On the Global Łojasiewicz inequality for polynomial functions

Huy Vui Ha *and Van Doat Dang †

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Abstract:. Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a polynomial in n variables. We study the global Łojasiewicz inequality of f :

$$|f(x)| \geq c.\min\{dist(x, f^{-1}(0))^{\alpha}, dist(x, f^{-1}(0))^{\beta}\}$$

for all $x \in \mathbb{R}^n$, where c, α, β are positive constants. We give a method to check for the existence of this inequality. In the case $n = 2$, our method gives complete results on some problems of interests: (a) Computating and estimating of the global Łojasiewicz exponents; (b) Studying the global Łojasiewicz inequality for polynomials which are non-degenerate at infinity; (c) Computation of the exponent involved in the Hörmander version of the global Łojasiewicz inequality.

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1 Introduction

This work is devoted to studying the global Łojasiewicz inequality of polynomial functions. The classical local Łojasiewicz inequality says that if $f\colon U \to \mathbb{R}$ is an analytic function in a bounded

*Institute of Mathematics, 18 Hoang Quoc Viet, Cau Giay Distict 10307, Ha noi, Viet nam. Email: hhvui@math.ac.vn. This author was partially supposted by Vietnam National Foundation for Science and Technology Development (NAFOSTED) grant number 101.04-2014.23.

†Thang Long high school for the gifted,10 Tran Phu, Da lat, Viet nam. Email: dangvandoat@yahoo.com.vn. This author was partially supposted by Vietnam National Foundation for Science and Technology Development (NAFOSTED) grant number 101.04-2016.05.

domain $U \subset \mathbb{R}^n$, then there exist $\alpha > 0$ and $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\alpha} \quad \text{for all } \; x \in U.$$

This deep result was born to solve one of the big problems of analysis (see [L], [Hor]). Later it become an important tool for studying many other problems not only of analysis, but also of algebra, geometry and other fields (see [T] and the references therein).

This inequality does not hold if instead of the bounded domain U, we take $\mathbb{R}^n$. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, in [DHT], the criterion for the existence of the global Łojasiewicz inequality for f was given:

(i) There exist $\delta > 0$ and $\mathcal{L}_0 > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0}, \quad \text{for all } \; x \in \mathbb{R}^n, |f(x)| \leq \delta$$

if only if there are no sequences of the first type

$$x^k \to \infty, |f(x^k)| \to 0, dist(x^k, f^{-1}(0)) \geq M_0 > 0;$$

(ii) There exist $\Delta > 0$ and $\mathcal{L}_\infty > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_\infty}, \quad \text{for all } \; x \in \mathbb{R}^n, |f(x)| \geq \Delta$$

if only if there are no sequences of the second type

$$x^k \to \infty, |f(x^k)| < M < +\infty, dist(x^k, f^{-1}(0)) \to +\infty;$$

(iii) There exist $c > 0, \mathcal{L}_0 > 0$ and $\mathcal{L}_\infty > 0$ such that

$$|f(x)| \geq c. \min\{dist(x, f^{-1}(0))^{\mathcal{L}_0}, dist(x, f^{-1}(0))^{\mathcal{L}_\infty}\} \quad \text{for all } \; x \in \mathbb{R}^n \tag{1}$$

if only if there are no sequences of the first and the second type.

In this paper, we will call inequalities of type (1) the global Łojasiewicz inequalities of f.

Among other thing, in this paper we consider two following questions

(i) How to check, whether condition (iii) holds or not?

(ii) How to compute the Łojasiewicz exponents?

We are able to give complete answers to these questions only for the case of two variables.
In the whole paper we assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of the form

$$f(x_1, \cdots, x_n) = a_0 x_n^d + a_1(x')x_n^{d-1} + \cdots + a_d(x') \tag{2}$$

where d is the degree of f and each of $a_i(x'), i = 0, \cdots, d$ is a polynomial (of degree $\leq i$) in $x' = (x_1, \cdots, x_{n-1})$.
We always put

$$V_1 \colon = \{x \in \mathbb{R}^n | \frac{\partial f}{\partial x_n} = 0\}.$$

Our key observation is following: The set V_1 can be considered as a testing set for the existence of the global Łojasiewicz inequality of f. That is, an inequality of forms (1) holds true for all $x \in \mathbb{R}^n$ if and only if an inequality of the same form (possibly with other exponents) holds for all points x from the subset V_1 of $\mathbb{R}^n$. This fact is rather usefull for studying the global Łojasiewicz inequality of polynomials, especially for the case of two variables.
Besides of Introduction, the paper consists of 7 sections. We begin our study with section 2, where the global Łojasiewicz inequality of f w.r.t. V_1 is investigated. Depending on the behavior of f on V_1, this inequality, if it exists, has one of 4 forms. Each of these forms clearly indicates which one of the Łojasiewicz exponents is involved. Section 3 is devoted to the global Łojasiewicz inequality of f. We will show that this inequality exists if and only if the global Łojasiewicz inequality of f w.r.t.V_1 exists. This justifies the name of V_1 as "the testing set" for the existence of the global Łojasiewicz inequality. Moreover, as in section 2, the global Łojasiewicz inequality of f has also 4 forms, each of them is dictated by the corresponding one's of the global Łojasiewicz inequality of f w.r.t.V_1. The Łojasiewicz exponents will be investigated in section 4. It turns out that the Łojasiewicz exponents of the global Łojasiewicz inequality possede two-sides estimation in terms of the Łojasiewicz exponents w.r.t. V_1 and the degree of the polynomial. Moreover, we will indicate some cases, where the Łojasiewicz exponents of the global Łojasiewicz inequality can be computed via the Łojasiewicz exponents w.r.t. V_1 and the degree. In section 5, the method of verifying whether the global Łojasiewicz inequality exists or not will be proposed for the case of two variables. In section 6, for $n = 2$, we give explicit formulas for the Łojasiewicz exponents in terms of Puiseux expansions. Section 7 deals with polynomials in two variables, which are non-degenerate at infinity. In this case, it turns out that the Łojasiewicz inequality near to the set $f^{-1}(0)$, i.e. in the domain $\{x \in \mathbb{R}^n | |f(x)| < \delta\}, \delta$ sufficiently small, always exists, while the global Łojasiewicz inequality of f may exist on not. It does exist, if, in addition, f is convenient. In this case, the Łojasiewicz exponents are computed explicitly and they depend only the part at infinity of the Newton polygon of f. Let us present breafly the content of section 8. In [Hor], Hörmander proved the following: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, then there exist $c > 0$ and $\mu > 0, \mu' > 0, \mu'' > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mu}, \quad \text{for all } x \in \mathbb{R}^n, \| x \| < 1;$$

and

$$(1+\parallel x\parallel)^{\mu'}|f(x)| \geq c.dist(x, f^{-1}(0))^{\mu''} \quad \text{for all } x \in \mathbb{R}^n, \parallel x \parallel \geq 1.$$

Clearly, the exponent μ is the Łojasiewicz exponent of f in the domain $\parallel x \parallel< 1$, and the factor $(1+\parallel x\parallel)^{\mu'}$ in the left-hand side of the second inequality is needed for controlling the "bad" behavior of the function $dist(x, f^{-1}(0))$, for $\parallel x\parallel$ sufficiently large. We will propose, always for $n = 2$, a version of the Hörmander inequality, in which a concrete value of the exponent μ' is computed explicitly.

Note that the role of the set V_1 in studying the global Łojasiewicz inequalities was investigated in [DHT], [DKL] and [HNS]. The fact that V_1 is the testing set for the existence of the Łojasiewicz inequality of a germ of a smooth function was observed in [H2].

2 Global Łojasiewicz inequality w.r.t.to V_1

Recall that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of the form

$$f(x', x_n) = a_0x_n^d + a_1(x')x_n^{d-1} + \cdots + a_d(x'),$$

where d is the degree of f and $x' = (x_1, \cdots, x_{n-1})$.

$$V_1\colon = \{x \in \mathbb{R}^n | \frac{\partial f}{\partial x_n} = 0\}$$

This section, we assume that V_1 is non-empty set, not contained in $f^{-1}(0)$.

Definition 2.1. *A sequence $\{x^k\} \subset \mathbb{R}^n$ is said to be a sequence of the first type of f, with respect to V_1, if the following conditions holds:*

$$\{x^k\} \subset V_1,$$

$$x^k \to \infty,$$

$$f(x^k) \to 0$$

and

$$dist(x, f^{-1}(0)) \geq M_0 > 0, \quad \textit{for some } M_0.$$

Proposition 2.2. *The following startements are equivalent*

(i) There are no sequences of the first type of f with respect to V_1;

(ii) There exist $\delta > 0$ such that either the set $\{x \in \mathbb{R}^n || f(x)| < \delta\} \cap V_1$ is empty or there are $c > 0$ and a positive rational number $\mathcal{L}_0(V_1)$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}$$

for all $x \in \{x \in \mathbb{R}^n || f(x)| \leq \delta\} \cap V_1$.

Proof. (i)$\Rightarrow$ (ii). Let

$$f_*: \; = \inf_{x \in V_1} |f(x)|,$$

if $f_* > 0$ then for $0 < \delta < f_*$, the set $\{x \in \mathbb{R}^n || f(x)| < \delta\} \cap V_1$ is empty.
Assume that $f_* = 0$. For $t > 0$, put

$$\varphi(t) = \sup_{|f(x)|=t, x \in V_1} dist(x, f^{-1}(0))$$

Since $f_* = 0$, the set $\{x \in \mathbb{R}^n || f(x)| = t\} \cap V_1$ not empty for each $t > 0$, sufficiently closed to zero. By (i), $\varphi(t)$ is well defined on $[0, \delta)$, with $\delta > 0$ sufficiently small. It follows from the Tarski - Seidenberg theorem that $\varphi(t)$ is a semi-algebraic function. Moreover, the condition (i) implies that $\varphi(t) \to 0$ as $t \to 0$. Hence, there exists $c_0 \neq 0$ and a positive rational number $\alpha(V_1)$ such that

$$\varphi(t) = c_0 t^{\alpha(V_1)} + o(t^{\alpha(V_1)}) \quad \text{for } t \to 0.$$

Let $\mathcal{L}_0(V_1): \; = \dfrac{1}{\alpha(V_1)}$, then we have

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}$$

for all $x \in \{x \in \mathbb{R}^n || f(x)| \leq \delta\} \cap V_1$, with sufficiently small δ.
The implication (ii)$\Rightarrow$(i) is straightforward. □

Remark 2.3. The exponent $\mathcal{L}_0(V_1)$ in the proof above satisfies the following equality:

$$\begin{aligned} \mathcal{L}_0(V_1) = \inf\{\alpha > 0 | \exists c_1 > 0, \delta > 0 \text{ such that} \\ |f(x)| \geq c.dist(x, f^{-1}(0))^{\alpha} \\ \text{for all } x \in V_1 \text{ and } |f(x)| < \delta\}. \end{aligned} \tag{3}$$

Definition 2.4. *We say that a sequence $\{x^k\}$ is of the second type of f w.r.t V_1, if*

$$\{x^k\} \subset V_1,$$

$$x^k \to \infty,$$

$$|f(x^k)| < M < +\infty \quad \textit{for some } M,$$

and

$$dist(x^k, f^{-1}(0)) \to +\infty.$$

Theorem 2.5. (The global Łojasiewicz inequality of f w.r.t. V_1)
The following statements are equivalent

(i) There are no sequences of the first and the second types of f w.r.t.V_1;

(ii) The following assertions hold true:

(a) If $f_* > 0$ and the function $dist(x, f^{-1}(0))$ is bounded on V_1, then for any $\rho > 0$, there exist $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho} \quad \textit{for all } x \in V_1;$$

(b) If $f_* > 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1, then there exist $c > 0$ and a positive rational number $\mathcal{L}_\infty(V_1) > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)} \quad \textit{for all } x \in V_1;$$

(c) If $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is bounded on V_1, then there exists $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)} \quad \textit{for all } x \in V_1;$$

(d) If $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1, then there exists $c > 0$ such that

$$|f(x)| \geq c.\min\{dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}, dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}\} \quad \textit{for all } x \in V_1.$$

Proof. (ii) $\Rightarrow$ (i) is straightforward.
Proof of (i) $\Rightarrow$ (ii):
Proof of (a): Assume that $f_* > 0$ and

$$dist(x, f^{-1}(0)) \leq D < +\infty \quad \text{for all } x \in V_1.$$

Let $\rho > 0$, then for any $x \in V_1$, we have

$$|f(x)| \geq f_* \geq \frac{f_*}{D^\rho}.dist(x, f^{-1}(0))^{\rho}$$

hence (a) holds.
Proof of (b): Assume that $f_* > 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1. Since there are no sequences of the second type w.r.t. V_1, it is easy to see that the set

$$\{x \in \mathbb{R}^n | |f(x)| = t\} \cap V_1$$

is not empty for $t > 0$ and t sufficiently large.
Put

$$\varphi(t) := \sup_{|f(x)|=t, x \in V_1} dist(x, f^{-1}(0))$$

It follows from (i) that, $\varphi(t)$ is a semi-algebraic function on $[\Delta, +\infty)$. Hence, there exists a rational number $\beta(V_1)$ and a positive constant c_0 such that

$$\varphi(t) = c_0 t^{\beta(V_1)} + o(t^{\beta(V_1)}), \quad \text{as } t \to \infty.$$

Moreover, since the function $dist(x, f^{-1}(0))$ is unbounded on V_1, $\varphi(t) \to \infty$ as $t \to \infty$, hence $\beta(V_1) > 0$.

