutions are defined by set criteria and to [19] for references of works where set order relations have been used outside the optimization community.

In optimization theory, the concepts of derivative and directional derivative of a function is an useful tool for deriving first-order necessary and/or sufficient optimality conditions.

In recent years, in connection with the numerous research in set optimization, much attention has been paid to extension of these concepts to set-valued maps. In the very first works, one takes a point in the graph of the set-valued map and assigns to it another set-valued map whose graph is some kind of tangent cone to the graph of the original one at the point in question, see the book [3]. Note that coderivative of a set-valued map is defined in a way of the same nature with a tangent cone being replaced by a normal cone, see [3, 26] for various types of derivatives and coderivatives. Later on, different concepts of directional derivative have been introduced depending on types of set differences involved (see [5] for a survey on possible set differences) and on types of order set relations using for defining optimal solutions. The first concept in this direction has been proposed by Kuroiwa in [23] where the lower less set relation and a special embedding technique have been involved. Hoheisel, Kanzow, Mordukhovich and Phan [17] use translation (the difference of a set and a point) instead of the difference of sets consisting of more than one point. Hamel-Schrage's approach in [13] is based on a residuation operation and on the solution concept of an infimizer. Pilecka [28] exploits the inf-residuation, a concept already used in [6, 12, 13], for a difference of sets in combination with the lower set less order relation. Jahn [19] develops a directional derivative from a computational point of view, and interprets it as a limit of difference quotients, which is adapted from Demyanov's difference (see [7, 29]) and is based on the concept of supporting points, in combination with a set less order relation. Recently, Dempe and Pilecka [8] use a slightly modified Demyanov difference to introduce a sort of directional derivative for a set-valued map and derive optimality conditions for efficient solutions defined by the set less order relation.

In this paper we introduce a Hausdorff-type distance relative to the ordering cone between two sets, which has nice properties regarding set order relations, and define a directional derivative as a limit of quotients of algebraic set difference. It turns out that the directional derivative enjoys most properties of the one of a scalar- single-valued function and can be used to derive necessary and/or sufficient conditions for various minimizers and maximizers of F some types of which are considered here in the first time.

The paper is organized as follows. Section 2 contains preliminaries. Next two sections are devoted to a Hausdorff-type distance and to a directional derivative, respectively. The last section contains conditions for several minimizers and maximizers of a set-valued map.

2 Notations and some auxiliary results

Throughout the paper, let X and Y be Banach spaces. Denote by X^* and Y^* the duals of X and Y, respectively, and by $\langle \cdot, \cdot \rangle$ the pairing between a space and its dual. By $\mathbb{B}$ we denote the unit ball in a normed space. For nonempty subsets A, B in Y, we define the algebraic sum (also called Hausdorff sum or Minkowski addition) and algebraic difference as follow $A + B := \{a + b \mid a \in A,\ b \in B\}$ and $A - B := \{a - b \mid a \in A,\ b \in B\}$. For a nonempty set $A \subset Y$ and $t \in \mathbb{R}$, let $tA := \{ta \mid a \in A\}$. The distance from a point u to a nonempty set U in the spaces X and Y are denoted by $d(u, U)$ or $d_U(u)$.

Let $K \subset Y$ be a pointed closed convex cone (pointedness means $K \cap (-K) = \{0\}$) and let $K^* := \{y^* \in Y^* \mid \langle y^*, k \rangle \geq 0,\ \forall\, k \in K\}$. The cone K induces a partial order in Y: for any $y_1, y_2 \in Y$

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K.$$

For the sake of simplicity, we will omit the subscript K in the notation $\leq_K$.

Definition 2.1 ([25]). Let $A \subset Y$ be a nonempty set and $a \in A$. We say that (i) A is K-*bounded* if there exists a bounded nonempty set $M \subset Y$ such that $A \subset M + K$; (ii) A is K-*compact* if any its cover of the form $\{U_\alpha + K \mid \alpha \in I,\ U_\alpha \text{ are open}\}$ admits a finite subcover;(iii) a is a *Pareto nondominated/efficient point* of A (denoted by $a \in \mathrm{Min}(A)$) if $a' \not\leq a$ for all $a' \in A$, $a' \neq a$.

Proposition 2.1 ([9, 25]). *Let $A \subset Y$ be a nonempty K-compact set. Then*

(i) $\mathrm{Min}(A) \neq \emptyset$ *and* $A \subseteq \mathrm{Min}(A) + K$ *(the nondomination property).*

(ii) $A + K = \mathrm{Min}(A) + K$ *and* $\mathrm{Min}(A)$ *is K-compact.*

(iii) If $B \subset Y$ also is a K-compact nonempty set and $A + K = B + K$, then $\mathrm{Min}(A) = \mathrm{Min}(B)$.

There are numerous set order relations, see e.g. [20, 22] but we will mainly use the following ones.

Definition 2.2. Let A and B be nonempty subsets of Y.

(i) The *l-type less order relation* $\preccurlyeq_l$ is defined by

$$A \preccurlyeq_l B \quad :\Longleftrightarrow \quad \big(\forall b \in B\ \exists\, a \in A : a \leq b\big) \Longleftrightarrow B \subseteq A + K.$$

(ii) The *u-type less order relation* $\preccurlyeq_u$ is defined by

$$A \preccurlyeq_u B \quad :\Longleftrightarrow \quad \big(\forall a \in A\ \exists\, b \in B : a \leq b\big) \Longleftrightarrow A \subseteq B - K.$$

(iii) The *set less order relation* $\preccurlyeq_s$ is defined by

$$A \preccurlyeq_s B \quad :\Longleftrightarrow \quad A \preccurlyeq_l B \text{ and } A \preccurlyeq_u B.$$

(iv) The *possibly less order relation* $\preccurlyeq_p$ is defined by

$$A \preccurlyeq_p B \quad :\Longleftrightarrow \quad \big(\exists a \in A\ \exists\, b \in B : a \leq b\big) \Longleftrightarrow (A - B) \cap (-K) \neq \emptyset.$$

(v) The *certainly less order relation* $\preccurlyeq_c$ is defined by

$$A \preccurlyeq_c B \;\; :\Longleftrightarrow \;\; (A = B) \text{ or } (A \neq B, \forall a \in A\, \forall b \in B : a \leq b) \Longleftrightarrow A - B \subseteq -K.$$

The set order relations $\preceq_l$, $\preceq_u$ and $\preceq_s$ have been introduced in [22]. For the set order relations $\preceq_p$ and $\preceq_c$, see [20]. Alongside with the set order relations, we will consider *strict set order relations* in the case K has a nonempty interior.

Definition 2.3. Assume that $\mathrm{int}K \neq \emptyset$. Let A and B be nonempty subsets of Y.

(i) $A \prec_l B \;\; :\Longleftrightarrow B \subseteq A + \mathrm{int}K$.

(ii) $A \prec_u B \;\; :\Longleftrightarrow A \subseteq B - \mathrm{int}K$.

(iii) $A \prec_p B \;\; :\Longleftrightarrow \;\; (A - B) \cap (-\mathrm{int}K) \neq \emptyset$.

(iv) $A \prec_c B \;\; :\Longleftrightarrow \;\; (A - B) \subseteq (-\mathrm{int}K)$.

It is immediate from the definitions the following implications.

Lemma 2.1. *Let A and B be nonempty subsets of Y. Then*

(i) $A \preceq_c B$ implies $A \preceq_p B$, and $A \preceq_c B$ implies $A \preceq_l B$ and *$A \preceq_u B$.*

(ii) $A \preceq_l B$ implies $A \preceq_p B$, and $A \preceq_u B$ implies $A \preceq_p B$.

Assume that $\mathrm{int}K \neq \emptyset$. *The assertions (i)-(ii) remain true if the involved set order relations are replaced by the corresponding strict ones.*

Throughout the paper, $F : X \rightrightarrows Y$ is a set-valued map. The domain and the graph of F are the sets $\mathrm{dom}F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\mathrm{gr}F := \{(x, y) \in X \times Y \mid y \in F(x)\}$, respectively. Recall that F is closed (convex) [2] if its graph is closed (convex, respectively).

3 A Hausdorff-type distance

The Hiriart-Urruty signed distance function Δ_U associated to a nonempty set $U \subset Y$ (see [15]) in the special case $U = -K$ plays an important role in our definition of a distance between two sets. Recall that

$$\Delta_{-K}(y) := d_{-K}(y) - d_{Y\setminus(-K)}(y) = \begin{cases} -d_{Y\setminus(-K)}(y) & \text{if } y \in -K \\ d_{-K}(y) & \text{otherwise.} \end{cases}$$

Some useful properties of Δ_{-K} are collected in the following proposition.

Proposition 3.1. *The function Δ_{-K} has the properties:*

(i) It is Lipschitz of rank 1 on Y, convex and positively homogenous.

(ii) It satisfies the triangle inequality: $\Delta_{-K}(y_1 + y_2) \leq \Delta_{-K}(y_1) + \Delta_{-K}(y_2)$ for any $y_1, y_2 \in Y$.

(iii) It is K-monotone: $\Delta_{-K}(y_1) \leq \Delta_{-K}(y_2)$ for any $y_1, y_2 \in Y,\; y_1 \leq y_2$.

(iv) *For any $y \in Y$ we have $\partial\Delta_{-K}(y) \subset K^* \cap \mathbb{B}$. Here, ∂ stands for the subdifferential of convex analysis.*

Proof. The properties (i)-(iii) are known, see e.g. [30], and the last one can be derived from [15, Prop. 2 and 5]. □

Below are some illustrating examples.

Example 3.1. (i) If $K = \{0\}$, then $\Delta_{-K}(y) = \|y\|$ for all $y \in Y$.