Putting $\mathcal{L}_\infty(V_1) = \dfrac{1}{\beta(V_1)}$, the above equality implies that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \geq \Delta\} \cap V_1$, with $\Delta > 0$ sufficiently large.

Since there are no sequences of the second type w.r.t.V_1, the function $dist(x, f^{-1}(0))$ must be bounded on the set

$$\{x \in \mathbb{R}^n | f_* \leq |f(x)| \leq \Delta\} \cap V_1.$$

Hence, we can extend the above inequality for all $x \in V_1$ and get

$$|f(x)| \geq c'.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)} \quad \text{for all } x \in V_1,$$

where c' is some positive constant.

Proof of (c): Assume that $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is bounded on V_1. Since there are no sequence of the first type w.r.t.V_1, according to Proposition 2.2, there exist $c > 0$ and $\delta > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \leq \delta\} \cap V_1$.

Since $dist(x, f^{-1}(0)) \leq D$, on the set $\{x \in \mathbb{R}^n | |f(x)| \geq \delta\} \cap V_1$. Choosing $c' = \dfrac{\delta}{D^{\mathcal{L}_0(V_1)}}$, we have

$$|f(x)| \geq c'.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \geq \delta\} \cap V_1$.

Thus we have

$$|f(x)| \geq \min(c, c').dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)} \quad \text{for all } x \in V_1.$$

Proof of (d): Assume that $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1.

Put

$$\varphi(t) := \sup_{|f(x)| = t, x \in V_1} dist(x, f^{-1}(0))$$

Since (i) holds, φ is a semi-algebraic function, which is well defined on $(0, \delta)$ and $[\Delta; +\infty)$, where $\delta > 0$ sufficiently small and $\Delta > 0$ sufficiently large. We have then

$$\varphi(t) = c_0 t^{\alpha(V_1)} + o(t^{\alpha(V_1)}), \quad \text{as } t \to 0$$

and

$$\varphi(t) = c_0' t^{\beta(V_1)} + o(t^{\beta(V_1)}), \quad \text{as } t \to +\infty.$$

Hence, there exist $\delta > 0$ sufficiently small and $\Delta > 0$ sufficiently large and $c' > 0$ such that

$$|f(x)| \geq c'.dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}$$

$$\text{for all } x \in \{x \in \mathbb{R}^n \| f(x)| < \delta\} \cap V_1$$

and

$$|f(x)| \geq c'.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}$$

$$\text{for all } x \in \{x \in \mathbb{R}^n \| f(x)| > \Delta\} \cap V_1.$$

Since there are no sequences of the second type of f w.r.t. V_1, the function $dist(x, f^{-1}(0))$ is bounded on the set $\{x \in \mathbb{R}^n | \delta \leq |f(x)| \leq \Delta\} \cap V_1$. Thus, it easy to see that there exist $c > 0$ such that

$$|f(x)| \geq c. \min\{dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}, dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}\} \quad \text{for all } x \in V_1.$$

Theorem 2.5 is proved. □

Remark 2.6. The exponent $\mathcal{L}_\infty(V_1)$ in the proof above can be defined by

$$\mathcal{L}_\infty(V_1) = \sup\{\beta > 0 | \exists c > 0, \Delta > 0 \text{ such that}$$

$$|f(x)| \geq c.dist(x, f^{-1}(0))^\beta \tag{4}$$

$$\text{for all } x \in V_1, |f(x)| \geq \Delta\}.$$

Definition 2.7. Exponents $\mathcal{L}_0(V_1)$ and $\mathcal{L}_\infty(V_1)$ are called the Łojasiewicz exponent *near to the set* $f^{-1}(0)$ and *far from the set* $f^{-1}(0)$ of f w.r.t. V_1, respectively.

3 Global Łojasiewicz inequality

In this section we show that the global Łojasiewicz inequality of f exists if and only if there exists the global Łojasiewicz inequality of f w.r.t.V_1. Moreover, the forms of the global Łojasiewicz inequality of f are dictated by the corresponding forms $(a) - (d)$ of the global Łojasiewicz inequality of f w.r.t.V_1. Firstly, we consider the case when V_1 is either empty or $V_1 \subset f^{-1}(0)$.

Theorem 3.1. *Assume that V_1 is either empty or $V_1 \subset f^{-1}(0)$. Then there exists $c > 0$ such that*

$$|f(x)| \geq c.dist(x, f^{-1}(0))^d \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. This theorem is a direct consequence of Theorem 2.1 of [HNS], which says that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^{(d)}$ such that

$$|\frac{\partial^d f}{\partial x_n^d}(x)| \geq \rho > 0 \quad \text{on } \mathbb{R}^n,$$

then

$$|f(x)| \geq \frac{\rho}{d! 2^{2d-1}} dist(x, V_1 \cup f^{-1}(0))^d \quad \text{for all } x \in \mathbb{R}^n.$$

□

The following technical result is crucial for our further investigation.

Lemma 3.2. *Let $f(x)$ be a polynomial of the form (2). Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be a point of $\mathbb{R}^n$ and $x \notin f^{-1}(0) \cup V_1$. Then, there exists a point $x^* = (x', x_n^*) \in \mathbb{R}^{n-1} \times \mathbb{R}$ satisfying the following conditions*

(i) $x^* \in f^{-1}(0) \cup V_1$;

(ii) $|f(x^*)| \leq |f(x)|$;

(iii) $\| x - x^* \| \leq (2e)[|a_0|.d!(d+1)!]^{\frac{1}{d}} |f(x)|^{\frac{1}{d}}$, *where* $e = \lim \left(1 + \frac{1}{n}\right)^n, n \to \infty$.

Proof. Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be a point of $\mathbb{R}^n$ such that $x \notin f^{-1}(0) \cup V_1$.
Put $\Omega(x') \colon = \{x_n \in \mathbb{R} | |f(x', x_n)| \leq |f(x)|\}$ and $\varphi(x_n) \colon = f(x', x_n)$.
Since $|\varphi^{(d)}(x_n)| = |a_0|.d!$ for all $x_n \in \mathbb{R}$, by the van der Corput Lemma (see [G], 2.6.2) we have

$$mes\Omega(x') \leq (2e)[|a_0|.d!(d+1)!]^{\frac{1}{d}} |f(x)|^{\frac{1}{d}} \tag{5}$$

Since $\Omega(x')$ is a compact semi-algebraic subset of $\mathbb{R}$, we have

$$\Omega(x') = \cup_{i=1}^k [a_i, b_i] \cup \{y_1, \cdots, y_s\},$$

where $a_i < b_i$ and

$$|f(x', a_i)| = |f(x', b_i)| = |f(x', y_j)| = |f(x)|,$$

for $i = 1, \cdots, k; j = 1, \cdots, s$.
Clearly $x_n \in \{a_1, \cdots, a_k, b_1, \cdots, b_k, y_1, \cdots, y_s\}$.
We claim that $x_n \in \{a_1, \cdots, a_k, b_1, \cdots, b_k\}$.
By contradition, assume that it is not the case, i.e. $x_n = y_j$ for some $j \in \{1, \cdots, s\}$. Since y_j is an isolated point of $\Omega(x')$, it is easy to see that the function $\varphi(x_n)$ attains its local extremum at the point $x_n = y_j$. Therefore we have

$$\frac{d\varphi}{dx_n}(x_n) = \frac{\partial f}{\partial x_n}(x) = 0$$

which means that $x \in V_1$, a contradiction.
Thus, without loss of generality, we may assume that $x_n = a_1$. Then, since

$$|f(x', a_1)| = |f(x', b_1)| = |f(x)| \neq 0,$$

either $f(x', a_1).f(x', b_1) < 0$ or $f(x', a_1).f(x', b_1) > 0$.
Firstly, assume that $f(x', a_1).f(x', b_1) < 0$. Then, there exists $t_1 \in (a_1, b_1)$ such that $f(x', t_1) = 0$. Hence the point $x^* = (x', t_1)$ belongs to $f^{-1}(0)$. By (5), we have

$$\| x - x^* \| = |a_1 - t_1| \leq mes\Omega(x') \leq (2e)[|a_0|.d!(d+1)!]^{\frac{1}{d}}|f(x)|^{\frac{1}{d}}$$

and x^* satisfies (i) - (iii).
Now, assume that $f(x', a_1).f(x', b_1) > 0$. We have then $f(x', a_1) = f(x', b_1) \in \{\pm|f(x)|\}$. By Rolle's theorem, there exists a point $t_0 \in (a_1, b_1)$ such that $\frac{\partial f}{\partial x_n}(x', t_0) = 0$.
Let $x^* := (x', t_0)$, then $x^* \in V_1$. Since $t_0 \in (a_1, b_1) \subset \Omega(x')$, $|f(x^*)| \leq |f(x)|$ and

$$\| x - x^* \| \leq (2e)[|a_0|.d!(d+1)!]^{\frac{1}{d}}|f(x)|^{\frac{1}{d}}.$$

Lemma 3.2 is proved. $\square$

Theorem 3.3. (The global Łojasiewicz inequality)
The following statements are equivalent

(i) There are no sequences of the first and the second types of f;

(ii) There exist $c > 0, \alpha > 0$ and $\beta > 0$ such that

$$|f(x)| \geq c.\min\{dist(x, f^{-1}(0))^{\alpha}, dist(x, f^{-1}(0))^{\beta}\} \quad \text{for all } x \in \mathbb{R}^n;$$

(iii) There are no sequences of the first and the second types of f w.r.t. V_1;

(iv) The following statements hold true:

(a) If $f_* > 0$ and the function $dist(x, f^{-1}(0))$ is bounded on V_1, then there exists $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^d \quad \text{for all } x \in \mathbb{R}^n;$$

(b) If $f_* > 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1, then there exists $c > 0$ such that

$$|f(x)| \geq c.\min\{dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}, dist(x, f^{-1}(0))^d\} \quad \text{for all } x \in \mathbb{R}^n;$$

(c) If $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is bounded on V_1, then there exists $c > 0$ such that

$$|f(x)| \geq c.\min\{dist(x, f^{-1}(0))^{\mathcal{L}_0(V_1)}, dist(x, f^{-1}(0))^d\} \quad \text{for all } x \in \mathbb{R}^n;$$

(d) *If $f_* = 0$ and the function $dist(x, f^{-1}(0))$ is unbounded on V_1, then there exists $c > 0$ such that*

$$|f(x)| \geq c. \min\{dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}, dist(x, f^{-1}(0))^{\mathcal{L}_\infty(V_1)}, dist(x, f^{-1}(0))^d\}$$

for all $x \in \mathbb{R}^n$.