(ii) If $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, then for all $y = (y_i) \in \mathbb{R}^n$

$$\Delta_{-\mathbb{R}^n_+}(y) = \begin{cases} -\min_i |y_i| & \text{if } y \in -\mathbb{R}^n_+ \\ \sqrt{\sum_{i=1}^n [y_i]_+^2} & \text{otherwise.} \end{cases}$$

(iii) If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, then $\Delta_{-K}(y) = y$ and $\partial\Delta_{-K}(y) = 1$ for all $y \in \mathbb{R}$.

Let A, B be nonempty subsets of Y. Denote

$$h_K(A, B) := \sup_{b \in B} \inf_{a \in A} \Delta_{-K}(a - b).$$

Lemma 3.1. *$h_K(A, B) > -\infty$ when A is K-bounded, $h_K(A, B) < +\infty$ when B is K-bounded and $h_K(A, B)$ is finite when both A and B are K-bounded.*

Proof. Consider the case A is K-bounded. Then $A \subset M + K$ for some nonempty bounded set $M \subset Y$. Fix $b \in B$. For any $a \in A$, there exist $m \in M$ and $k \in K$ such that $a = m + k$. Then $m \leq a$. Since the function Δ_{-K} is K-monotone 1-Lipschitz, we have $\Delta_{-K}(a - b) \geq \Delta_{-K}(m - b) \geq -\|m - b\| \geq -\|m\| - \|b\|$. Then $h_K(A, B) \geq -\sup_{m \in M} \|m\| - \|b\| > -\infty$. The remaining cases can be checked similarly. □

Now we can define a special distance in the family of nonempty K-bounded sets.

Definition 3.1. Let A, B be nonempty K-bounded subsets of Y. A *Hausdorff-type distance* relative to the ordering cone K between A and B, denoted by $d_K(A, B)$, is defined as follows:

$$d_K(A, B) := \max\{h_K(A, B), h_K(B, A)\}.$$

Remark 3.1. The name "Hausdorff-type distance" is originated from the fact that this distance coincides with the classical Hausdorff distance given by

$$d(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

when $K = \{0\}$ because in this case $\Delta_{-K}(y) = \|y\|$ for all $y \in Y$.

In what follows, when no confuse occurs, we **abbreviate** $d_K(A, B)$ and $h_K(A, B)$ to $d(A, B)$ and $h(A, B)$, respectively.

As the reader will see, the functions h and d have nice properties. Let us consider first the functions h.

Lemma 3.2. *Let A and A' be nonempty subsets of Y.*

(i) If A' is K-compact, then for any $a \in Y$ the function $\Delta_{-K}(. - a)$ attains its finite infimum on A'.

(ii) If A' is K-bounded and A is K-compact, then the function $\inf_{a' \in A'} \Delta_{-K}(a' - .)$ attains its finite maximum on A.

(iii) If A and A' are K-compact, then

$$h(A, A') = \max_{a' \in A'} \min_{a \in A} \Delta_{-K}(a - a').$$

Proof. (i) Suppose that A' is K-compact, then it is K-bounded [25]. Let $a \in Y$ be given. One can see from the proof of Lemma 3.1 that $t := \inf_{a' \in A'} \Delta_{-K}(a' - a) > -\infty$. Suppose to the contrary that $\Delta_{-K}(. - a)$ does not attain its infimum on A'. Then for any $a' \in A'$ there exists a positive scalar $\epsilon(a')$ depending on a' such that $\Delta_{-K}(a' - a) > t + \epsilon(a')$. For each $a' \in A'$, let $U_{a'} := \{v \in Y \mid \Delta_{-K}(v - a) > t + \epsilon(a')\}$. Note that since $0 \in K$, we have $U_{a'} \subset U_{a'} + K$ and since $\Delta_{-K}(v + k - a) \geq \Delta_{-K}(v - a) > t + \epsilon(a')$ for any $v \in U_{a'}$ and $k \in K$, we get $U_{a'} + K \subset U_{a'}$. Therefore, $U_{a'} = U_{a'} + K$. Further, since the function Δ_{-K} is Lipschitz, the sets $U_{a'}$ are open and since $a' \in U_{a'}$ we have $A' \subset \cup_{a' \in A'} U_{a'}$. The K-compactness of A' implies the existence of finite vectors $a'_1, \ldots, a'_i$ such that $a'_j \in A'$ for all $j = 1, \ldots, i$ and $A' \subset \cup_{j=1}^{i} (U_{a'_j} + K)$. Hence, $A' \subset \cup_{j=1}^{i} U_{a'_j}$ and we get $t = \inf_{a' \in A'} \Delta_{-K}(a' - a) > t + \inf\{\epsilon(a'_j) \mid j = 1, \ldots, i\} > t$, a contradiction.

(ii) Exploiting the properties of the function Δ_{-K} stated in Proposition 3.1, one can easily check that the function $\inf_{a' \in A'} \Delta_{-K}(a' - .)$ is 1-Lipschitz and monotone in the following sense

$$a_2 \leq_K a_1 \Longleftrightarrow \inf_{a' \in A'} \Delta_{-K}(a' - a_1) \leq \inf_{a' \in A'} \Delta_{-K}(a' - a_2).$$

Next, according to Lemma 3.1, for any $a \in A$ we have $\inf_{a' \in A'} \Delta_{-K}(a' - a) > -\infty$ and $t := h(A', A) < +\infty$. Suppose to the contrary that the function $\inf_{a' \in A'} \Delta_{-K}(a' - .)$ does not attain its maximum on A. Fix $a \in A$. Then there exists a positive scalar $\epsilon(a)$ depending on a such that $\inf_{a' \in A'} \Delta_{-K}(a' - a) < t - \epsilon(a)$. Set $U_a := \{v \in Y \mid \inf_{a' \in A'} \Delta_{-K}(a' - v) < t - \epsilon(a)\}$. One can check that $U_a = U_a + K$. By the same arguments as in the proof of (i) and taking into account the mentioned above properties of the function $\inf_{a' \in A'} \Delta_{-K}(a' - .)$, we can find finite numbers of vectors $a_1, \ldots, a_i$ such that $a_j \in A$ for all $j = 1, \ldots, i$ and $A \subset \cup_{j=1}^{i} (U_{a_j} + K) = \cup_{j=1}^{i} U_{a_j}$. Then we obtain that $t = \sup_{a \in A} \inf_{a' \in A'} \Delta_{-K}(a' - a) < t - \inf\{\epsilon(a_j) \mid j = 1, \ldots, i\} < t$, a contradiction.

(iii) The assertion follows from the assertions (i)-(ii). □

The following characterization of the set order relation $\preceq_l$ in term of the function h is an important tool in our arguments.

Lemma 3.3. *Let A and A' be nonempty subsets of Y. Assume that A' is K-bounded. Then*

$$A' \preceq_l A \Longleftrightarrow h(A', A) \leq 0$$

(the implication "$\Longleftarrow$" holds under an additional condition that A' is K-compact).

Proof. Suppose that $A' \preceq_l A$ or $A \subset A' + K$. For any $a \in A$ there is $a'_0 \in A'$ such that $a'_0 - a \in -K$ and hence, $\Delta_{-K}(a'_0 - a) \leq 0$. We get $\inf_{a' \in A} \Delta_{-K}(a' - a) \leq \Delta_{-K}(a'_0 - a) \leq 0$. As $a \in A$ is arbitrarily chosen, we get $h(A', A) \leq 0$.

Next, suppose that A is K-compact and $h(A', A) \leq 0$. Suppose to the contrary that $A' \not\preceq_l A$ or $A \not\subseteq A' + K$. Then there exists $a \in A$ such that $a \notin A' + K$. For all $a' \in A'$ one has $a' - a \notin -K$ and hence, $\Delta_{-K}(a' - a) > 0$. Since A' is K-compact, Lemma 3.2 (i) implies that $\Delta_{-K}(. - a)$ attains its minimum on A' and, therefore, $\min_{a' \in A'} \Delta_{-K}(a' - a) > 0$ and we obtain $h(A', A) > 0$, a contradiction. □

Lemma 3.4. *Assume that A, A' are nonempty subsets of Y and A' is K-bounded. Then*

(i) $h(A', A) = h(A' + K, A + K)$.

(ii) $h(A, A) = 0$ *if* $\text{int}K = \emptyset$ *or* $\text{Min}(A) \neq \emptyset$ *(for instance, if A is K-compact).*

Proof. (i) Observe that since $A \subset A + K$, we have $h(A', A) \leq h(A', A + K)$. Further, let $a \in A$, $k \in K$ and $a' \in A'$ be arbitrary vectors. The K-monotonicity of the function Δ_{-K} implies $\Delta_{-K}(a' - (a + k)) = \Delta_{-K}(a' - a - k)) \leq \Delta_{-K}(a' - a)$ and therefore, $h(A', A) \geq h(A', A + K)$. Thus, $h(A', A) = h(A', A + K)$. Applying this equality to the set $A' + K$ in the place of A', we get $h(A' + K, A) = h(A' + K, A + K)$. By a similar argument we can show that $h(A', A) = h(A' + K, A)$. The desired equality follows.

(ii) If $\text{int}K = \emptyset$, then $\Delta_{-K}(y) \geq 0$ for all $y \in Y$ and hence, $h(A, A) \geq 0$. If $\text{Min}(A) \neq \emptyset$ (which happens, for instance, when A is K-compact, see Proposition 2.1), then for $\bar{a} \in \text{Min}(A)$ one has $a \not\leq_K \bar{a}$ and $\Delta_{-K}(a - \bar{a}) \geq 0$ for all $a \in A$, which gives $h(A, A) \geq 0$. Finally, Lemma 3.3 gives $h(A, A) \leq 0$. Hence, $h(A, A) = 0$. □

Lemma 3.5. *Assume that A, B and C are nonempty K-bounded subsets of Y. Then the triangle inequality holds:*

$$h(A, B) \leq h(A, C) + h(C, B).$$

Proof. Recall that the triangle inequality of the function Δ_{-K} yields

$$\Delta_{-K}(a - b) \leq \Delta_{-K}(a - c) + \Delta_{-K}(c - b), \forall a \in A, \forall b \in B, \forall c \in C$$

and the desired inequality follows from the definition of h. □

We list useful properties of the function d in the following.

Proposition 3.2. *Assume that A and B are nonempty K-bounded sets. Then*

(i) $d(A, B) = d(B, A)$.

(ii) $d(A, B) = d(A + K, B + K)$.

(iii) $d(\lambda A, \lambda B) = \lambda d(A, B)$ *for any* $\lambda \geq 0$.

(iv) *The triangle inequality holds: for any nonempty K-bounded set C, we have*

$$d(A, B) \leq d(A, C) + d(C, B).$$

(v) *Assume that A and B are K-compact. Then $d(A, B) \geq 0$ and $d(A, B) = 0$ iff $A + K = B + K$.*

Proof. Note that the assertion (i) follows from the definition of d, the assertion (ii) follows from Lemma 3.4 and the assertion (iii) is immediate from the definitions of h, d and Proposition 3.1 (i). Further, from Lemma 3.5 we get

$$\begin{aligned} h(A, B) &\leq h(A, C) + h(C, B) \\ &\leq \max\{h(A, C), h(C, A)\} + \max\{h(C, B), h(B, C)\} \\ &= d(A, C) + d(B, C). \end{aligned}$$

Similarly, we have $h(B, A) \leq d(A, C) + d(B, C)$. The assertion (iv) follows. It remains to prove the last assertion. If at least one relation, say $A \not\preceq_l B$ holds, then Lemma 3.3 yields $h(A, B) > 0$ and therefore, $d(A, B) > 0$. Suppose that both the relations $A \preceq_l B$ and $B \preceq_l A$ hold. Then $B + K \subseteq A + K \subseteq B + K$ and hence, $A+K = B+K$. Lemma 3.4 implies $h(A, B) = h(A+K, B+K) = h(A+K, A+K) = 0$. Similarly, we have $h(B, A) = 0$. Therefore, $d(A, B) = 0$. □

It is immediate from Propositions 2.1 (ii) and 3.2 (ii) the following useful result.

Corollary 3.1. *Let A and B be nonempty K-compact subsets of Y. Then*

$$d(A, B) = d(\mathrm{Min}(A), \mathrm{Min}(B)).$$

It turns out that the function d has useful properties regarding the limit operation. Firstly, we show that the limit set is "unique" w.r.t cone extensions. Recall that for a set $A \subset Y$, its cone extension is the set $A + K$.

Proposition 3.3. *Let A_t ($t \in \mathbb{R}_+$ sufficiently small), A and B be nonempty K-compact subsets of Y. Suppose that*

$$\lim_{t \downarrow 0^+} d(A_t, A) = 0.$$

Then

$$\lim_{t \to 0} d(A_t, B) = 0 \Longleftrightarrow A + K = B + K.$$

Proof. Observe that by Proposition 3.2 (iv) and (v) we have

$$0 \leq d(A, B) \leq d(A, A_t) + d(A_t, B)$$

and

$$0 \leq d(A_t, B) \leq d(A_t, A) + d(A, B),$$

which imply that $\lim_{t \downarrow 0^+} d(A_t, B) = 0$ if and only if $d(A, B) = 0$. Finally, recall that in view of Proposition 3.2 (v), $d(A, B) = 0$ if and only if $A + K = B + K$. □

Furthermore, the limit operation reserves the order relations $\preceq_l$ and $\preceq_p$. This property is very useful in deriving optimality conditions.

Proposition 3.4. *Assume that A_t ($t \in \mathbb{R}_+$), A and B are nonempty K-compact subsets of Y and that*

$$\lim_{t \downarrow 0^+} d(A_t, A) = 0.$$

Then the following assertions hold:

(i) If $A_t \preceq_l B$ ($B \preceq_l A_t$), then $A \preceq_l B$ (resp., $B \preceq_l A$).

(ii) If $A_t \preceq_p B$ and B is compact, then $A \preceq_p B$.

Proof. (i) Since $A_t \preceq_l B$, Lemma 3.3 gives $h(A_t, B) \le 0$. Further, Lemma 3.5 yields

$$h(A, B) \le h(A, A_t) + h(A_t, B) \le d(A, A_t) + h(A_t, B).$$

Since $\lim_{t\to 0} d(A_t, A) = 0$ and $h(A_t, B) \le 0$, we obtain $h(A, B) \le 0$ and Lemma 3.3 gives $A \preceq_l B$. Similarly, if $B \preceq_l A_t$ then $h(B, A_t) \le 0$ and we deduce from the relations $\lim d(A_t, A) = 0$ and

$$h(B, A) \le h(B, A_t) + h(A_t, A) \le h(B, A_t) + d(A_t, A)$$

that $h(B, A) \le 0$. Therefore, $B \preceq_l A$.

(ii) Let $t_i := 1/i$ and $A_i := A_{t_i}$ for $i = 1, 2, ...$ Without loss of generality, we may assume that $A_i \preceq_p B$ for all i. For each $i = 1, 2, ...$, let $b_i \in B \cap (A_i + K)$ be given. Then $A_i \preceq_l \{b_i\}$ and Lemma 3.3 implies $h(A_i, \{b_i\}) \le 0$. Since B is compact, we may assume that b_i converges to $b \in B$. Further, Lemma 3.5 and Proposition 3.1 (i) give

$$\begin{aligned} h(A, \{b\}) &\le h(A, A_i) + h(A_i, \{b_i\}) + h(\{b_i\}, \{b\}) \\ &\le d(A, A_i) + h(A_i, \{b_i\}) + \|b_i - b\|. \end{aligned}$$

Since $\lim_{i\to\infty} d(A, A_i) = 0$, $h(A_i, \{b_i\}) \le 0$ and $\lim_{i\to\infty} \|b_i - b\| = 0$, it follows that $h(A, \{b\}) \le 0$. Lemma 3.3 implies $A \preceq_l \{b\}$. Then $\{b\} \subset A + K$ or $(A - B) \cap (-K) \neq \emptyset$. Hence, $A \preceq_p B$. □

Next, we use the Hausdorff-type distance to characterize the concept of a K-*Lipschitz continuous set-valued map* used in set optimization (see e.g. [1]).

Proposition 3.5. *Assume that F has K-compact values. Then F is K-Lipschitz continuous with the constant L near a point $x \in \mathrm{dom}F$ in the sense that there is a neighborhood U of x such that*

$$F(x_1) \subseteq F(x_2) + L\mathbb{B}\|x_1 - x_2\| + K, \quad \forall x_1, x_2 \in U \cap \mathrm{dom}F \tag{1}$$

if and only if

$$d(F(x_1), F(x_2)) \le \eta\|x_1 - x_2\|, \quad \forall x_1, x_2 \in U \cap \mathrm{dom}F, \tag{2}$$

where $\eta := \rho L$ and $\rho := \sup\{\Delta_{-K}(e) \mid e \in \mathbb{B}\}$.

Proof. Note that since K is closed, we have $\rho > 0$.

The " if" part: Assume that (2) holds. It suffices to show that (1) holds for $\hat{L} = \eta/\bar{\rho}$ for any $\bar{\rho}$ satisfying $0 < \bar{\rho} < \rho$. Assume to the contrary that $F(x_1) \not\subseteq$

$F(x_2) + \hat{L}\|x_1 - x_2\|\mathbb{B} + K$ for some $x_1, x_2 \in U \cap \mathrm{dom}F$. Then there exists $v_1 \in F(x_1)$ such that $v_1 \notin F(x_2) + \hat{L}\|x_1 - x_2\|\mathbb{B} + K$. Let ρ' be a constant such that $\bar{\rho} < \rho' < \rho$. We can find $e \in \mathbb{B}$ such that $\rho' \leq \Delta_{-K}(e)$. Then for any $v_2 \in F(x_2)$ we have $v_2 - v_1 - \hat{L}\|x_1 - x_2\|e \notin -K$. Proposition 3.1 gives

$$\Delta_{-K}(v_2 - v_1) - \Delta_{-K}(\hat{L}\|x_1 - x_2\|e) \geq \Delta_{-K}(v_2 - v_1 - \hat{L}\|x_1 - x_2\|e) > 0.$$

Hence,

$$\Delta_{-K}(v_2 - v_1) > \hat{L}\|x_1 - x_2\|\Delta_{-K}(e) \geq \eta\rho'/\bar{\rho}\|x_1 - x_2\|.$$

Therefore, we get

$$d(F(x_1), F(x_2)) \geq \sup_{v_1 \in F(x_1)} \inf_{v_2 \in F(x_2)} \Delta_{-K}(v_2 - v_1) \geq \eta\rho'/\bar{\rho}\|x_1 - x_2\| > \eta\|x_1 - x_2\|,$$

which is a contradiction to (2). Thus (1) holds for $\hat{L} = \eta/\bar{\rho}$.