Proof. (i)$\Leftrightarrow$ (ii) was proved in ([DHT], Proposition 3.10).
(i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are straightfoward.
It rests to prove (iii) $\Rightarrow$ (iv).
We need

Lemma 3.4. *Assume that there are $c_0 > 0$ and $\rho_1 > 0, \cdots, \rho_s > 0$ such that*

$$|f(x)| \geq c_0. \min\{dist(x, f^{-1}(0)^{\rho_i}, i = 1, \cdots, s\} \quad \text{for all } x \in V_1. \tag{6}$$

Then, there exists $c > 0$ such that

$$|f(x)| \geq c. \min\{dist(x, f^{-1}(0)^{\rho_i}, dist(x, f^{-1}(0)^d, i = 1, \cdots, s\} \quad \text{for all } x \in \mathbb{R}^n. \tag{7}$$

Proof of Lemma 3.4. Let $x = (x', x_n)$ be an abritrary point of $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$. If $x \in f^{-1}(0)$ then (7) holds automatically. Also, if $x \in V_1$, then (7) follows from (6).
Assume that $x \notin f^{-1}(0) \cup V_1$. Then by Lemma 3.2, there exists $x^* \in \mathbb{R}^n$ such that

$$x^* \in V_1 \cup f^{-1}(0), |f(x^*)| \leq |f(x)| \tag{8}$$

and

$$(2e)^{-1}[|a_0|.d!(d+1)!]^{\frac{-1}{d}} \parallel x - x^* \parallel \leq |f(x)|^{\frac{1}{d}} \tag{9}$$

Clearly, if $x^* \in f^{-1}(0)$, (7) follows from (9).
Assume that $x^* \in V_1$. Let us denote by $H(x^*)$ the point of $f^{-1}(0)$ such that

$$dist(x^*, f^{-1}(0)) = \parallel x^* - H(x^*) \parallel .$$

Consider two possibilities. If

$$\parallel x - x^* \parallel \leq \parallel x^* - H(x^*) \parallel,$$

then

$$dist(x, f^{-1}(0)) \leq \parallel x - H(x^*) \parallel \leq 2 \parallel x^* - H(x^*) \parallel = 2dist(x^*, f^{-1}(0)).$$

Since $\rho_i > 0$, for $i = 1, 2, \cdots, s$. We have

$$2^{-\rho_i} dist(x, f^{-1}(0))^{\rho_i} \leq dist(x^*, f^{-1}(0))^{\rho_i}$$

Then (7) follows easily from (6) and (8).
Now, assume that $\| x - x^* \| \geq \| x^* - H(x^*) \|$, then

$$dist(x, f^{-1}(0)) \leq 2 \| x - x^* \|$$

Hence, by (9), there exists $c > 0$ such that

$$c.dist(x, f^{-1}(0))^d \leq |f(x)|$$

and (7) hold true. Lemma 3.4 is proved.
Proof (iii)$\Rightarrow$ (iv): Assertions (b), (c) and (d) follow from Theorem 2.5 and Lemma 3.4.
Let us prove (a). By Theorem 2.5, if $f_* > 0$ and if the function $dist(x, f^{-1}(0))$ is bounded on the set V_1, then for any $\rho > 0$, there exists $c_0 > 0$ such that

$$|f(x)| \geq c_0 dist(x, f^{-1}(0))^\rho \tag{10}$$

for all $x \in V_1$.
By choosing $\rho = d$ in (10), then the assertion (a) of (iv) follows from (10) and Lemma 3.4. Theorem 3.3 is proved. □

4 Łojasiewicz exponents of the global Łojasiewicz inequality

Through all this section we assume that

- f is a polynomial of the form (2);
- The global Łojasiewicz inequality of f holds true.

Then we can define two exponents $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$, are called respectively the Łojasiewicz exponent *near to the set* $f^{-1}(0)$ and the Łojasiewicz exponent *far from the set* $f^{-1}(0)$.
Before recalling the definitions of $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$, we begin with a simple result. First, we need the following notation: if $h(t)$ and $g(t)$ are positive functions in $t \in \mathbb{R}$, by $h(t) \asymp g(t)$ we mean that

$$\lim_{t \to \infty} \frac{h(t)}{g(t)} = c \neq 0.$$

Lemma 4.1. *Let f be of the form (2) and $(0, a) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then we have*

$$|f(0, a)| \asymp |a|^d \asymp dist((0, a), f^{-1}(0))^d$$

Proof. Clearly, $|f(0,a)| \asymp |a|^d$. We will prove that $|a|^d \asymp dist((0,a), f^{-1}(0))^d$, i.e. $|a| \asymp dist((0,a), f^{-1}(0))$.

Claim. There exist $r > 0$ and $c > 0$ such that if $(x', \lambda(x')) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is a point of $f^{-1}(0)$ with $\| x' \| \geq r$, then

$$|\lambda(x')| \leq c \parallel x' \parallel .$$

Proof of Claim: By contradiction, there exists a sequence $x^k = (x'^k, \lambda(x'^k)) \to \infty$ such that $f(x^k) = 0$ and $\lim \dfrac{|\lambda(x'^k)|}{\| x'^k \|} = \infty$. Then, since

$$f(x', x_n) = a_0 x_n^d + a_1(x') x_n^{d-1} + \cdots + a_d(x')$$

and $deg(a_i(x')) \leq i$, we have

$$\lim \frac{|a_i(x'^k)|}{|\lambda(x'^k)|^i} \to 0 \quad \text{as} \quad \| x'^k \| \to \infty.$$

Since $(x'^k, \lambda(x'^k)) \in f^{-1}(0)$,

$$a_0 + \frac{a_1(x'^k)}{\lambda(x'^k)} + \cdots + \frac{a_d(x'^k)}{\lambda(x'^k)^d} = 0,$$

which implies that $a_0 = 0$, a contradiction.

Let us use the l^1-norm in $\mathbb{R}^n$: If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ then

$$\| (x_1, \cdots, x_n) \|_{l^1} := \sum_{i=1}^{n} |x_i|.$$

Now, let c and r be as in the claim. Let (x_0', x_{n0}) be a point of $f^{-1}(0)$, we have

$$dist((0,a), f^{-1}(0)) \leq \| (0,a) - (x_0', x_{n0}) \|_{l^1} = \| x_0' \|_{l^1} + |a - x_{n0}| \sim |a|. \tag{11}$$

Let $(x'(a), \lambda(x'(a))) \in \mathbb{R}^{n-1} \times \mathbb{R}$ be the point of $f^{-1}(0)$ such that $dist((0,a), f^{-1}(0)) = \| (x'(a), \lambda(x'(a))) - (0,a) \|_{l^1}$. Since the distance from the point $(x'(a), \lambda(x'(a)))$ to the hyperplane $x_n = 0$ is equal to $|\lambda(x'(a))|$, we have

$$|a| = \| (0,a) - (0,0) \|_{l^1} \leq \| (0,a) - (x'(a), \lambda(x'(a))) \|_{l^1} + |\lambda(x'(a))|. \tag{12}$$

Firstly we assume that $\| x'(a) \|_{l^1} \leq r$. Then, since f is of the form (2), the set

$$\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | \ \| x' \|_{l^1} \leq r, f(x', x_n) = 0\}$$

is compact. Therefore there exists a constant $\varphi(r) > 0$ such that $|x_n| \leq \varphi(r)$ if $f(x', x_n) = 0$ and $\| x' \|_{l^1} \leq r$.

It follows from (12) that

$$|a| \leq \| (0,a) - (x'(a), \lambda(x'(a))) \|_{l^1} + \varphi(r) = dist((0,a), f^{-1}(0)) + \varphi(r) \asymp dist((0,a), f^{-1}(0)) \tag{13}$$

As consequence of (11) and (13) we have

$$|a| \asymp dist((0,a), f^{-1}(0))$$

if $\| x'(a) \|_{l^1} \leq r$.
Assume that $\| x'(a) \|_{l^1} \geq r$, then it follows from (12) that

$$|a| \leq \| x'(a) \|_{l^1} + |a - \lambda(x'(a))| + |\lambda(x'(a))| \leq \| x'(a) \|_{l^1} + |a - \lambda(x'(a))| + c \| x'(a) \|_{l^1}$$

$$\leq (c+1)(\| x'(a) \|_{l^1} + |a - \lambda(x'(a))|) = (c+1) dist((0,a), f^{-1}(0)).$$

This inequality, together with (11), implies that $|a| \asymp dist((0,a), f^{-1}(0))$ and Lemma 4.1 is proved. □

Assume that there are no sequences of the first and of the second types of f. Then the function

$$\varphi(t) = \sup_{|f(x)|=t} dist(x, f^{-1}(0))$$

is well defined on $(0, +\infty)$ and is semi-algebraic. Hence, there exist $c_0 > 0, c_\infty > 0$ and rational numbers α and β such that

$$\varphi(t) = c_0 t^\alpha + o(t^\alpha), \quad \text{as} \quad t \to 0$$

and

$$\varphi(t) = c_\infty t^\beta + o(t^\beta), \quad \text{as} \quad t \to \infty.$$

Since there are no sequences of the first type, $\varphi(t) \to 0$ as $t \to 0$, hence $\alpha > 0$. It follows from Lemma 4.1 that $\varphi(t) \to +\infty$ as $t \to +\infty$, which implies that $\beta > 0$.
Putting $\mathcal{L}_0(f) = \frac{1}{\alpha}$, $\mathcal{L}_\infty(f) = \frac{1}{\beta}$, we can show that there exist $\delta \in (0,1)$ and $\Delta \gg 1$ and $c_1 > 0, c_2 > 0$ such that

$$|f(x)| \geq c_1.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)},$$

for all $x \in \{x \in \mathbb{R}^n || f(x)| < \delta\}$, and

$$|f(x)| \geq c_2.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(f)},$$

for all $x \in \{x \in \mathbb{R}^n || f(x)| > \Delta\}$.
Moreover, the exponents $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$ statisfies respectively the following formula

$$\begin{aligned} \mathcal{L}_0(f) = \inf\{\rho > 0 | \exists \delta > 0 \ \text{ and } \ c > 0 \ \text{such that} & \\ |f(x)| \geq c.dist(x, f^{-1}(0))^\rho & \\ \text{for all } \ x \in \{x \in \mathbb{R}^n |, |f(x)| \leq \delta\}\}. & \end{aligned} \tag{14}$$

and

$$\mathcal{L}_\infty(f) = \sup\{\rho > 0 | \exists \Delta \gg 1 \ \text{ and } \ c > 0 \ \text{ such that}$$
$$|f(x)| \geq c.dist(x, f^{-1}(0))^\rho \tag{15}$$
$$\text{for all } \ x \in \{x \in \mathbb{R}^n |, |f(x)| \geq \Delta\}\}.$$

Following [H1] we call $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$, respectively the Łojasiewicz exponent *near to the set* $f^{-1}(0)$ and the Łojasiewicz exponent *far from the set* $f^{-1}(0)$.
In this section we estimate $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$ via d and the exponients $\mathcal{L}_0(V_1)$ and $\mathcal{L}_\infty(V_1)$, where the exponents $\mathcal{L}_0(V_1)$ and $\mathcal{L}_\infty(V_1)$ are defined in section 2.
Put

$$\Omega_1 \colon \ = \{x \in \mathbb{R}^n | dist(x, f^{-1}(0)) < 1\}$$

and

$$\Omega_2 \colon \ = \{x \in \mathbb{R}^n | dist(x, f^{-1}(0)) \geq 1\}$$

Put

$$\mathcal{L}(f, \Omega_1) \colon \ = \inf\{\rho > 0 | \exists c > 0 \ \text{ such that}$$
$$|f(x)| \geq c.dist(x, f^{-1}(0))^\rho$$
$$\text{for all } \ x \in \Omega_1\}$$

and

$$\mathcal{L}(f, \Omega_2) \colon \ = \sup\{\rho > 0 | \exists c > 0 \ \text{ such that}$$
$$|f(x)| \geq c.dist(x, f^{-1}(0))^\rho$$
$$\text{for all } \ x \in \Omega_2\}.$$

Proposition 4.2. *If there are no sequences of the first and the second type of f then*

$$\mathcal{L}_0(f) = \mathcal{L}(f, \Omega_1)$$

$$\mathcal{L}_\infty(f) = \mathcal{L}(f, \Omega_2).$$

Proof. .
Proof of $\mathcal{L}_0(f) \geq \mathcal{L}(f, \Omega_1)$.
Since there are no sequences of the first type, there exist $c > 0$ and $\delta > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)}$$

for all $x \in \{x \in \mathbb{R}^n || f(x)| < \delta\}$. Choose $\delta_0 > 0$ such that $\delta_0 < \min\{\delta, c\}$, we see that $\{x \in \mathbb{R}^n || f(x)| \leq \delta_0\} \subset \Omega_1$. In fact, since $\delta_0 < \delta$

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \leq \delta_0\}$.
This implies that

$$\delta_0 > c.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)}$$

hence

$$1 > \frac{\delta_0}{c} > dist(x, f^{-1}(0))^{\mathcal{L}_0(f)}$$

therefore $dist(x, f^{-1}(0)) < 1$.
Clearly, if $x \in \Omega_1 \setminus \{x \in \mathbb{R}^n | |f(x)| < \delta_0\}$ then

$$|f(x)| \geq \delta_0 > \delta_0.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)}.$$