The "only if" part: Assume that (1) holds. For any $v_1 \in F(x_1)$ there exist $v_2 \in F(x_2)$, $e \in \mathbb{B}$ and $k \in K$ such that $v_1 = v_2 - L\|x_1 - x_2\|e + k$ or $v_2 - v_1 = L\|x_1 - x_2\|e - k$. Proposition 3.1 gives $\Delta_{-K}(v_2 - v_1) \leq L\|x_1 - x_2\|\Delta_{-K}(e)$ or $\Delta_{-K}(v_2 - v_1) \leq \rho L\|x_1 - x_2\|$. Therefore, we get

$$h(F(x_2), F(x_1)) = \sup_{v_1 \in F(x_1)} \inf_{v_2 \in F(x_2)} \Delta_{-K}(v_2 - v_1) \leq \rho L\|x_1 - x_2\|.$$

Similarly, we have $h(F(x_1), F(x_2)) \leq \rho L\|x_1 - x_2\|$. Hence, (2) holds with $\eta = \rho L$. □

Proposition 3.2 implies that d has all properties of a metric. In fact, this distance induces a metric on the family

$$\mathcal{V} := \{[A] \mid A \subset Y \text{ is } K-\text{compact}\},$$

where for any nonempty K-compact set $A \subset Y$,

$$[A] := \{A' \subset Y \mid A' \text{ is } K-\text{compact and } A' + K = A + K\}.$$

Observe that $\mathcal{V}$ is a semi-linear space with the addition and multiplication operations given by $[A] + [B] := [A + B]$ and $t[A] := [tA]$ for any pair $[A], [B] \in \mathcal{V}$ and any nonnegative scalar t. We define a function $d_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ by

$$d_{\mathcal{V}}([A], [B]) := d(A, B).$$

By Proposition 3.2, $d_{\mathcal{V}}$ is well-defined and it induces a metric on $\mathcal{V}$. Proposition 3.3 shows that the following limit operation in $\mathcal{V}$ is well-defined: For $[A_i] \in \mathcal{V}$ $(i = 1, 2, ...)$ and $[A] \in \mathcal{V}$, we write

$$\lim_{i \to +\infty} [A_i] = [A] \text{ if and only if } \lim_{i \to +\infty} d_{\mathcal{V}}([A_i], [A]) = 0.$$

Proposition 3.5 states that the set-valued map F with K-compact values is K-Liptschitz continuous at $\bar{x}$ if and only if the single-valued map $[F] : \mathrm{dom}F \to \mathcal{V}$ defined by $[F](x) := [F(x)]$ is Lipschitz continuous at this point, namely,

$$d_{\mathcal{V}}([F](x_1), [F](x_2)) \leq \eta\|x_1 - x_2\|, \quad \forall x_1, x_2 \in U \cap \mathrm{dom}F.$$

4 A concept of directional derivative

In this section, we introduce a new concept of directional derivative for the set-valued map F and study its properties.

From now on, we assume that F has **compact values**. Recall that $d \in Y$ is an admissible direction of F at $x \in \text{dom}F$ if $x + td \in \text{dom}F$ for $t > 0$ sufficiently small.

Definition 4.1. Let $x \in \text{dom}F$ and d be an admissible direction of F at x. Denote

$$W(x, d) := \{A \subset Y \mid \ A \text{ is } K\text{−compact and } \lim_{t \downarrow 0^+} d(\frac{F(x+td) - F(x)}{t}, A) = 0\}.$$

The *directional derivative* $DF(x, d)$ of F at x in the direction d is defined by

$$DF(x, d) := \begin{cases} \text{Min}(A) \text{ for some } A \in W(x, d) & \text{if } W(x, d) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

We say that F has the directional derivative $DF(x, d)$ at x in the direction d if $DF(x, d) \neq \emptyset$.

Proposition 4.1. *Assume that $W(x, d) \neq \emptyset$.*

(i) The directional derivative is well-defined in the sense that $DF(x, d)$ is nonempty and it does not depend on the choice of $A \in W(x, d)$. Moreover, we have

$$\lim_{t \downarrow 0^+} d(\frac{F(x+td) - F(x)}{t}, DF(x, d)) = 0.$$

(ii) Let $B \subset Y$ be a nonempty K-compact set. Then $DF(x, d) = B$ if and only if

$$\lim_{t \downarrow 0^+} d(\frac{F(x+td) - F(x)}{t}, B) = 0 \text{ and } \text{Min}(B) = B.$$

Proof. (i) The non-emptiness of $DF(x, d)$ follows from the K-compactness of the set A and Proposition 2.1. Further, let be given a pair $A_1, A_2 \in W(x, d)$. By Proposition 3.3, we have $A_1 + K = A_2 + K$ and by Proposition 2.1 (iii) we have $\text{Min}(A_1) = \text{Min}(A_2)$. Hence, $DF(x, d) = \text{Min}(A_1) = \text{Min}(A_2)$, which means that the directional derivative is well-defined. Next, assume that $DF(x, d) = \text{Min}(A)$ for some $A \in W(x, d)$. By Proposition 2.1 (ii), $DF(x, d)$ is K-compact and $DF(x, d) + K = \text{Min}(A) + K = A + K$. The desired equality follows from Proposition 3.3.

(ii) The "if" part follows from the definition and the "only if" part follows from the assertion (i) and the fact that $B = DF(x, d) = \text{Min}(A)$ for some $A \in W(x, d)$ and $\text{Min}(\text{Min}(A)) = \text{Min}(A)$. □

We provide some illustrating examples.

Example 4.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$.

(i) Let $F(x) := \{(x, 0), (0, x)\}$. One has $DF(0, 1) = \{(1, 0), (0, 1)\}$ and $DF(0, -1) = \{(-1, 0), (0, -1)\}$.

(ii) Let $F(x) := \{(|x|, 0), (0, |x|)\}$. Then $DF(0, 1) = DF(0, -1) = \{(1, 0), (0, 1)\}$.

(iii) Let

$$F(x) := \begin{cases} \{(x, 1), (x, 2)\} & \text{if } x \neq 0 \\ \{(0, 0), (0, 1)\} & \text{if } x = 0 \end{cases}$$

We will calculate $DF(x, d)$ at $x = 0$.

Let $d = 1$ and $t > 0$. Then $F(x + td) = F(t) = \{(t, 1), (t, 2)\}$, $F(t) - F(0) = \{(t, 1), (t, 2), (t, 0)\}$ and

$$A_t := (F(t) - F(0))/t = \{(1, 1/t), (1, 2/t), (1, 0)\}.$$

It is clear that $\mathrm{Min}(A_t) = \{(1, 0)\}$. Set $A := \mathrm{Min}(A_t)$. Corollary 3.1 implies that $d(A_t, A) = d(\mathrm{Min}(A_t), \mathrm{Min}(A_t)) = 0$. Therefore, $DF(0, 1) = \{(1, 0)\}$.

Let $d = -1$ and $t > 0$. Then $F(x + td) = F(-t) = \{(-t, 1), (-t, 2)\}$, $F(t) - F(0) = \{(-t, 1), (-t, 2), (-t, 0)\}$ and

$$A_t := (F(t) - F(0))/t = \{(-1, 1/t), (-1, 2/t), (-1, 0)\}.$$

Then $\mathrm{Min}(A_t) = \{(-1, 0)\}$. Set $A := \mathrm{Min}(A_t)$. Corollary 3.1 implies that $d(A_t, A) = d(\mathrm{Min}(A_t), \mathrm{Min}(A_t)) = 0$. Therefore, $DF(0, -1) = \{(-1, 0)\}$.

(iv) Let

$$F(x) := \begin{cases} \{(2x^2, 1), (3x, 2)\} & \text{if } x > 0 \\ \{(0, 0), (0, 1)\} & \text{if } x = 0 \\ \{(x^4, 1), (-x^3, 2), \} & \text{if } x < 0 \end{cases}$$

We will calculate $DF(x, d)$ at $x = 0$.

Let $d = 1$ and $t > 0$. Then $F(x+td) = F(t) = \{(2t^2, 1), (3t, 2)\}$, $F(t)-F(0) = \{(2t^2, 1), (3t, 2), (2t^2, 0), (3t, 1)\}$ and

$$A_t := (F(t) - F(0))/t = \{(2t, 1/t), (3, 2/t), (2t, 0), (3, 1/t)\}.$$

It is clear that $\mathrm{Min}(A_t) = \{(2t, 0)\}$ for $t < 1$. Set $A := \{(0, 0)\}$. Corollary 3.1 implies that $d(A_t, A) = d(\mathrm{Min}(A_t), A) = 2t$. Therefore, $DF(0, 1) = \{(0, 0)\}$.

Let $d = -1$ and $t > 0$. Then $F(x + td) = F(-t) = \{(t^4, 1), (-t^3, 2)\}$, $F(-t) - F(0) = \{(t^4, 1), (t^3, 2), (t^4, 0), (t^3, 1)\}$ and

$$A_t := (F(t) - F(0))/t = \{(t^3, 1/t), (t^2, 2/t), (t^3, 0), (t^2, 1/t)\}.$$

It is clear that $\mathrm{Min}(A_t) = \{(t^3, 0)\}$ for $t < 1$. Set $A := \{(0, 0)\}$. Corollary 3.1 implies that $d(A_t, A) = d(\mathrm{Min}(A_t), A) = t^3$. Therefore, $DF(0, -1) = \{(0, 0)\}$.

(v) Let $F : [0, +\infty[\rightrightarrows \mathbb{R}^2$ be defined by:

$$F(x) := \{(u, v) \mid u^2 + v^2 \leq x^2\}.$$

Let $x = 0$ and $d = 1$. We claim that

$$DF(0, 1) = \{(u, v) \in -\mathbb{R}^2_+ \mid u^2 + v^2 = 1\}$$

Let $t > 0$. We have $F(x+td) = F(t) = \{(u,v) \mid u^2+v^2 \leq t^2\}$, $F(t) - F(0) = F(t)$ and

$$A_t := (F(t) - F(0))/t = \{(u/t, v/t) \mid u^2+v^2 \leq t^2\} = \{(u,v)) \mid u^2+v^2 \leq 1\}.$$

Then $\mathrm{Min}(A_t) = \{(u,v) \in -\mathbb{R}^2_+ \mid u^2+v^2 = 1\}$. Set $A := \mathrm{Min}(A_t)$. Corollary 3.1 implies that $d(A_t, A) = d(\mathrm{Min}(A_t), \mathrm{Min}(A_t)) = 0$. Therefore, $DF(0,1) = \{(u,v) \in -\mathbb{R}^2_+ \mid u^2+v^2 = 1\}$.

We show that the directional derivative given in Definition 4.1 enjoys most properties of the one of a scalar- single-valued function and it is closely related to the coderivative of the considered set-valued map F.

Proposition 4.2. *Assume that d is an admissible direction of F at x and $DF(x,d) \neq \emptyset$. Then for any scalar $\lambda > 0$, we have $DF(x, \lambda d) = \lambda DF(x,d)$ and $D(\lambda F)(x,d) = \lambda DF(x,d)$.*

Proof. By Proposition 3.2 we have

$$d(\frac{F(x+\frac{t}{\lambda}\lambda d) - F(x)}{\frac{t}{\lambda}}, \lambda DF(x,d)) = \lambda d(\frac{F(x+td) - F(x)}{t}, DF(x,d)).$$

Therefore, setting $t' = \frac{t}{\lambda}$, we get

$$\lim_{t' \to 0^+} d(\frac{F(x+t'\lambda d) - F(x)}{t'}, \lambda DF(x,d)) = \lim_{t \to 0^+} d(\frac{F(x+td) - F(x)}{\frac{t}{\lambda}}, \lambda DF(x,d))$$

$$= \lambda \lim_{t \to 0^+} d(\frac{F(x+td) - F(x)}{t}, DF(x,d)) = 0.$$

To prove the first equality, it suffices to apply Proposition 4.1 (ii): we have $DF(x,\lambda d) = \lambda DF(x,d)$ because $\mathrm{Min}(\lambda DF(x,d)) = \lambda \mathrm{Min}(DF(x,d)) = \lambda DF(x,d)$. The second equality can be proved by similar arguments. □

Next, we establish a property of the directional derivative in the case F is **Lipschitz** in the sense of set-valued analysis, see e.g. [3].

Proposition 4.3. *Let $Y = \mathbb{R}^n$. Suppose that F is Lipschitz with the constant L on the neighborhood $U(x)$ of $x \in \mathrm{int\ dom}F$, i.e.*

$$F(x_1) \subseteq F(x_2) + L\mathbb{B}\|x_1 - x_2\|, \quad \forall x_1, x_2 \in U(x)$$

and F has the directional derivative $DF(x,d)$ at x in a direction d. Then

$$DF(x,d) \preceq_p L\|d\|\mathbb{B}.$$

Proof. Let $t > 0$ be sufficiently small so that $x+td \in U(x)$. The relation $F(x+td) \subset F(x) + Lt\|d\|\mathbb{B}$ implies that $(A_t - L\|d\|\mathbb{B}) \cap (-K) \neq \emptyset$ or $A_t \preceq_p L\|d\|\mathbb{B}$, where $A_t := (F(x+td) - F(x))/t$. Since $\lim_{t \to 0^+} d(A_t, DF(x,d)) = 0$, Proposition 3.4 (ii) yields that $DF(x,d) \preceq_p L\|d\|\mathbb{B}$. □

We consider now properties of the directional derivative in the **convex case**. Recall that F is *K-convex* [21, 24] if its domain is convex and for any $x_1, x_2 \in \mathrm{dom}F$ and $\lambda \in [0,1]$ one has $\lambda F(x_1)+(1-\lambda)F(x_2) \subseteq F(\lambda x_1+(1-\lambda)x_2)+K$ or, equivalently,

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_l \lambda F(x_1) + (1-\lambda)F(x_2).$$

One can check that if F is convex, then it is K-convex.

It is well-known that for a convex function $f : X \to \mathbb{R}$, the quotient $\frac{f(x+td)-f(x)}{t}$ is decreasing w.r.t. t and the inequality $f'(x,d) \le f(x+d) - f(x)$ holds. We show that similar results hold in the set-valued case.

Proposition 4.4. *Suppose that F is K-convex, $x \in \mathrm{dom}F$ and d is an admissible direction.*

(i) Let $r > 0$ be a scalar such that $x + rd \in \mathrm{dom}F$. Then for any scalar t such that $0 < t \le r$ we have

$$\frac{F(x+td) - F(x)}{t} \preceq_l \frac{F(x+rd) - F(x)}{r}. \tag{3}$$

(ii) Assume that $x + d \in \mathrm{dom}F$ and F has the directional derivative $DF(x,d)$ at x in the direction d. Then

$$DF(x,d) \preceq_l F(x+d) - F(x). \tag{4}$$

Proof. (i) Since $x + td = \frac{r-t}{r}x + \frac{t}{r}(x+rd)$, we get

$$F(x+td) \preceq_l \frac{r-t}{r}F(x) + \frac{t}{r}F(x+rd)$$

or

$$\frac{r-t}{r}F(x) + \frac{t}{r}F(x+rd) \subseteq F(x+td) + K.$$

Then for any $u \in F(x)$, $v \in F(x+rd)$ there exist $z \in F(x+td)$ and $k \in K$ such that $\frac{r-t}{r}u + \frac{t}{r}v = z + k$. It follows that $\frac{v-u}{r} = \frac{z-u}{t} + \frac{k}{t}$. Therefore, $(F(x+rd) - F(x))/r \subseteq (F(x+td) - F(x))/t + K$, which means that (3) is satisfied.

(ii) The relation (3) with $r = 1$ gives $A_t \preceq_l F(x+d) - F(x)$ for any $t \in]0,1[$, where $A_t := (F(x+td) - F(x))/t$. Since $\lim_{t\to 0^+} d(A_t, DF(x,d)) = 0$, Proposition 3.4 (ii) implies (4). □

Remark 4.1. Let us return to the map given in Example 4.1 (v). This map is $\mathbb{R}^2_+$-convex and (4) is satisfied for $DF(0,1)$, namely, $DF(0,1) \preceq_l F(1) - F(0)$, where $F(1) - F(0) = \{(u,v) \mid u^2 + v^2 \le 1\}$ and $DF(0,1) = \{(u,v) \in -\mathbb{R}^2_+ \mid u^2 + v^2 = 1\}$.

It is well-known the following relation between the directional derivative and the subdifferential of a scalar- single-valued convex function, see e.g. [31, Prop. 3.2]

Proposition 4.5. *Assume that $f : X \mapsto \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous (lsc) convex function. If f has the directional derivative at $x \in \mathrm{dom} f$ in any admissible direction, then*

$$\sup_{f(x+d)\le f(x)} \frac{-f'(x,d)}{\|d\|} = d(0, \partial f(x)).$$

We will extend the above result to the set-valued case. For this end, we need some notions and auxiliary results. Let us recall the concept of coderivative of convex analysis. Assume that F is convex and closed. The coderivative of convex analysis $D^*F(x, y)$ of F at $(x, y) \in \mathrm{gr}F$ is defined as follows: for any $y^* \in Y^*$,

$$D^*F(x,y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((x,y); \mathrm{gr}F)\}$$

[2]. Here, for a nonempty closed convex set Ω in Y, the normal cone $N(\bar{v}; \Omega)$ to Ω at $\bar{v} \in \Omega$ is defined by $N(\bar{v}; \Omega) = \{v^* \in Y^* \mid \langle v^*, v - \bar{v}\rangle \le 0 \text{ for all } v \in \Omega\}$.

Let $z \in Y$ and $x \in \mathrm{dom}F$. Define a function $g_{F,z} : X \to \mathbb{R}$ and a map $V_{F,z} : X \rightrightarrows Y$ by

$$g_{F,z}(x) := \inf_{y\in F(x)} \Delta_{-K}(y - z)$$

and

$$V_{F,z}(x) := \{y \in F(x) \mid \Delta_{-K}(y - z) = g_{F,z}(x)\}.$$

Recall that F is upper semicontinuous (in brief, usc) at $\bar{x} \in \mathrm{dom}F$ if for any neighborhood V of $F(\bar{x})$ there exists an neighborhood U of $\bar{x}$ such that $F(x) \subset V$ for any $x \in \mathrm{dom}F \cap U$. We say that F is usc if it is usc everywhere on its domain. Recall that if F is compact-valued and usc, then it is closed.

Lemma 4.1. *Let $x \in \mathrm{dom}F$.*

(i) If F is compact-valued, then $g_{F,z}(x) > -\infty$ and $V_{F,z}(x) \ne \emptyset$.

(ii) If F is compact-valued and usc on dom$F$, then $g_{F,z}$ is lsc on dom$F$.

(iii) If F is compact-valued and convex, then $g_{F,z}$ is convex and for any $y_x \in V_{F,z}(x)$ one has

$$\partial g_{F,z}(x) = \cup_{y^*\in\partial\Delta_{-K}(y_x - z)} D^*F(x, y_x)(y^*). \tag{5}$$

Proof. The first assertion follows from Lemma 3.2, the two others follow from [11, Prop. 2.2] and [10, Prop. 3.3]. □

We will use the following notations. Fix $x \in \mathrm{dom}F$. We define a function g as follows

$$g(u) := h(F(u), F(x)$$

for any $u \in \mathrm{dom}F$. By the definitions of h and $g_{F,y}$ we have

$$g(u) = \sup_{y\in F(x)} \inf_{v\in F(u)} \Delta_{-K}(v - y) = \sup_{y\in F(x)} g_{F,y}(u).$$

For a given $u \in X$, let

$$J(u) := \{z \in F(x) \mid g_{F,z}(u) = g(u)\}.$$

Proposition 4.6. *Assume that* $\mathrm{dom}F = X$*,* F *is convex usc compact-valued on* X *and* F *has directional derivative* $DF(x,.)$ *at* $x \in X$ *in any direction.*

(i) Assume that the set $\Theta_1(x)$ *defined by*

$$\Theta_1(x) := \cup_{y,z\in F(x),y^*\in\partial\Delta_{-K}(z-y)} D^*F(x,z)(y^*)$$

is nonempty. Then for any $d \in X$ *one has*

$$\frac{\sup_{v\in DF(x,d)}(-\Delta_{-K}(v))}{\|d\|} \leq d(0,\Theta_1(x)). \tag{6}$$