Thus, we can conclude that the inequality

$$|f(x)| \geq c_*.dist(x, f^{-1}(0))^{\mathcal{L}_0(f)},$$

with $c_* = \min\{c, \delta_0\} = \delta_0$, holds true for all $x \in \Omega_1$. Hence, we have $\mathcal{L}_0(f) \geq \mathcal{L}(f, \Omega_1)$.
Proof of $\mathcal{L}_0(f) \leq \mathcal{L}(f, \Omega_1)$.
Let $\rho > 0$. Assume that there exists $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho} \quad \text{for all } \ x \in \Omega_1.$$

The inequality $\mathcal{L}_0(f) \leq \mathcal{L}(f, \Omega_1)$ will be proved, if we show that $\rho \geq \mathcal{L}_0(f)$.
Let us take δ_0 the constant defined in the proof above. Then

$$\{x \in \mathbb{R}^n | |f(x)| < \delta_0\} \subset \Omega_1.$$

Therefore,

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| < \delta_0\}$. Since $\mathcal{L}_0(f)$ satisfies (14), this inequality implies that $\mathcal{L}_0(f) \leq \rho$.
Thus, $\mathcal{L}_0(f) = \mathcal{L}(f, \Omega_1)$.
Proof of $\mathcal{L}_\infty(f) \leq \mathcal{L}(f, \Omega_2)$.
Since there are no sequeces of the second type, there exist $c > 0$ and $\Delta \gg 1$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(f)} \tag{16}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \geq \Delta\}$.
Moreover, the function $dist(x, f^{-1}(0))$ is bounded on the set

$$\Omega_2 \setminus \{x \in \mathbb{R}^n | |f(x)| \geq \Delta\},$$

i.e

$$D: \ = \sup\{dist(x, f^{-1}(0)) | x \in \Omega_2 \ \text{ and } \ |f(x)| \leq \Delta\} < +\infty.$$

Since there are no sequence of the first type, $|f(x)| \geq \delta > 0$ on the set $\Omega_2 \setminus \{x \in \mathbb{R}^n | |f(x)| \geq \Delta\}$. Hence, if $x \in \Omega_2 \setminus \{x \in \mathbb{R}^n | |f(x)| \geq \Delta\}$ we have

$$|f(x)| \geq \delta \geq \frac{\delta}{D^{\mathcal{L}_\infty(f)}} dist(x, f^{-1}(0))^{\mathcal{L}_\infty(f)} \tag{17}$$

It follows from (16) and (17) that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mathcal{L}_\infty(f)}$$

for all $x \in \Omega_2$. Hence, by the definition of $\mathcal{L}(f, \Omega_2)$, we have

$$\mathcal{L}_\infty(f) \leq \mathcal{L}(f, \Omega_2).$$

Proof of $\mathcal{L}_\infty(f) \geq \mathcal{L}(f, \Omega_2)$.
Let $\rho > 0$ and there exists $c > 0$ such that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho} \tag{18}$$

for all $x \in \Omega_2$. Let $x \in \mathbb{R}^n$ such that $|f(x)| \geq c$ and $dist(x, f^{-1}(0)) < 1$, then since $\rho > 0$, we have

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho}.$$

This fact, together with (18) implies that

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\rho}$$

for all $x \in \{x \in \mathbb{R}^n | |f(x)| \geq c\}$.
Hence, by (15), we get $\rho \leq \mathcal{L}_\infty(f)$, which, by the definition of $\mathcal{L}(f, \Omega_2)$ implies that $\mathcal{L}_\infty(f) \geq \mathcal{L}(f, \Omega_2)$. The equality $\mathcal{L}_\infty(f) = \mathcal{L}(f, \Omega_2)$ is proved. □

Now, we study the Łojasiewicz exponents $\mathcal{L}_0(f)$ and $\mathcal{L}_\infty(f)$ by investigating the relationship between them and the exponents $d, \mathcal{L}_0(V_1)$ and $\mathcal{L}_\infty(V_1)$.
According to Theorem 3.3, if the global Łojasiewicz inequality of f holds true, then it has one of 4 forms, described in (iv) of Theorem 3.3. We will consider each of these cases saparately. Before doing that, let us state an easy result.

Lemma 4.3. *We have*

(i)

$$\mathcal{L}_\infty(f) \leq \min\{\mathcal{L}_\infty(V_1), d\};$$

(ii)

$$\mathcal{L}_0(f) \geq \mathcal{L}_0(V_1).$$

Proof. .

Proof of (i): It follows from Lemma 4.1 that $\mathcal{L}_\infty(f) \leq d$.

The relation (4) implies that $\mathcal{L}_\infty(f) \leq \mathcal{L}_\infty(V_1)$. Hence, (i) is true.

Proof of (ii): This follows from (3). □

Let cases $(a) - (d)$ be as in Theorem 3.3 (iv).

Proposition 4.4. *Assume that there are no sequences of the first and the second types of f w.r.t.V_1 then the following hold*

Case (a):

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_0(f) \leq d$$

$$\mathcal{L}_\infty(f) = d.$$

Case (b):

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_0(f) \leq \max\{\mathcal{L}_\infty(V_1), d\}$$

$$\mathcal{L}_\infty(f) = \min\{\mathcal{L}_\infty(V_1), d\}$$

Case (c):

(i) If $\mathcal{L}_0(V_1) \geq d$ then

$$\mathcal{L}_\infty(f) = d$$

$$\mathcal{L}_0(f) = \mathcal{L}_0(V_1).$$

(ii) If $\mathcal{L}_0(V_1) < d$ then

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_0(f) \leq d$$

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_\infty(f) \leq d.$$

Case (d):

(i) If $\mathcal{L}_0(V_1) \leq \min\{\mathcal{L}_\infty(V_1), d\}$ then

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_0(f) \leq \max\{\mathcal{L}_\infty(V_1), d\}$$

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_\infty(f) \leq \min\{\mathcal{L}_\infty(V_1), d\}.$$

(ii) If $\min\{\mathcal{L}_\infty(V_1), d\} \leq \mathcal{L}_0(V_1) \leq \max\{\mathcal{L}_\infty(V_1), d\}$ then

$$\mathcal{L}_0(V_1) \leq \mathcal{L}_0(f) \leq \max\{\mathcal{L}_\infty(V_1), d\}$$

and

$$\mathcal{L}_\infty(f) = \min\{\mathcal{L}_\infty(V_1), d\}$$

(iii) If $\mathcal{L}_0(V_1) \geq \max\{\mathcal{L}_\infty(V_1), d\}$ *then*

$$\mathcal{L}_0(f) = \mathcal{L}_0(V_1)$$

and

$$\mathcal{L}_\infty(f) = \min\{\mathcal{L}_\infty(V_1), d\}.$$

Proof. The proof follows from Lemma 4.3 and Proposition 4.2. □

5 Method of checking for the existence of the global Łojasiewicz inequality of polynomials in two variables

In this section we consider the following question: How to check for the condition of the existence of the global Łojasiewicz inequality?

If $n > 2$, these problems seem still difficult, but if $n = 2$, it is easy.

Let $f(x, y)$ be a polynomial in two variables. We assume f is of the form

$$f(x, y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x) \tag{19}$$

where d is the degree of f.

Then, according to the Puiseux theorem [GV], f can be factored as

$$f(x, y) = a_0 \prod_{j=1}^{d} (y - \lambda_j(x)),$$

where each expression $\lambda_j(x)$ is a function of $x^{\frac{1}{p}}$ for certain integer p in the neighborhood of $x = \infty$. The function $\lambda_j(x)$ have no essential singularity at infinity, hence they can be written as fractional power series

$$\lambda_j(x) = \sum_{-\infty}^{k_j} a_{jk} x^{\frac{k}{p}} \tag{20}$$

for $|x| \gg 1$, where k_j are integer numbers. Each $\lambda_j(x)$ is called *a Puiseux series at infinity of* f. Since f is of the specific form (19),it is important to note that, $k_j \leq p$ for $j = 1, \cdots, d$. We also call $\lambda_j(x)$ *a root at infinity of* f.

Let $\lambda_j(x)$ be one of the roots of the form (20), then $\lambda_j(x)$ is called *real root* if all coefficients a_{jk} of $\lambda_j(x)$ are reals. Note that for a non real root $\lambda^*(x)$, the curve $y = \lambda^*(x)$ has no locus in $x > 0$, x is closed to $+\infty$. This is because for $x > 0$, all fractional power series of x are real numbers, and so $\lambda^*(x)$ is not a real number. In order to consider the real locus of $f(x, y) = 0$

for $x < 0$, we must take the real roots of $\overline{f}(x, y) := f(-x, y)$.
Let

$$\frac{\partial f}{\partial y}(x, y) = a_0 d \prod_{i=1}^{d-1} (y - \overline{\lambda_i}(x)),$$

where $\overline{\lambda_i}(x), i = 1, 2, \cdots, d-1$, are Puiseux's roots at infinity of $\dfrac{\partial f}{\partial y}(x, y)$.
We will consider the domain $x > 0$ and $x < 0$ separately. Assume that $x > 0$, we denote

$$\mathcal{P}(f) = \{\lambda_1(x), \cdots, \lambda_d(x)\}$$

and

$$\mathcal{P}(\frac{\partial f}{\partial y}) = \{\overline{\lambda_1}(x), \cdots, \overline{\lambda_{d-1}}(x)\}$$

the set of Puiseux's roots at infinity of f and of $\dfrac{\partial f}{\partial y}$, respectively.
For $g \in \{f, \dfrac{\partial f}{\partial y}, \overline{f}, \dfrac{\partial \overline{f}}{\partial y}\}$, we denote $\mathcal{P}_{\mathbb{R}}(g)$ the set of all real roots at infinity of g.
Let $\psi(x)$ be a fractional power series,

$$\psi(x) = b_0 x^{\rho} + o(x^{\rho})$$

where $b_0 \neq 0$. Then the exponent ρ will be called *the order of ψ at infinity* and we will denote it by $v(\psi(x))$.

Proposition 5.1. *The following statements are equivalent*

(i) There are no sequences $(x^k, y^k), x^k > 0$ of the first type of f w.r.t. V_1;

(ii) There are no $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ such that

$$v(f(x, \overline{\lambda}(x))) < 0;$$

and

$$\min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) \geq 0.$$

Proposition 5.2. *The following statements are equivalent*

(i) There are no sequences $(x^k, y^k), x^k > 0$ of the second type of f w.r.t. V_1;

(ii) There are no $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ such that

$$v(f(x, \overline{\lambda}(x))) \leq 0;$$

and

$$\min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) > 0.$$

Proposition 5.3. *Two conditions are equivalent*

(i)

$$f_* = \inf_{(x,y)\in V_1} |f(x,y)| > 0;$$

(ii) $f^{-1}(0) \cap \left(\dfrac{\partial f}{\partial y}\right)^{-1}(0) = \emptyset$ *and there are no* $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \cup \mathcal{P}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$ *such that*

$$v(f(x,\overline{\lambda}(x)) < 0, \;\; \textit{if} \;\; \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$$

and

$$v(\overline{f}(x,\overline{\lambda}(x)) < 0, \;\; \textit{if} \;\; \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y}).$$

The following facts help us to know which form of 4 form (a) - (d) of Theorem 3.3 will have the global Łojasiewicz inequality.

Proposition 5.4. *Two conditions are equivalent*

(i) The function $dist(x, f^{-1}(0))$ *is unbounded on* V_1*;*

(ii) There is $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \cup \mathcal{P}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$ *such that*

$$\min_{\lambda(x)\in\mathcal{P}_{\mathbb{R}}(f)} \{v(\overline{\lambda}(x) - \lambda(x))\} > 0, \;\; \textit{if} \;\; \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$$

and

$$\min_{\lambda(x)\in\mathcal{P}_{\mathbb{R}}(\overline{f})} \{v(\overline{\lambda}(x) - \lambda(x))\} > 0, \;\; \textit{if} \;\; \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y}).$$

Similarly, replacing $f(x,y), \mathcal{P}_{\mathbb{R}}(f)$ and $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ respectively by $\overline{f}(x,y) = f(-x,y), \mathcal{P}_{\mathbb{R}}(\overline{f})$ and $\mathcal{P}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$ in Propositions 5.1 and 5.2, we obtain the criterion for the non-existence of sequences $(x^k, y^k), x^k < 0$, of the first and the second types of f, w.r.t. V_1.
The proof of Propositions 5.1, 5.2, 5.3 and 5.4 follows from

Lemma 5.5. *Let* $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ *and* $\lambda(x) \in \mathcal{P}_{\mathbb{R}}(f)$*. Let*

$$V_\lambda : \; = \{(x,y) \in \mathbb{R}^2 | x > 0, y = \lambda(x)\}.$$

Then

$$dist((x,\overline{\lambda}(x)), V_\lambda) \asymp x^{v(\lambda(x)-\overline{\lambda}(x))},$$

i.e. there are $c_1 > 0$ and $c_2 > 0$ such that the inequalities

$$c_1 x^{v(\lambda(x)-\overline{\lambda}(x))} \leq dist((x, \overline{\lambda}(x)), V_\lambda) \leq c_2 x^{v(\lambda(x)-\overline{\lambda}(x))}$$

holds for $x > 0$ sufficiently large.