(ii) Assume that K *has a nonempty interior and* $d(0,\partial g(x)) > 0$*. Then*

$$\xi d(0,\partial g(x)) \leq \sup_{F(x+d)\preceq_l F(x)} \frac{\sup_{v\in DF(x,d)}(-\Delta_{-K}(v))}{\|d\|}. \tag{7}$$

If in addition $X = \mathbb{R}^n$ *and* $\mathrm{dom}F = \mathbb{R}^n$*, then*

$$\partial g(x) = \mathrm{co}\{\cup_{z\in J(x)} \cup_{y^*\in\partial\Delta_{-K}(y_x-z),y_x\in V_{F,z}} D^*F(x,y_x)(y^*)\}. \tag{8}$$

Here, $\xi := \sup\{d_{Y\setminus K}(k_o) \mid k_0 \in \mathrm{int}K\ , d_{-K}(k_0) = 1\}$ *and "*co*" stands for the convex hull of a set.*

Proof. (i) Let $x^* \in \Theta_1(x)$. Then one can find $y,z \in F(x)$ and $y^* \in \partial\Delta_{-K}(z-y)$ such that $x^* \in D^*F(x,z)(y^*)$. By the definition of the coderivative of convex anaalysis, $(x^*,-y^*) \in N((x,z);\mathrm{gr}F)$. Therefore, the inequality $\langle x^*, x'-x\rangle - \langle y^*, y'-z\rangle \leq 0$ holds for any $(x',y') \in \mathrm{gr}F$. Further, since $y^* \in \partial\Delta_{-K}(z-y)$ and the signed distance function satisfies the triangle inequality, we have

$$\langle y^*, y'-z\rangle \leq \Delta_{-K}(y'-y) - \Delta_{-K}(z-y) \leq \Delta_{-K}((y'-y)-(z-y)) = \Delta_{-K}(y'-z).$$

Then one has the inequality

$$\langle x^*, x'-x\rangle \leq \Delta_{-K}(y'-z), \qquad \forall (x,z),(x',y') \in \mathrm{gr}F. \tag{9}$$

Recall that by Proposition 4.1 we have $\lim_{t\downarrow 0^+} d(A_t, DF(x,d)) = 0$, where $A_t := (F(x+td)-F(x))/t$. Then for any $\epsilon > 0$ there exists $t > 0$ such that $d(A_t, DF(x,d)) \leq 2\epsilon$. Hence $h(A_t, DF(x,d)) \leq 2\epsilon$ or

$$\sup_{v\in DF(x,d)} \inf_{u\in A_t} \Delta_{-K}(u-v) \leq 2\epsilon.$$

For any $v \in DF(x,d)$ there exist $u \in A_t$ such that $\Delta_{-K}(u-v) \leq \epsilon$ and hence

$$\Delta_{-K}(u) \leq \Delta_{-K}(v) + \epsilon. \tag{10}$$

Choose $u_1 \in F(x+td)$ and $u_2 \in F(x)$ such that $u_1 - u_2 = tu$. Applying the inequality (9) to the pairs (x,u_1) and $(x+td,u_2)$, we get $\langle x^*, td\rangle \leq \Delta_{-K}(u_2-u_1)$ and taking (10) into account, we get

$$\langle x^*, d\rangle \leq \Delta_{-K}(\frac{u_2-u_1}{t}) = \Delta_{-K}(u) \leq \Delta_{-K}(v) + \epsilon.$$

Since $\epsilon > 0$ and $v \in DF(x,d)$ are arbitrary, we obtain

$$\langle x^*, d\rangle \leq \inf_{v\in DF(x,d)} \Delta_{-K}(v).$$

It is clear that the following relations hold

$$\frac{\sup_{v\in DF(x,d)}(-\Delta_{-K}(v))}{\|d\|} = -\frac{\inf_{v\in DF(x,d)} \Delta_{-K}(v)}{\|d\|} \leq -\frac{\langle x^*, d\rangle}{\|d\|} \leq \|x^*\|.$$

As $x^* \in \Theta_1(x)$ is arbitrarily chosen, the inequality (6) follows.

(ii) Let us prove (7). Note that $g(u)$ is finite on X by Lemma 3.2. Next, Lemma 4.1 (iii) yields that $g_{F,y}$ is convex and so is g. Since $d(0, \partial g(x)) > 0$ by the assumption, we have $0 \notin \partial g(x)$. Then x is not a global minimizer of the convex function g, which means that there exists at least a vector $d \in X$ such that $g(x+d) < g(x) = 0$ or $h(F(x+d), F(x)) < 0$. Lemma 3.3 yields that $F(x+d) \preceq_l F(x)$. Thus, the right-hand part in the inequality (7) is meaningful.

Let η be a scalar such that

$$0 < \eta < d(0, \partial g(x)) \tag{11}$$

and let $k_0 \in \text{int}K$ be such that $\Delta_{-K}(k_0) = d_{-K}(k_0) = 1$. Define a set-valued map $G : u \in X \rightrightarrows G(u) := F(u) + \eta\|u - x\|k_0$. We claim that x is not a global $\preceq_l$-minimizer of G. Suppose to the contrary that x is a $\preceq_l$ minimizer of G. Observe that $G(x) = F(x)$. Then, for any $u \in X$, we have either $G(u)+K = G(x)+K = F(x)+K$ or $G(u) \not\preceq_l G(x) = F(x)$. In the first case, Proposition 3.4 yields $h(G(u), F(x)) = h(F(x), F(x)) = 0$. In the second case, Proposition 3.4 yields $h(G(u), F(x)) > 0$. So, for any u we have $h(G(u), F(x)) \geq 0$. Since $G(u) = F(u) + \eta\|u - x\|k_0$, one can easily derive from the triangle inequality property of the function Δ_{-K} that

$$h(F(u), F(x)) + \eta\|x - u\|\Delta_{-K}(k_0) \geq h(G(u), F(x)) \geq 0.$$

Thus, x is a minimizer of the function $g(.) + \eta\|x - .\|$ and therefore, $0 \in \partial(g(.) + \eta\|x - .\|)(x)$. The exact sum rule for subdifferential of convex analysis gives

$$0 \in \partial(g(.) + \eta\|x - .\|)(x) = \partial g(x) + \eta\{x^* \in X^* \mid \|x^*\| \leq 1\}.$$

It follows that $d(0, \partial g(x)) \leq \eta$, which is a contradiction to (11).

We have showed that x is not a global $\preceq_l$ minimizer of G. Then there exists $d \in X$ such that $G(x + d) \preceq_l G(x) = F(x)$ or

$$F(x + d) + \eta\|d\|k_0 \preceq_l F(x).$$

Take an arbitrary element $y \in F(x)$. Then there exists $z \in F(x + d)$ and $k_1 \in K$ such that $z - y + \eta\|d\|k_0 + k_1 = 0$. On the other hand, $F(x+d) - F(x) \subseteq DF(x, d) + K$ by Proposition 4.4. Therefore, there exist $v \in DF(x, d)$ and $k_2 \in K$ such that $z - y = v + k_2$. Then $v + k_2 + \eta\|d\|k_0 + k_1 = 0$ and we get $\Delta_{-K}(v + \eta\|d\|k_0) \leq 0$. Therefore, $\Delta_{-K}(v) - \eta\|d\|\Delta_{-K}(-k_0) \leq 0$ and

$$\frac{-\Delta_{-K}(v)}{\|d\|} \geq -\eta\Delta_{-K}(-k_0) = \eta d_{Y\setminus(-K)}(-k_0) = \eta d_{Y\setminus K}(k_0).$$

We get

$$\frac{\sup_{v\in DF(x,d)}(-\Delta_{-K}(v))}{\|d\|} \geq \eta d_{Y\setminus K}(k_0).$$

It is clear that $F(x+d) \preceq_l F(x)$ and since $k_0 \in K$ satisfying $\Delta_{-K}(k_0) = d_{-K}(k_0) = 1$ is arbitrarily chosen, it follows that

$$\sup_{F(x+d)\preceq_l F(x)} \frac{\sup_{v\in DF(x,d)}(-\Delta_{-K}(v))}{\|d\|} \geq \xi\eta.$$

As η satisfying (11) is arbitrarily chosen, we obtain (7).

It remain to prove (8). Observe that $J(u) \neq \emptyset$ by Lemma 3.2. If for each $u \in \mathbb{R}^n$ the function $y \mapsto g_{F,y}(u)$ is upper semicontinuous at any $y \in F(x)$, i.e.

$$\limsup_{y'\to y, y'\in F(x)} g_{F,y'}(u) \leq g_{F,y}(u), \tag{12}$$

then we can apply Theorem 4.4.2 in [16] to get

$$\partial g(u) = \mathrm{co}\{\cup \partial g_{F,y}(u) \mid y \in J(u)\},$$

which together with (5) imply (8). Let us prove (12). For any $v \in F(u)$ and $y' \in F(x)$, we have $\Delta_{-K}(v - y') \leq \Delta_{-K}(v - y) + \Delta_{-K}(y - y')$. Then we obtain $\inf_{v\in F(u)} \Delta_{-K}(v - y') \leq \inf_{v\in F(u)} \Delta_{-K}(v - y) + \Delta_{-K}(y - y')$ or $g_{F,y'}(u) \leq g_{F,y}(u) + \Delta_{-K}(y - y')$. From the last inequality, one can easily deduce that (12) holds. $\square$

Remark 4.2. Note that $\xi > 0$ and, since $d_{Y\setminus K}(k_o) \leq d_{-K}(k_0) = 1$, we have $\xi \leq 1$. In particular, $\xi = \sqrt{2}/2$ when $K = \mathbb{R}^2_+$ and $\xi = 1$ when $K = \mathbb{R}_+$.