Proof. For $r > 0$ sufficiently large, we denote

$$U\colon = \{(x, y) \in \mathbb{R}^2 | x > r\}.$$

Let $\lambda(x) = bx^\rho + o(x^\rho)$, for $x > 0$ sufficiently large, with $b \neq 0$. Since $\lambda(x) \in \mathcal{P}_\mathbb{R}(f)$, $\rho \leq 1$. Firstly, assume that $\rho < 1$. Consider the map

$$\phi\colon U \to U, \ \ (x, y) \mapsto (x, y - \lambda(x)).$$

We will prove that ϕ is a bilipschitz mapping, i.e. there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \parallel \phi(x, y) - \phi(x_0, y_0) \parallel_{l^1} \leq \parallel (x, y) - (x_0, y_0) \parallel_{l^1} \leq c_2 \parallel \phi(x, y) - \phi(x_0, y_0) \parallel_{l^1},$$

for all (x, y) and (x_0, y_0) from U.
We have

$$\begin{aligned} \parallel \phi(x, y) - \phi(x_0, y_0) \parallel_{l^1} &= |x - x_0| + |y - y_0 + \lambda(x_0) - \lambda(x)| \\ &\leq |x - x_0| + |y - y_0| + |\lambda(x) - \lambda(x_0)|. \end{aligned}$$

Since $\lambda(x) = bx^\rho + o(x^\rho)$ with $\rho < 1$, it is easy to see that

$$|\lambda(x) - \lambda(x_0)| \leq \frac{1}{2}|x - x_0|$$

if x and x_0 sufficiently large. Hence

$$\parallel \phi(x, y) - \phi(x_0, y_0) \parallel_{l^1} \leq \frac{3}{2}(|x - x_0| + |y - y_0|) = \frac{3}{2} \parallel (x, y) - (x_0, y_0) \parallel_{l^1} .$$

Conversly

$$\begin{aligned} \parallel \phi(x, y) - \phi(x_0, y_0) \parallel_{l^1} &\geq |x - x_0| + |y - y_0| - |\lambda(x) - \lambda(x_0)| \\ &\geq |x - x_0| + |y - y_0| - \frac{1}{2}|x - x_0| \geq \frac{|x - x_0| + |y - y_0|}{2} \\ &= \frac{1}{2} \parallel (x, y) - (x_0, y_0) \parallel_{l^1} . \end{aligned}$$

Hence, ϕ is bilipschitz if $\rho < 1$.
Assume now that

$$\lambda(x) = bx + \widehat{\lambda}(x)$$

where $b \neq 0$ and $\widehat{\lambda}(x) = b_1 x^{\rho_1} + o(x^{\rho_1})$, with $\rho_1 < 1$ and $|x|$ sufficiently large.
Put

$$\phi_1 \colon U \to U, \ \ (x, y) \mapsto (x, y - bx)$$

and

$$\phi_2 \colon U \to U, \ \ (x, y) \mapsto (x, y - \widehat{\lambda}(x))$$

Clearly, ϕ_1 is a bilipschitz equivalent. Since $\rho_1 < 1$, ϕ_2 is also a bilipschitz equivalent. Therefore, the map $\phi = \phi_2 \circ \phi_1$ is a bilipschitz equivalent. Then we have

$$dist((x, \overline{\lambda}(x)), V_\lambda) \asymp dist(\phi(x, \overline{\lambda}(x)), \phi(V_\lambda)).$$

Moreover, since

$$\phi(V_\lambda) = \{(u, v) \in \mathbb{R}^2 | u > r, v = 0\}$$

and

$$\phi((x, \overline{\lambda}(x))) = (u, \overline{\lambda}(u) - \lambda(u)) = (x, \overline{\lambda}(x) - \lambda(x)).$$

we have

$$dist(\phi(x, \overline{\lambda}(x)), \phi(V_\lambda)) \asymp |\overline{\lambda}(x) - \lambda(x)| \asymp x^{v(\lambda(x) - \overline{\lambda}(x))}$$

and Lemma 5.5 is proved. □

Remark 5.6. Theorem 3.3 and Propositions 5.1, 5.2 provide the following method of checking for the existence of the global Łojasiewicz inequality of polynomials in two variables.

- We compute the real roots at infinity of $f(x, y)$, $\dfrac{\partial f}{\partial y}(x, y)$ and of $\overline{f}(x, y)$ and $\dfrac{\partial \overline{f}}{\partial y}(x, y)$.
- Next, we verify if the conditions (ii) in Propositions 5.1 and 5.2 holds or not. If they hold, then f has the global Łojasiewicz inequality.

6 Computation of the Łojasiewicz exponents

In this section we will compute explicitly the exponents $\mathcal{L}_0(V_1)$, $\mathcal{L}_\infty(V_1)$ and $\mathcal{L}_0(f)$, $\mathcal{L}_\infty(f)$, where f is a polynomial in two variables. We will keep all the notations of Section 5.

6.1 Computation of $\mathcal{L}_0(V_1)$ and $\mathcal{L}_\infty(V_1)$

We start with the computation of $\mathcal{L}_\infty(V_1)$. We denote by $\mathcal{P}_{\mathbb{R}}^\infty(\dfrac{\partial f}{\partial y})$ (resp., $\mathcal{P}_{\mathbb{R}}^\infty(\dfrac{\partial \overline{f}}{\partial y})$) the set of all $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})$ (resp., $\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$) such that

$$\min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) > 0$$

$$\left(\text{resp.,} \quad \min_{\lambda\in\mathcal{P}_{\mathbb{R}}(\overline{f})} v(\overline{\lambda}(x)-\lambda(x))>0\right)$$

Let $\overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})\cup\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$.
We put

$$D_{+}(\overline{\lambda})=\begin{cases}1 & \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \quad \text{and } \mathcal{P}_{\mathbb{R}}(f)=\emptyset\\ \min_{\lambda\in\mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x)-\lambda(x)) & \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial f}{\partial y}) \quad \text{and } \mathcal{P}_{\mathbb{R}}(f)\neq\emptyset\end{cases}$$

and

$$D_{-}(\overline{\lambda})=\begin{cases}1 & \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y}) \quad \text{and } \mathcal{P}_{\mathbb{R}}(\overline{f})=\emptyset\\ \min_{\lambda\in\mathcal{P}_{\mathbb{R}}(\overline{f})} v(\overline{\lambda}(x)-\lambda(x)) & \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y}) \quad \text{and } \mathcal{P}_{\mathbb{R}}(\overline{f})\neq\emptyset\end{cases}$$

We also put

$$\mathcal{L}^{+}_{\infty}(\overline{\lambda}) \colon = \frac{v(f(x,\overline{\lambda}(x))}{D_{+}(\overline{\lambda})}, \quad \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\frac{\partial f}{\partial y});$$

and

$$\mathcal{L}^{-}_{\infty}(\overline{\lambda}) \colon = \frac{v(\overline{f}(x,\overline{\lambda}(x))}{D_{-}(\overline{\lambda})}, \quad \text{if } \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y});$$

$$\mathcal{L}^{+}_{\infty}(V_1) \colon = \min\{\mathcal{L}^{+}_{\infty}(\overline{\lambda}) | \ \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\frac{\partial f}{\partial y})\};$$

$$\mathcal{L}^{-}_{\infty}(V_1) \colon = \min\{\mathcal{L}^{-}_{\infty}(\overline{\lambda}) | \ \overline{\lambda}\in\mathcal{P}^{\infty}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})\}.$$

Theorem 6.1. *If f has no sequences of the second type w.r.t. V_1 and if the set $\mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial f}{\partial y})\cup \mathcal{P}^{\infty}_{\mathbb{R}}(\dfrac{\partial \overline{f}}{\partial y})$ is not empty then the minimum of $\mathcal{L}^{+}_{\infty}(V_1)$ and $\mathcal{L}^{-}_{\infty}(V_1)$ is a positive rational number and equal to $\mathcal{L}_{\infty}(V_1)$.*

Proof. The proof of Theorem 6.1 follows from definitions of $\mathcal{L}^{+}_{\infty}(V_1)$, $\mathcal{L}^{-}_{\infty}(V_1)$ and Lemma 5.5. □

We have seen that the computation of $\mathcal{L}_{\infty}(V_1)$ is a problem purely "at infinity". This exponent can be expressed in terms of the Puiseux roots at infinity of f and $\dfrac{\partial f}{\partial y}$. For computing the exponent $\mathcal{L}_0(V_1)$, the Puiseux roots at infinity are not enough and we have to use also the Puiseux series of f and of $\dfrac{\partial f}{\partial y}$ at the points of intersections of the curves $\{\dfrac{\partial f}{\partial y}=0\}$ and $\{f=0\}$. We recall that

$$\mathcal{L}_0(V_1)=\inf\{\alpha || f(x)| \geq c.dist(x,f^{-1}(0))^{\alpha} \text{ for some } c>0$$

$$\text{and} \quad \forall (x,y) \in \{(x,y) \in \mathbb{R}^2 || f(x,y)| < \delta\} \cap V_1\}.$$

Firstly, we compute the Łojasiewicz exponent of f in the domain

$$\{(x,y) \in \mathbb{R}^2|, |f(x,y)| < \delta, |x| > r\} \cap V_1,$$

where r is sufficiently large, and δ is sufficiently small.

For $\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) \cup \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$, we put

$$D_+(\overline{\lambda}) \colon = \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) \ \text{ if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y});$$

$$D_-(\overline{\lambda}) \colon = \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(\overline{f})} v(\overline{\lambda}(x) - \lambda(x)) \ \text{ if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y}).$$

$$\mathcal{L}^+_{0,\infty}(\overline{\lambda}) \colon = \frac{v(f(x, \overline{\lambda}(x)))}{D_+(\overline{\lambda})}$$

$$\mathcal{L}^-_{0,\infty}(\overline{\lambda}) \colon = \frac{v(\overline{f}(x, \overline{\lambda}(x)))}{D_-(\overline{\lambda})}$$

Let us denote by $\mathcal{P}^0_{\mathbb{R}}(\frac{\partial f}{\partial y})$ (resp.,$\mathcal{P}^0_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$) the set of all $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$, (resp., $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$) such that $v(f(x, \overline{\lambda}(x))) < 0$, (resp.,$v(\overline{f}(x, \overline{\lambda}(x))) < 0$).

For $\overline{\lambda} \in \mathcal{P}^0_{\mathbb{R}}(\frac{\partial f}{\partial y}) \cup \mathcal{P}^0_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$, we put

$$\mathcal{L}^+_{0,\infty}(V_1) \colon = \max\{\mathcal{L}^+_{0,\infty}(\overline{\lambda})| \ \ \overline{\lambda} \in \mathcal{P}^0_{\mathbb{R}}(\frac{\partial f}{\partial y})\};$$

$$\mathcal{L}^- +_{0,\infty} (V_1) \colon = \max\{\mathcal{L}^-_{0,\infty}(\overline{\lambda})| \ \ \overline{\lambda} \in \mathcal{P}^0_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})\}.$$

Theorem 6.2. *Assume that there are no sequences of the first type of f w.r.t.V_1 then the following statements hold*

(i) If $\mathcal{P}^0_{\mathbb{R}}(\frac{\partial f}{\partial y})$ is not empty, then there exist $c > 0, r > 0$ and $\delta > 0$ such that

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\mathcal{L}^+_{0,\infty}(V_1)},$$

$$\textit{for all } (x,y) \in \{(x,y) \in \mathbb{R}^2 || f(x,y)| < \delta, x > r\} \cap V_1.$$

(ii) If $\mathcal{P}^0_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$ is not empty, then there exist $c > 0, r > 0$ and $\delta > 0$ such that

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\mathcal{L}^-_{0,\infty}(V_1)},$$

$$\textit{for all } (x,y) \in \{(x,y) \in \mathbb{R}^2 | |f(x,y)| < \delta, x < -r\} \cap V_1.$$

(iii) If there are no sequences of the first type of f w.r.t.V_1 and the set $\mathcal{P}^0_{\mathbb{R}}(\frac{\partial f}{\partial y}) \cup \mathcal{P}^0_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$ is not empty, then there exist $c > 0, \delta > 0$ and $r > 0$ such that

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\mathcal{L}_{0,\infty}(V_1)}$$

$$\textit{for all } (x,y) \in V_1 \cap \{(x,y) \in \mathbb{R}^2 | |f(x,y)| < \delta, |x| > r\},$$

where $\mathcal{L}_{0,\infty}(V_1) = \max\{\mathcal{L}^+_{0,\infty}(V_1), \mathcal{L}^-_{0,\infty}(V_1)\}$.

Proof. This follows from definitions of the exponents $\mathcal{L}^+_{0,\infty}(V_1)$, $\mathcal{L}^-_{0,\infty}(V_1)$ and Lemma 5.5. □

Now, let c, δ, and r be as in Theorem 6.2, (iii). Then, we see that

$$V_1 \cap \{(x,y) \in \mathbb{R}^2 | |f(x,y)| \leq \delta, |x| \leq r\} = S_1 \cup S_2 \cup \cdots \cup S_k,$$

where $S_i, i = 1, 2, \cdots, k$, are compact semi-algebraic curves.
Let $\mathcal{A}$ be the set of all isolated points in the intersection of set $S_1 \cup S_2 \cup \cdots \cup S_k$ and $f^{-1}(0)$. Clearly, we can choose δ and r such that each of points of $\mathcal{A}$ is contained in the interior of the set

$$\{(x,y) \in \mathbb{R}^2 | |f(x,y)| < \delta, |x| \leq r\}.$$

Let $\epsilon > 0$ sufficiently small such that each ball $B(A, \epsilon) = \{(x,y) \in \mathbb{R}^2 | \parallel (x,y) - A \parallel < \epsilon\}, A \in \mathcal{A}$, is contained in the interior of the set $\{|x| < r, |f(x,y)| \leq \delta\}$.
We define

$$\mathcal{L}_0(A, V_1) = \inf\{\alpha > 0 | |f(x,y)| \geq c.dist((x,y), f^{-1}(0))^\alpha$$
$$\text{for all } (x,y) \in V_1 \cap B(A,\epsilon) \text{ with some } c > 0\}.$$

By the classical Łojasiewicz inequality, the exponent $\mathcal{L}_0(A, V_1)$ is well defined.
With the notations above, we have

Proposition 6.3.
$$\mathcal{L}_0(V_1) = \max\{\mathcal{L}_{0,\infty}(V_1), \mathcal{L}_0(A, V_1) | A \in \mathcal{A}\}.$$

Proof. This fomula is a consequence of the following facts

(i) In the domain

$$\{(x,y) \in \mathbb{R}^2 || f(x,y)| \leq \delta, |x| \geq r\} \cap V_1.$$

we have

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\mathcal{L}_{0,\infty}(V_1)}$$

(ii) In each domain $V_1 \cap B(A, \epsilon)$, $A \in \mathcal{A}$, we have

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\mathcal{L}_0(A,V_1)}$$

(iii) Moreover, the exponents $\mathcal{L}_{0,\infty}(V_1)$ and $\mathcal{L}_0(A, V_1)$ are best possible.

(iv) There exist $\rho > 0$ such that $|f(x,y)| > \rho$, provided

$$(x,y) \in \{(x,y) \in \mathbb{R}^2 || f(x,y)| < \delta, |x| \leq r\} \cap V_1 \setminus \bigcup_{A \in \mathcal{A}} B(A, \epsilon).$$

Thus, to compute $\mathcal{L}_0(V_1)$, it rests to compute the exponents $\mathcal{L}_0(A, V_1)$, $A \in \mathcal{A}$.
Computation of $\mathcal{L}_0(A, V_1)$, with $A \in \mathcal{A}$.
We will work in the following situation. Given polynomials $f(x,y)$ and $\phi(x,y)$. Let A be an isolated point of $f^{-1}(0) \cap \phi^{-1}(0)$. It is required to compute

$$\mathcal{L}_0(A, \phi^{-1}(0)) = \inf\{\alpha > 0 || f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\alpha},$$

for all (x,y) closed to A and $(x,y) \in \phi^{-1}(0)\}$.
The computation of exponents $\mathcal{L}_0(A, \phi^{-1}(0))$ is carrying out in two steps.
Step 1: We compute $\mathcal{L}_0(A, \phi^{-1}(0))$, with the assumption that the polynomials $f(x,y)$ and $\phi(x,y)$ satisfy the following condition

(i) $A = (0,0)$;

(ii) $\dfrac{\partial^m f}{\partial y^m}(0,0) \neq 0$, where m is the multiplicity of f at the point $(0,0)$;

(iii) $\dfrac{\partial^l \phi}{\partial y^l}(0,0) \neq 0$, where l is the multiplicity of ϕ at the point $(0,0)$.

With these conditions, locally in a neighborhood of $(0,0) \in \mathbb{C}$, the polynomials $f(x,y)$ and $\phi(x,y)$ can be factorized as follows

$$f(x,y) = u(x,y) \prod_{j=1}^{m} (y - \lambda_j(x))$$

$$\phi(x,y) = v(x,y) \prod_{i=1}^{l} (y - \overline{\lambda_i}(x))$$

where $u(0,0) \neq 0$ and $v(0,0) \neq 0$, $\lambda_j(x)$ and $\overline{\lambda_i}(x)$, $j = 1, 2, \cdots, m$; $i = 1, 2, \cdots, l$ are fractional power series of the form

$$\lambda_j(x) = b_{j1}x^{\frac{p_{j1}}{m}} + b_{j2}x^{\frac{p_{j2}}{m}} + \cdots$$
$$\overline{\lambda_i}(x) = c_{i1}x^{\frac{q_{i1}}{l}} + c_{i2}x^{\frac{q_{i2}}{l}} + \cdots$$

Here $p_{jk} \in \mathbb{N}, q_{ik} \in \mathbb{N}, k = 1, 2, \cdots$ and $p_{j1} < p_{j2} < \cdots$, $q_{i1} < q_{i2} < \cdots$. The series $\lambda_j(t^m)$ and $\overline{\lambda}_i(t^l)$, $j = 1, 2, \cdots, m$; $i = 1, 2, \cdots, l$ converge in a small disk centered in $(0,0)$. (See, for example [W]).

It is important to note that Condition (ii) implies that $\dfrac{p_{j1}}{m} \geq 1$, for all $j \in \{1, 2, \cdots, m\}$.

The series $\lambda_1(x), \lambda_2(x), \cdots, \lambda_m(x)$ and $\overline{\lambda}_1(x), \overline{\lambda}_2(x), \cdots, \overline{\lambda}_l(x)$ are called *Puiseux's roots at* $(0,0)$ of f and ϕ, respectively.

As in the case of Puiseux's roots at infinity, a Puiseux's root $\lambda(x)$ is called *real at* $(0,0)$ if all the coefficients of $\lambda(x)$ are real numbers.

Notation: Let $\overline{f}(x,y) \colon = f(-x,y)$ and $\overline{\phi}(x,y) \colon = \phi(-x,y)$.

Let $g(x,y) \in \{f, \phi, \overline{f}, \overline{\phi}\}$. We denote by $\mathcal{P}_{\mathbb{R}}(g,0)$ the set of all real Puiseux's roots of g at $(0,0)$.

For $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\phi,0) \cup \mathcal{P}_{\mathbb{R}}(\overline{\phi},0)$, we put

$$D_+(\overline{\lambda},0) \colon = \max_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} (v(\overline{\lambda}(x) - \lambda(x))) \text{ if } \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\phi,0);$$

$$D_-(\overline{\lambda},0) \colon = \max_{\lambda \in \mathcal{P}_{\mathbb{R}}(\overline{f})} (v(\overline{\lambda}(x) - \lambda(x))) \text{ if } \overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\overline{\phi},0).$$

$$\mathcal{L}_+((0,0),\overline{\lambda}) = \frac{v(f(x,\overline{\lambda}(x)))}{D_+(\overline{\lambda},0)}, \quad \text{if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\phi,0);$$

$$\mathcal{L}_-((0,0),\overline{\lambda}) = \frac{v(\overline{f}(x,\overline{\lambda}(x)))}{D_-(\overline{\lambda},0)}, \quad \text{if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\overline{\phi},0).$$

Finally, we put

$$\mathcal{L}_+((0,0),\phi^{-1}(0)) \colon = \max_{\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\phi,0)} \mathcal{L}_+((0,0),\overline{\lambda})$$

and

$$\mathcal{L}_-((0,0),\phi^{-1}(0)) \colon = \max_{\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\overline{\phi},0)} \mathcal{L}_-((0,0),\overline{\lambda})$$

Proposition 6.4. *Assume that*

(i) $A = (0,0)$*;*

(ii) $\dfrac{\partial^m f}{\partial y^m}(0,0) \neq 0$, *where* m *is the multiplicity of* f *at the point* $(0,0)$*;*

(iii) $\dfrac{\partial^l \phi}{\partial y^l}(0,0) \neq 0$, *where* l *is the multiplicity of* ϕ *at the point* $(0,0)$.

Then we have

$$\mathcal{L}_0(A, \phi^{-1}(0)) = \max\{\mathcal{L}_+((0,0), \phi^{-1}(0)), \mathcal{L}_-((0,0), \phi^{-1}(0))\}.$$

Proposition 6.4 follows from the definitions of the exponents $\mathcal{L}_+((0,0), \phi^{-1}(0)), \mathcal{L}_-((0,0), \phi^{-1}(0))$ and the following analog of Lemma 5.5.

Lemma 6.5. *Assume that*

(i) $(0,0) \in f^{-1}(0) \cap \phi^{-1}(0)$*;*

(ii) $\dfrac{\partial^m f}{\partial y^m}(0,0) \neq 0$, *where* m *is the multiplicity of* f *at the point* $(0,0)$.

Let $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\phi, 0)$ *and* $\lambda(x) \in \mathcal{P}_{\mathbb{R}}(f, 0)$. *Then, for all* x *sufficiently close to* 0, *we have*

$$dist((x, \overline{\lambda}(x)), V_\lambda) \asymp x^{v(\overline{\lambda}(x) - \lambda(x))},$$

where

$$V_\lambda \colon = \{(x,y) \in \mathbb{R}^2 | y = \lambda(x)\}.$$

Proof. Since $\dfrac{\partial^m f}{\partial y^m}(0,0) \neq 0$ and m is the multiplicity of f at the point $(0,0)$, if $\lambda(x)$ is a Puiseux root of f at $(0,0)$, then

$$\lambda(x) = bx^{\frac{p}{m}} + o(x^{\frac{p}{m}})$$

with $\dfrac{p}{m} \geq 1$. Using this fact, the proof of lemma 6.5 is completely analogous to that of lemma 5.5. $\square$

Step 2: Let us consider the general case: A is an isolated point of $f^{-1}(0) \cap \phi^{-1}(0)$, not necessarily coincided to $(0,0)$. Let denote m and l respectively the multiplicity of f and of ϕ at A.

The following lemma is evident

Lemma 6.6. *There exists an affine isomorphism*

$$L \colon \mathbb{R}^2 \to \mathbb{R}^2, \ \ (x,y) \mapsto L(x,y) = (u,v) \ \text{ such that}$$

(i) $L(A) = (0,0)$;

(ii) If we denote by $f_0 = f \circ L^{-1}$ *and* $\phi_0 = \phi \circ L^{-1}$, *then* $\dfrac{\partial^m f_0}{\partial v^m}(0,0) \neq 0$ *and* $\dfrac{\partial^l \phi_0}{\partial v^l}(0,0) \neq 0$.

Lemma 6.7. *Let* L, f_0 *and* ϕ_0 *be as in Lemma 6.6. Then we have*

$$\mathcal{L}_0(A, \phi^{-1}(0)) = \mathcal{L}_0((0,0), \phi_0^{-1}(0)).$$

Proof. It is easy to see that the following statement hold

(i) There exist $c_1 > 0$ and $c_2 > 0$ such that

$$\| z - z' \| \leq c_1 \| L(z) - L(z') \|,$$

and

$$\| z - z' \| \leq c_2 \| L^{-1}(z) - L^{-1}(z') \|,$$

for all $z, z' \in \mathbb{R}^2$;

(ii) The maps

$$L^{-1} \colon f_0^{-1}(0) \to f^{-1}(0)$$
$$L^{-1} \colon \phi_0^{-1}(0) \to \phi^{-1}(0)$$

are bijections.