Let us consider the case F is a convex single-valued function $f : X \mapsto \mathbb{R}$. Here, $K = \mathbb{R}_+$ and $\partial\Delta_{-\mathbb{R}_+}(y) = \{1\}$ for any $y \in \mathbb{R}$ (see Example 3.1). Further, for $y^* \in \partial\Delta_{-\mathbb{R}_+}(f(x)) = \{1\}$ we have $D^*f(x,1) = \{x^* \mid (x^*, -1) \in N((x, f(x)), \mathrm{epi}f)\} = \partial f(x)$, and hence, $\Theta_1(x) = \partial f(x)$. We also have $\partial g(x) = \partial f(x)$. Recall that $\xi = 1$. Thus, Proposition 4.6 reduces to Proposition 4.5.

5 Necessary and/or sufficient conditions for minimizers and maximizers of a set-valued map

In the existing literature, there have been obtained necessary and/or sufficient conditions in term of directional derivatives for minimizers defined by the set order relations $\preceq_l$, $\preceq_u$ and $\preceq_s$. In this section, we obtain these conditions for some maximizers and minimizers defined by set order relations $\preceq_l$, $\preceq_u$, $\preceq_c$, $\preceq_p$ as well as by strict set order relations.

Let $\preceq$ denote one of the order relations in Definitions 2.2 and 2.3.

Definition 5.1. Let $x \in \mathrm{dom}F$. We say that

(i) x is a *local $\preceq$-minimizer* of F if there is a neighborhood U of x such that for any $x' \in U \cap \mathrm{dom}F$, $x' \neq x$, one has

$$F(x') \preceq F(x) \text{ implies } F(x) \preceq F(x').$$

(ii) x is a *local strict $\preceq$-minimizer* of F if there is a neighborhood U of x such that for any $x' \in U \cap \mathrm{dom}F$, $x' \neq x$ one has

$$F(x') \not\preceq F(x).$$

(iii) x is a *local ideal $\preceq$-minimizer* of F if there is a neighborhood U of x such that for any $x' \in U \cap \mathrm{dom}F$, $x' \neq x$ one has

$$F(x) \preceq F(x').$$

(iv) x is a *local ideal $\preceq$-maximizer* of F if there is a neighborhood U of x such that for any $x' \in U \cap \mathrm{dom}F$, $x' \neq x$ one has

$$F(x') \preceq F(x).$$

When $U = X$ in the above definitions, we have the corresponding global concepts.

Lemma 2.1 implies that the concepts of maximizer/minimizer are “weakest” when they are defined by the set order relation $\preceq_p$ and are “strongest” when they are defined by the set order relation $\preceq_c$.

Let us formulate necessary conditions.

Proposition 5.1. *Suppose that F has the directional derivative $DF(x, d)$ at $x \in \mathrm{dom}F$ in an admissible direction d.*

(i) If x is a local ideal $\preceq_c$-minimizer of F, then

$$\{0\} \preceq_c DF(x, d). \tag{13}$$

(ii) If x is a local ideal $\preceq$-maximizer of F, then

$$DF(x, d) \preceq_p \{0\}, \tag{14}$$

where $\preceq$ denotes one of the relations $\preceq_p$, $\preceq_l$, $\preceq_u$ and $\preceq_c$.

Note that the relations (13) and (14) are equivalent to $DF(x, d) \subset K$ and $DF(x, d) \cap (-K) \neq \emptyset$, respectively. When $Y = \mathbb{R}$ and F is a scalar- single-valued function f, they become $f'(x, d) \geq 0$ and $f'(x, d) \leq 0$, respectively.

Proof. Let $t > 0$ such that $x + td \in U \cap \mathrm{dom}F$, where U is the neighborhood of x mentioned in Definition 5.1.

(i) As x is a local ideal $\preceq_c$-minimizer, we have $F(x+td)-F(x) \subseteq K$ or $\{0\} \preceq_l A_t$, where $A_t := (F(x+td) - F(x))/t$. Proposition 3.4 (i) applied to $A := DF(x, d)$ and $B := \{0\}$ gives $\{0\} \preceq_l DF(x, d)$. Hence, $\{0\} \preceq_c DF(x, d)$.

(ii) By Lemma 2.1, x is a local ideal $\preceq_p$-maximizer in all cases. Hence, we have $(F(x + td) - F(x)) \cap (-K) \neq \emptyset$ or $A_t \preceq_p \{0\}$, where $A_t := (F(x + td) - F(x))/t$. Proposition 3.4 (ii) applied to $A = DF(x, d)$ and $B = \{0\}$ gives $DF(x, d) \preceq_p \{0\}$. $\square$

Remark 5.1. Observe that $DF(x,d) \preceq_p \{0\}$ is equivalent to $DF(x,d) \preceq_l \{0\}$. In the case with $\preceq_c$ and $\preceq_u$ in Proposition 5.1 (ii), we do not know yet whether the relation $DF(x,d) \preceq_p \{0\}$ could be replaced by the stronger one $DF(x,d) \preceq_c \{0\}$, which is equivalent to $DF(x,d) \preceq_u \{0\}$ and $DF(x,d) \subset K$, or not.

Example 5.1. (i) Let F be the map in Example 4.1 (ii). Then $x = 0$ is an ideal $\preceq_c$-minimizer of F. Recall that $DF(0,1) = DF(0,-1) = \{(1,0),(0,1)\}$. The necessary condition (13) is satisfied as $\{(0,0)\} \preceq_c \{(1,0),(0,1)\}$.

(ii) Let F be the map in Example 4.1 (iii). Then $x = 0$ is not an ideal $\preceq_c$-minimizer of F because the necessary condition (13) is not satisfied: $DF(0,-1) = \{(-1,0)\}$ and $\{(0,0)\} \not\preceq_c \{(-1,0)\}$.

(iii) Let F be the map in Example 4.1 (iv). One can check that $x = 0$ is an ideal $\preceq_c$-minimizer of F. We have $DF(0,1) = DF(0,-1) = \{(0,0)\}$ and $\{(0,0)\} \preceq_c \{(0,0)\}$, which means that the necessary condition (13) is satisfied.

(iv) Let F be the map in Example 4.1 (v). One can see that $x = 0$ is an ideal $\preceq_l$-maximizer of F and the necessary condition (14) is satisfied because $DF(0,1) = \{(u,v) \in -\mathbb{R}^2_+ \mid u^2 + v^2 = 1\}$ and $\{(u,v) \in -\mathbb{R}^2_+ \mid u^2 + v^2 = 1\} \preceq_l \{(0,0)\}$.

Let us formulate sufficient conditions for several types of global minimizers under a convexity assumption.

Proposition 5.2. *Suppose that F is K-convex and F has the directional derivative $DF(x,.)$ at $x \in \operatorname{dom}F$ in any admissible direction. Then*

(i) x is a global ideal $\preceq_c$-minimizer of F if for any admissible direction d one has

$$\{0\} \preceq_c DF(x,d).$$

The assertion remains true if we replace the set order relation $\preceq_c$ by the strict one $\prec_c$ (assuming that $\operatorname{int}K \neq \emptyset$).

(ii) x is a global strict $\preceq$-minimizer of F if for any admissible direction d one has

$$DF(x,d) \not\preceq_p \{0\},$$

where $\preceq$ denotes one of the relations $\preceq_p$, $\preceq_l$, $\preceq_u$ and $\preceq_c$.

The assertion remains true if we replace the involved set order relations by the corresponding strict ones (assuming that $\operatorname{int}K \neq \emptyset$).

Proof. Since F is K-convex, Proposition 4.4 yields that for any admissible direction d such that $x + d \in \operatorname{dom}F$ one has

$$F(x+d) - F(x) \subseteq DF(x,d) + K.$$

(i) Since $\{0\} \preceq_c DF(x,d)$ gives $DF(x,d) \subseteq K$, we deduce from $F(x+d)-F(x) \subseteq DF(x,d) + K \subseteq K$ that x is a global ideal $\preceq_c$-minimizer of F. Next, assume that $\{0\} \prec_c DF(x,d)$ or $DF(x,d) \subseteq \text{int}K$. Then we have $F(x+d) - F(x) \subseteq DF(x,d) + K \subseteq \text{int}K$. This means that x is a global ideal $\prec_c$-minimizer of F.

(ii) Suppose to the contrary that x is not a global strict $\preceq$-minimizer of F. By Lemma 2.1, x is not a global strict $\preceq_p$-minimizer of F in all cases. Then there exists d such that $F(x+d) \preceq_p F(x)$ or $(F(x+d) - F(x)) \cap (-K) \neq \emptyset$. On the other hand, $DF(x,d) \npreceq_p \{0\}$ implies $DF(x,d) \cap (-K) = \emptyset$ and hence $(DF(x,d)+K) \cap (-K) = \emptyset$. The inclusion $F(x+d) - F(x) \subseteq DF(x,d) + K$ yields $(F(x+d) - F(x)) \cap (-K) = \emptyset$, a contradiction.

Next, suppose to the contrary that x is not a global strict $\prec$-minimizer of F. By Lemma 2.1, x is not a global strict $\prec_p$-minimizer of F in all cases.Then $F(x+d) \prec F(x)$ for some d or $(F(x+d)-F(x)) \cap (-\text{int}K) \neq \emptyset$. On the other hand, $DF(x,d) \not\prec_p \{0\}$ implies $DF(x,d) \cap (-\text{int}K) = \emptyset$ and hence $(DF(x,d)+K) \cap (-\text{int}K) = \emptyset$. The inclusion $F(x+d) - F(x) \subseteq DF(x,d) + K$ yields $(F(x+d) - F(x)) \cap (-\text{int}K) = \emptyset$, a contradiction. $\square$

Remark 5.2. The assertions (i) in Propositions 5.1 and 5.2 provide necessary and (in the convex case) sufficient conditions for a $\preceq_c$-minimizer of F.