Now, firstly, we show that

$$\mathcal{L}_0((0,0), \phi_0^{-1}(0)) \leq \mathcal{L}_0(A, \phi^{-1}(0)).$$

Let ρ be any number such that

$$\rho > \mathcal{L}_0(A, \phi^{-1}(0)).$$

Let w be an abritrary point of $\phi_0^{-1}(0)$, closed to $(0,0)$. Then,

$$dist(w, f_0^{-1}(0)) \leq \| w - w' \|$$

for any $w' \in f_0^{-1}(0)$. Since L is a bijection, the above inequality follows that

$$dist(w, f_0^{-1}(0)) \leq \| w - L(z') \|,$$

for all $z' \in f^{-1}(0)$. Hence, we have

$$dist(w, f_0^{-1}(0)) \leq \| L(L^{-1}(w)) - L(z') \| \leq c_1 \| L^{-1}(w) - z' \|$$

provided $z' \in f^{-1}(0)$. Hence, we obtain

$$dist(w, f_0^{-1}(0)) \leq c_2 . dist(L^{-1}(w), f^{-1}(0)).$$

Since $L^{-1}(w) \in \phi^{-1}(0)$ and by assumption $\rho > \mathcal{L}_0(A, \phi^{-1}(0))$, we have

$$dist(w, f_0^{-1}(0)) \leq c_2 . dist(L^{-1}(w), f^{-1}(0)) \leq c . |f(L^{-1}(w))|^{\frac{1}{\rho}} = c . |f_0(w)|^{\frac{1}{\rho}},$$

for each w close to $(0,0)$. This means that $\mathcal{L}_0((0,0),\phi_0^{-1}(0)) \leq \rho$. Since this holds for any $\rho > \mathcal{L}_0(A,\phi^{-1}(0))$ we conclude that

$$\mathcal{L}_0((0,0),\phi_0^{-1}(0)) \leq \mathcal{L}_0(A,\phi^{-1}(0)).$$

The proof of the converse inequality

$$\mathcal{L}_0(A,\phi^{-1}(0)) \leq \mathcal{L}_0((0,0),\phi_0^{-1}(0))$$

is similar. □

Now, using Lemma 6.6 and Lemma 6.7, for f and $\phi = \dfrac{\partial f}{\partial y}$, we can compute the exponent $\mathcal{L}_0(A_j,V_1), j = 1,2,\cdots,k$. Thus, the exponent $\mathcal{L}(V_1)$ is computed. □

6.2 Computation of $\mathcal{L}_0(f)$

Let

$$\begin{aligned}\mathcal{L}_0(f)\colon = \inf\{\rho > 0 | \exists c > 0, \delta > 0 \quad &\text{such that} \\ |f(x,y)| \geq c.dist((x,y), f^{-1}(0))^\rho& \\ \text{for } (x,y) \in \mathbb{R}^2, f(x,y)| \leq \delta\};&\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{0,\infty}(f)\colon = \inf\{\rho > 0 | \exists c > 0, \delta > 0, r > 0 \quad &\text{such that} \\ |f(x,y)| \geq c.dist((x,y), f^{-1}(0))^\rho& \\ \text{for } (x,y) \in \mathbb{R}^2, |x| > r, f(x,y)| \leq \delta\};&\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{0,0}(f)\colon = \inf\{\rho > 0 | \exists c > 0, \delta > 0, r > 0 \quad &\text{such that} \\ |f(x,y)| \geq c.dist((x,y), f^{-1}(0))^\rho& \\ \text{for } (x,y) \in \mathbb{R}^2, |x| \leq r, f(x,y)| \leq \delta\}.&\end{aligned}$$

Clearly

$$\mathcal{L}_0(f) = \max\{\mathcal{L}_{0,\infty}(f), \mathcal{L}_{0,0}(f)\}$$

The computation of $\mathcal{L}_{0,0}(f)$ is based on the work of Kuo [Ku]. Firstly, we note that if $f^{-1}(0)$ has no singular points in the set $\{(x,y) \in \mathbb{R}^2 || f(x,y)| < \delta, |x| \leq r\}$, then

$$\mathcal{L}_{0,0}(f) = 1.$$

If $f^{-1}(0)$ has singular points in this domain, then

$$\mathcal{L}_{0,0}(f) = \max\{\mathcal{L}(f,A)|\ A \text{ is a singular point of } \ f^{-1}(0)\},$$

where $\mathcal{L}(f,A)$ is the Łojasiewicz exponent of f at A, i.e.

$$\mathcal{L}(f,A)\colon = \inf\{\rho > 0 | \exists c > 0, \epsilon > 0 \quad \text{such that}$$

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\rho}$$

$$\text{for all } \ (x,y) \in \mathbb{R}^2, \| \ (x,y) - A \ \| \leq \epsilon\}.$$

The exponent $\mathcal{L}(f,A)$ can be computed explicitly by Kuo's formula [Ku]. It seems that this formular well-known, so we prefer do not to go in details.

The exponent $\mathcal{L}_{0,\infty}(f)$ was computed in [VD]. We will recall this formula.

Let $\lambda_i(x) \in \mathcal{P}(f) \setminus \mathcal{P}_{\mathbb{R}}(f)$

$$\lambda_i(x) = a_1 x^{\alpha_1} + \cdots + a_{s-1}x^{\alpha_{s-1}} + a_s x^{\alpha_s} + \cdots$$

where $\alpha_1 > \alpha_2 > \cdots$ and $a_s \in \mathbb{C}$.

Assume that $a_1, a_2, \cdots, a_{s-1} \in \mathbb{R}$ and $a_s \notin \mathbb{R}$, then the series

$$\lambda_i^{\mathbb{R}}(x)\colon = a_1 x^{\alpha_1} + \cdots + a_{s-1}x^{\alpha_{s-1}} + c x^{\alpha_s}$$

where $c \notin \mathbb{R}$ and c is generic is called the real appoximation of $\lambda_i(x)$.

Let $\lambda_i(x), \lambda_j(x) \in \mathcal{P}(f) \setminus \mathcal{P}_{\mathbb{R}}(f)$ and

$$\rho_{ij} = v(\lambda_i^{\mathbb{R}}(x) - \lambda_j^{\mathbb{R}}(x))$$

Let

$$\lambda_i^{\mathbb{R}}(x) = a_1 x^{\alpha_1} + \cdots + a_{t-1}x^{\alpha_{t-1}} + a x^{\rho_{ij}} + o(x^{\rho_{ij}})$$

$$\lambda_j^{\mathbb{R}}(x) = a_1 x^{\alpha_1} + \cdots + a_{t-1}x^{\alpha_{t-1}} + b x^{\rho_{ij}} + o(x^{\rho_{ij}})$$

We put

$$\lambda_{ij}^{\mathbb{R}}(x) = a_1 x^{\alpha_1} + \cdots + a_{t-1}x^{\alpha_{t-1}} + c x^{\rho_{ij}}$$

where c is a generic real number. We put

$$D(\lambda_{ij}^{\mathbb{R}}) = \begin{cases} 1 \quad , & \text{if } \ \mathcal{P}_{\mathbb{R}}(f) = \emptyset \\ \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\lambda(x) - \lambda_{ij}^{\mathbb{R}}(x)), & \text{if } \ \mathcal{P}_{\mathbb{R}}(f) \neq \emptyset \end{cases}$$

and

$$L(\lambda_{ij}^{\mathbb{R}}) = \frac{v(f(x, \lambda_{ij}^{\mathbb{R}}(x)))}{D(\lambda_{ij}^{\mathbb{R}})}$$

Let $\lambda(x) \in \mathcal{P}_{\mathbb{R}}(f)$. We denote $t(\lambda)$ the multiplicity of λ as a root of the equation $f(x, \lambda(x)) = 0$.

Put $t(f) = \max\{t(\lambda) | \lambda \in \mathcal{P}_{\mathbb{R}}(f)\}$

Proposition 6.8. ([VD], Theorem 2.5)
If f has no sequences $(x_k, y_k), x_k > 0,$ of the first type, then $\mathcal{L}^+_{0,\infty}(f) > 0$ and

$$\mathcal{L}^+_{0,\infty}(f) = \inf\{\alpha > 0 | \exists c > 0, \delta > 0, r > 0 \ \ \textit{such that}$$

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\alpha}$$

$$\textit{for all} \ \ (x,y) \in \{(x,y) \in \mathbb{R}^2, x > r, |f(x,y)| < \delta\}\}$$

$$= \max\{t(f), L(\lambda^{\mathbb{R}}_{ij}) \ | \ v(f(x, \lambda^{\mathbb{R}}_{ij}(x)) < 0\}$$

where λ_i and λ_j run through all elements of the set $\mathcal{P}(f) \setminus \mathcal{P}_{\mathbb{R}}(f)$.

Similary, for $\mathcal{P}(\overline{f}) \setminus \mathcal{P}_{\mathbb{R}}(\overline{f}), \overline{f}(x,y) = f(-x,y)$, we can define the exponent $\mathcal{L}^-_{0,\infty}(f)$. We have

$$\mathcal{L}_{0,\infty}(f) = \max\{\mathcal{L}^+_{0,\infty}(f), \mathcal{L}^-_{0,\infty}(f)\}$$

i.e. $\mathcal{L}_{0,\infty}(f)$ can be computed via the real approximations of Puiseux's roots at infinity of f and $\overline{f}$. The exponent $\mathcal{L}_0(f)$ is computed.

6.3 Computation of $\mathcal{L}_\infty(f)$

We need

Lemma 6.9. ([VD], Proposition 2.7) *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function of the form*

$$f(x,y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x).$$

where $d = degf$. Then there are $c, R, r > 0$ and $\rho \in \mathbb{R}$ such that

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\rho},$$

for all $(x,y) \in \{(x,y) \in \mathbb{R}^2 | |x| > r, dist((x,y), f^{-1}(0)) > R\}$.
Moreover, if $\widehat{\mathcal{L}}_\infty(f)$ denotes the suppremum of the exponent ρ satisfying this inequality, then

$$\widehat{\mathcal{L}}_\infty(f) = \min\{L(\lambda^{\mathbb{R}}_{ij}) | D(\lambda^{\mathbb{R}}_{ij}) > 0\}$$

where λ_i and λ_j are Puiseux roots at infinity of f and $\overline{f}$, and either $\lambda_i, \lambda_j \in \mathcal{P}(f) \setminus \mathcal{P}_{\mathbb{R}}(f)$ or $\lambda_i, \lambda_j \in \mathcal{P}(\overline{f}) \setminus \mathcal{P}_{\mathbb{R}}(\overline{f})$.

Lemma 6.10. *If f has no sequences of the first and the second types, then*

$$\mathcal{L}_\infty(f) = \widehat{\mathcal{L}}_\infty(f)$$

Proof. Since there are no sequences of the second type, $\widehat{\mathcal{L}}_\infty(f) > 0$ and

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\widehat{\mathcal{L}}_\infty(f)}$$

for all $(x,y) \in \mathbb{R}^2$ such that $|x| > r$ and $dist((x,y), f^{-1}(0)) > R$.
Since there are no sequences of the first type we have

$$\widehat{f}_*: = \inf\{|f(x,y)||1 \leq dist((x,y), f^{-1}(0)) \leq R\} > 0.$$

Hence, we can extend the inequality (28) to a larger domain and get

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\widehat{\mathcal{L}}_\infty(f)}$$

for all $(x,y) \in \mathbb{R}^2$ such that $|x| > r$ and $dist((x,y), f^{-1}(0)) > 1$.
Now, equality $\mathcal{L}_\infty(f) = \widehat{\mathcal{L}}_\infty(f)$ is a consiquence of Proposition 4.2. □

7 Non-degenerate polynomials at infinity

In this section we study the global Łojasiewicz inequality for polynomials in two variables, which are non-degenerate w.r.t. the Newton part at infinity of their Newton polygons.
As before let

$$f(x,y) = a_0y^d + a_1(x)y^{d-1} + \cdots + a_d(x),$$

with $d = degf$.
To define the Newton polygon of f, we write f in the form

$$f(x,y) = \sum_{i+j \leq d} c_{ij}x^iy^j.$$

We put

$$supp(f): = \{(i,j)|c_{ij} \neq 0\},$$

the support of f and

$$\Gamma(f) = co(supp(f)),$$

the convex hull of $supp(f)$.
According to [K], [AGV], we will call $\Gamma(f)$ the *Newton polygon of f*. Clearly, the point $(0,d)$ is a vertex of $\Gamma(f)$ (the highest ones). Moreover, $\Gamma(f)$ is situated lower the line

$$L: i + j = d$$

Let denote by $\partial\Gamma(f)$ the boundary of $\Gamma(f)$. We denote by σ_* the edge of $\partial\Gamma(f)$ which is the nearest ones to the line L (hence, σ_* contains the point (0,d) as one of its vertex). We denote (a_{**}, b_{**}) the vertex of $\Gamma(f)$ such that

$$b_{**} = \min\{b|\exists a \text{ such that } (a,b) \in \Gamma(f)\}$$

i.e, (a_{**}, b_{**}) is the lowest vertex of $\Gamma(f)$.
Let $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}, \sigma_k$ be the sequence of edges of $\partial\Gamma(f)$ such that

$$\sigma_1 = \sigma_*$$

$$(a_{**}, b_{**}) \in \sigma_k$$

and

$$\sigma_i \cap \sigma_{i+1} \neq \emptyset, \quad i = 1, \cdots, k-1$$

We put

$$\partial_\infty \Gamma(f) = \{\sigma_1, \sigma_2, \cdots, \sigma_{k-1}, \sigma_k\}.$$

Definition 7.1. *The set $\partial_\infty \Gamma(f)$ is called the Newton part at infinity of f.*

Remark 7.2. Usually, (see [K], [AGV]), for studying the behavior of f in a neighborhood of the infinity, one uses the notion of the Newton diagram at infinity of f, which is defined as follows. Let $\Gamma_\infty(f)$ denote the convex hull of the union $supp(f) \cup \{(0,0)\}$. Let $\partial(\Gamma_\infty(f))$ be the boundary of $\Gamma_\infty(f)$. Then, by definition, the Newton diagram at infinity of f, denoted by $D(\Gamma_\infty(f))$, is the union of all edges of $\partial\Gamma_\infty(f)$, which do not contain the origin.