Example 5.2. Let F be the map in Example 4.1 (i). This map $\mathbb{R}^2_+$-convex and satisfies the relation $DF(x,d) \not\prec_p \{0\}$ because $DF(0,1) = \{(1,0),(0,1)\}$ and $DF(0,-1) = \{(-1,0),(0,-1)\}$. Therefore, $x = 0$ is a global strict $\prec_c$ minimizer of F.

When F is not assumed to be K-convex, the relation (4) may not be satisfied. Nevertheless, we have the following sufficient conditions for local minimizers which hold in a finite dimensional space settings under an additional condition.

Proposition 5.3. *Assume that X is a finite dimensional space and* dom$F = X$. *Suppose that F has the directional derivative $DF(x,d)$ at x in any direction $d \in X$, $\|d\| = 1$ and possesses the following property with respect to d: any sequences $\{t_i\}$ satisfying $t_i \downarrow 0^+$ and $\{d_i\}$ satisfying $\|d_i\| = 1$, $d_i \to d$ contain subsequences $\{t_{i_j}\}$ and $\{d_{i_j}\}$ such that*

$$DF(x,d) \preceq_l \frac{F(x + t_{i_j} d_{i_j}) - F(x)}{t_{i_j}}. \tag{15}$$

Then the assertions of Proposition 5.2 with "global" replaced by "local" hold true.

Proof. We will use some arguments similar to the ones used in the proof of Proposition 5.2. Observe that (15) is equivalent to

$$\frac{F(x + t_{i_j} d_{i_j}) - F(x)}{t_{i_j}} \subseteq DF(x,d) + K.$$

(i) Suppose to the contrary that x is not a local ideal $\preceq_c$-minimizer of F. Then there exists a sequence $\{x_i\}$ such that $x_i \to x$ and $F(x) \npreceq_c F(x_i)$ or $F(x_i) - F(x) \nsubseteq K$ for all i. On the other hand, let $d_i = \frac{x_i - x}{\|x_i - x\|}$ and $t_i = \|x_i - x\|$. Clearly $t_i \downarrow 0^+$ and we may assume that $d_i \to d$ for some $d \in X$ because $\|d_i\| = 1$ for all i and X is a finite dimensional space. By the assumptions, one can find some subsequence $\{x_{i_j}\}$ such that

$$\frac{F(x_{i_j}) - F(x)}{\|x_{i_j} - x\|} = \frac{F(x + t_{i_j} d_{i_j}) - F(x)}{t_{i_j}} \subseteq DF(x,d) + K.$$

Since $\{0\} \preceq_c DF(x,d)$ is equivalent to $DF(x,d) \subseteq K$, it follows that $F(x_{i_j}) - F(x) \subseteq K$, a contradiction.

(ii) Suppose to the contrary that x is not a local strict $\preceq$-minimizer of F. By Lemma 2.1, x is not a local strict $\preceq_p$-minimizer of F in all cases. Then there exists a sequence $\{x_i\}$ such that $x_i \to x$ and $(F(x_i) - F(x)) \cap (-K) \neq \emptyset$ for all i. On the other hand, similarly to the case (i), we can find a subsequence $\{x_{i_j}\}$ and $d \in X$ such that $\frac{F(x_{i_j}) - F(x)}{\|x_{i_j} - x\|} \subseteq DF(x,d) + K$. Since $DF(x,d) \not\preceq_p \{0\}$, we have $DF(x,d) \cap (-K) = \emptyset$ and therefore, $(DF(x,d) + K) \cap (-K) = \emptyset$. Then $(F(x_{i_j}) - F(x)) \cap (-K) = \emptyset$, a contradiction.

The proof in the case with strict order relations can be proved by similar arguments and is then omitted. $\square$

Example 5.3. Let F be the map in Example 4.1 (iv). This map satisfies all conditions of Proposition 5.3 at $x = 0$ and satisfies $\{0\} \preceq_c DF(x,d)$ because $DF(x,d) = \{(0,0)\}$. Therefore, $x = 0$ is a local ideal $\preceq_c$ minimizer of F.

Acknowledgement: The research started during my stay at the Department of Mathematics, University of Erlangen-Nuremberg, Germany, under a Georg Forster grant from the Alexander von Humboldt Foundation, and was partially supported by the Vietnam National Foundation for Science and Technology Development.

I would like to express my deepest gratitude to Professor J. Jahn for his kindness, hospitality and discussions.

References

[1] Anh Tuan, V., Tammer, C., Zalinescu, C.: The Lipschitzianity of convex vector and set-valued functions. *TOP* 24, no. 1, 273-299 (2016).

[2] Aubin, J.-P., Ekeland, I.: *Applied nonlinear analysis* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984.

[3] Aubin, J.-P., Frankowska, H.: *Set-Valued Analysis. Systems & Control: Foundations & Applications*, 2. Birkhäuser Boston, Inc., Boston, MA, 1990.

[4] Bao, Truong Q., Mordukhovich, B. S.: Set-valued optimization in welfare economics. *Advances in mathematical economics* Volume 13, 113-153, Adv. Math. Econ., 13, Springer, Tokyo, 2010.

[5] Baier R., Farkhi, E.: Regularity of set-valued maps and theor selections through set differences: Part 1: Lipschitz continuity. *Serdica Math. J.* 39, 365-390 (2013).

[6] Crespi, Giovanni P.; Schrage, C.: Set optimization meets variational inequalities. *Set optimization and applicationsthe state of the art*, 213-247, Springer Proc. Math. Stat., 151, Springer, Heidelberg, 2015.

[7] Demyanov V. F., Rubinov, A. M.: *Foundations of Nonsmooth Analysis and Quasidifferential Calculus* (in Russian), Nauka, Moscow, 1990.

[8] Dempe, S., Pilecka, M.: Optimality conditions for set-valued optimization problems using a modified Demyanov difference. *J. Optim. Theory Appl.* 171, no. 2, 402-421 (2016).

[9] Ha, T. X. D.: Existence and density results for proper efficiency in cone compact sets. *J. Optim. Theory Appl.* 111, no. 1, 173-194 (2001).

[10] Ha, T. X. D.: Lagrange multipliers for set-valued optimization problems associated with coderivatives. *J. Math. Anal. Appl.* 311, no. 2, 647-663 (2005).

[11] Ha, T. X. D.: Estimates of error bounds for some sets of efficient solutions of a set-valued optimization problem. *Set optimization and applicationsthe state of the art*, 249-273, Springer Proc. Math. Stat., 151, Springer, Heidelberg, 2015.

[12] Hamel, A., Schrage, C.: Notes on extended real- and set-valued functions. *J. Convex Anal.* 19, no. 2, 355-384 (2012).

[13] Hamel, A., Schrage, C.: Directional derivatives, subdifferentials and optimality conditions for set-valued convex functions. *Pac. J. Optim.* 10, no. 4, 667-689 (2014).

[14] Hernández, E.: A survey of set optimization problems with set solutions. *Set optimization and applications - the state of the art* 143-158, Springer Proc. Math. Stat., 151, Springer, Heidelberg, 2015.

[15] Hiriart-Urruty, J. B.: New concepts in nondifferentiable programming. *Bull. Soc. Math. France* 60, 57-85 (1979).

[16] Hiriart-Urruty, J. B., Lemaréchal, C.: *Convex Analysis and Minimization Algorithms I*, Springer, New York, 1993.

[17] Hoheisel, T., Kanzow, C., Mordukhovich, B. S., Phan, H.: Generalized Newtons method based on graphical derivatives. *Nonlinear Anal.* 75, 1324-1340 (2012).

[18] Jahn, J.: *Vector Optimization - Theory, Applications and Extensions* Springer, Heidelberg, 2011.

[19] Jahn, J., Directional derivative in set optimization with the less oder relation. *Taiwanese J. Math.* 19, no. 3, 737-757 (2015).

[20] Jahn J., Ha, T. X. D.: New order relations in set optimization. *J. Optim. Theory Appl.* 148, 209-236 (2011).

[21] Kuroiwa, D.: Convexity for set-valued maps. *Appl. Math. Lett.* 9, no. 2, 97-101 (1996).

[22] Kuroiwa, D.: Natural Criteria of Set-valued Optimization, manuscript, Shimane University, Japan, 1998.

[23] Kuroiwa, D.: On derivatives and convexity of set-valued maps and optimality conditions in set optimization. *J. Nonlinear Convex Anal.* 10, 41-50 (2009).

[24] Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Proceedings of the Second World Congress of Nonlinear Analysts, Part 3 (Athens, 1996). *Nonlinear Anal.* 30, no. 3, 1487-1496 (1997).

[25] Luc, D.T.: *Theory of Vector Optimization.* Lecture Notes in Economics and Mathematical Systems, 319. Springer-Verlag, Berlin, 1989.

[26] Mordukhovich, B. S.: *Variational Analysis and Generalized differentiation, I: Basic Theory.* Grundlehren Series (Fundamental Principles of Mathematical Sciences) 330, Springer, Berlin, 2006.

[27] Neukel, N.: Order relations of sets and its application in socio-economics. *Appl. Math. Sci. (Ruse)* 7, Vol. 113-113, 5711-5739 (2013).

[28] Pilecka, M.: *Optimality Conditions in Set-valued Programming Using the Set Criterion*, Preprint 2014-02, Technical University of Freiberg, Germany, 2014.

[29] Rubinov, A. M., Akhundov, I. S.: Difference of compact sets in the sense of Demyanov and its application to non-smooth analysis. *Optimization* 23, 179-188 (1992).

[30] Zaffaroni, A.: Degrees of efficiency and degrees of minimality. *SIAM J. Control Optim.* 42, no. 3, 1071-1086 (2003).

[31] Zalinescu, C.: Weak sharp minima, well behaving functions and global error bounds for convex inequalities in Banach spaces, in *Optimization Methods and their Applications* (V. Bulatov and V. Baturin Eds.) Irkutsk, Baikal (2001) 272-284.