It is easy to see that if f is convenient, i.e. if $\Gamma(f)$ intersects all the coordinate axises in points different from the origin, then

$$\partial_\infty(\Gamma(f)) = D(\Gamma_\infty(f)).$$

But in general, these two sets are different. We can see this on the following example. Let

$$f(x, y) = y^{14} + x^4 y^{10} + x^8 y^4 - 2x^5 y^3 + x^2 y^2 + y^4,$$

then

$$D(\Gamma_\infty(f)) = \sigma_1 \cup \sigma_2$$

and

$$\partial_\infty(\Gamma(f)) = \sigma_1 \cup \sigma_2 \cup \sigma_3,$$

where $\sigma_1 = [(0, 14), (4, 10)]$, the edge joining the points $(0, 14)$ and $(4, 10)$, and $\sigma_2 = [(4, 10), (8, 4)], \sigma_3 = [(8, 4), (2, 2)]$.

For $\sigma \in \partial_\infty(\Gamma(f))$ we put

$$f_\sigma(x, y) = \sum_{(i,j) \in \sigma} a_{ij} x^i y^j.$$

For each edge $\sigma = [(a_1, b_1), (a_2, b_2)]$ with $b_1 > b_2$, we put

$$d(\sigma) = b_1 - b_2$$

and

$$v(\sigma) = \frac{a_2 - a_1}{b_1 - b_2}$$

Definition 7.3. *We say that f is non-degenerate w.r.t the Newton part at infinity of f (f is non-degenerate at infinity, for short), if the following condition holds: for any $\sigma \in \partial_\infty(\Gamma(f))$, the system*

$$\frac{\partial f_\sigma}{\partial x}(x, y) = \frac{\partial f_\sigma}{\partial y}(x, y) = 0$$

has no solutions in $(\mathbb{R} \setminus 0)^2$.

Lemma 7.4. *The following statements hold*

(i) $d = d(\sigma_1) + \cdots + d(\sigma_k) + b_{**}$;

(ii) $1 \geq v(\sigma_1) > v(\sigma_2) > \cdots > v(\sigma_k)$;

(iii) $y = 0$ *is the Puiseux root at infinity of f, with multiplicity b_{**};*

(iv) For $i \in \{1, 2, \cdots, k\}$, there are exactly $d(\sigma_i)$ Puiseux's roots at infinity (counting with multiplicities), each of the roots is of the form

$$y(x) = c.x^{v(\sigma_i)} + o(x^{v(\sigma_i)})$$

where c is a non-zero root of the polynomial $h_{\sigma_i}(u) \colon = f_{\sigma_i}(1, u)$.

(v) If f is non-degenerate at infinity, then the polynomial $h_{\sigma_i}(u)$ has no non-zero real root of multiplicity ≥ 2.

Proof. The statement (i)-(iii) are straightforward. Statement (iv) is well-known. We will prove (v).

Since $f_{\sigma_i}(x, y)$ is a quasi-homogeneous polynomial, i.e. there are p, q and $m \in \mathbb{Z}$, such that

$$ip + jq = m$$

for any $(i, j) \in supp(f_{\sigma_i})$. It is easy to show that $f_{\sigma_i}(x, y)$ can be written in the form

$$f_{\sigma_i}(x, y) = x^{\rho_1} y^{\rho_2} \left(x^{\frac{-p}{q}} y - c_1\right) \left(x^{\frac{-p}{q}} y - c_2\right) \cdots \left(x^{\frac{-p}{q}} y - c_s\right)$$

where $c_1, c_2, \cdots, c_s$ are non-zero roots of polynomial $h_\sigma(u) = f_\sigma(1, u)$.

By contradiction, assume that $h_\sigma(u)$ has c as a non-zero real root of multiplicity ≥ 2, i.e. f_{σ_i} has $\left(x^{\frac{-p}{q}} y - c\right)^2$ as a factor with $c \in \{\mathbb{R} \setminus 0\}$.

Then, the point $(1, c) \in (\mathbb{R} \setminus 0)^2$ is a root of the system

$$\frac{\partial f_\sigma}{\partial x}(x, y) = \frac{\partial f_\sigma}{\partial y}(x, y) = 0,$$

a contradiction. □

Let us describe the Newton part at infinity of $\frac{\partial f}{\partial y}$, which we denote by $\partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$. Assume that

$$\partial_\infty(\Gamma(f)) = \{\sigma_1, \sigma_2, \cdots, \sigma_k\}$$

denotes the Newton part at infinity of f, where $\sigma_1 = [(0,d),(a_1,b_1)], \sigma_2 = [(a_1,b_1),(a_2,b_2)], \cdots, \sigma_{k-1} = [(a_{k-2},b_{k-2}),(a_{k-1},b_{k-1})]$ and $\sigma_k = [(a_{k-1},b_{k-1}),(a_{**},b_{**})]$.
It is easy to see that the first $(k-1)$ edges of $\partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$ are $\sigma'_1, \sigma'_2, \cdots, \sigma'_{k-1}$, where $\sigma'_1 = [(0,d-1),(a_1,b_1-1)], \sigma'_2 = [(a_1,b_1-1),(a_2,b_2-1)], \cdots, \sigma'_{k-1} = [(a_{k-2},b_{k-2}-1),(a_{k-1},b_{k-1}-1)]$.
Clearly, $d(\sigma'_i) = d(\sigma_i)$ and $v(\sigma'_i) = v(\sigma_i), i = 1,2,\cdots,k-1$.
Next, if the edge $\sigma_k = [(a_{k-1},b_{k-1}),(a_{**},b_{**})]$ with $b_{**} \geq 1$, then the $k-th$ edge of $\partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$ will be $\sigma'_k = [(a_{k-1},b_{k-1}-1),(a_{**},b_{**}-1)]$ and

$$\partial_\infty(\Gamma(\frac{\partial f}{\partial y})) = \{\sigma'_1, \sigma'_2, \cdots, \sigma'_k\}.$$

Clearly, $d(\sigma'_k) = d(\sigma_k)$ and $v(\sigma'_k) = v(\sigma_k)$.
If $b_{**} = 0$, then $\partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$ can has $k-1, k$ or more edges. The fact which is most important for our further investigation is the following
Claim: If

$$\partial_\infty(\Gamma(\frac{\partial f}{\partial y})) = \{\sigma'_1, \sigma'_2, \cdots, \sigma'_{k-1}, \sigma'_k, \sigma'_{k+1}, \cdots, \sigma'_s\},$$

then for $i \geq k$, we have

$$v(\sigma'_i) \leq v(\sigma_k).$$

Proof. This follows from the construction of $\partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$. □

Lemma 7.5. *Assume that f is non-degenerate at infinity and $\overline{\lambda}_{i_0}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$ is of the form*

$$\overline{\lambda}_{i_0}(x) = c.x^{v(\sigma'_{i_0})} + o(x^{v(\sigma'_{i_0})})$$

then we have

$$v(f(x,\overline{\lambda}_{i_0}(x))) = \sum_{l=1}^{i_0} d(\sigma_l)v(\sigma_l) + \left[\sum_{m=i_0+1}^{k} d(\sigma_m)\right] v(\sigma_{i_0}) = a_{i_0} + \left[\sum_{m=i_0+1}^{k} d(\sigma_m)\right] v(\sigma_{i_0})$$

if $i_0 \in \{1,2,\cdots,k-1\}$, where σ_{i_0} is the edge of $\partial_\infty(\Gamma(f))$, joining the points (a_{i_0-1},b_{i_0-1}) and (a_{i_0},b_{i_0}), and

$$v(f(x,\overline{\lambda}_{i_0}(x))) = \sum_{l=1}^{k} d(\sigma_l)v(\sigma_l) = a_{**} \quad \text{if} \quad i_0 \geq k.$$

Proof. Firstly, we show that if $\overline{\lambda}_(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$, and $\lambda(x) \in \mathcal{P}_{\mathbb{R}}(f)$, then we have

$$v(\overline{\lambda}_(x) - \lambda(x)) = \max\{v(\overline{\lambda}_(x)), v(\lambda(x)\} \tag{21}$$

In fact, since $\overline{\lambda}_(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$, and $\lambda(x) \in \mathcal{P}_{\mathbb{R}}(f)$, we see that $\overline{\lambda}_(x)$ is of the form

$$\overline{\lambda}(x) = c.x^{v(\sigma'_{i_0})} + o(x^{v(\sigma'_{i_0})})$$

for some $\sigma'_{i_0} \in \partial_\infty(\Gamma(\frac{\partial f}{\partial y}))$, where c is a non-zero real root of the polynomial $\overline{h}_{\sigma'_{i_0}}(u) = \left(\frac{\partial f}{\partial y}\right)_{\sigma'_{i_0}}(1,u)$.
and

$$\lambda(x) = c'x^{v(\sigma_i)} + o(x^{v(\sigma_i)})$$

for some $\sigma_i \in \partial_\infty(\Gamma(f))$, where c' is a non-zero root of the polynomial $h_{\sigma_i}(u) = f_{\sigma_i}(1,u)$.
Clearly, if $v(\sigma'_{i_0}) \neq v(\sigma_i)$, then (21) is true.
Assume that $v(\sigma'_{i_0}) = v(\sigma_i)$, this happens if only if $\sigma'_{i_0} = \sigma'_i$, with $i \in \{1, 2, \cdots, k\}$.
By contradiction, assume $v(\overline{\lambda}_(x) - \lambda(x)) = \max\{v(\overline{\lambda}_(x)), v(\lambda(x)\} = v(\sigma_i)$. This implies that $c = c'$. Clearly, we have

$$\left(\frac{\partial f}{\partial y}\right)_{\sigma'_i} = \frac{\partial f_{\sigma_i}}{\partial y}.$$

Hence, the equality implies that $c = c'$, then $\left(\frac{\partial f}{\partial y}\right)_{\sigma'_i}(1,c) = \frac{\partial f_{\sigma_i}}{\partial y}(1,c) = 0$, which means that the polynomial $h_{\sigma_i}(u) := f_{\sigma_i}(1,u)$ has non-zero real root of multiplicity ≥ 2. This contradict Lemma 7.4, (v).
Lemma 7.5 follows easily from (21) and the following equality the proof of which is straighfoward:

$$\sum_{l=1}^{i_0} d(\sigma_l)v(\sigma_l) = a_{i_0}.$$

□

Let us recall that f is convenient, if $\Gamma(f)$ intersects both coordinate axises.

Theorem 7.6. *Let $f(x,y)$ be a polynomial which non-degenerate at infinity. Then the following statements holds*

(i) There exist $r \gg 1$ and $\epsilon > 0$ such that

$$|f(x,y)| \geq \epsilon$$

for all $(x,y) \in \{(x,y) \in \mathbb{R}^2 | |x| \geq r\} \cap V_1$;

(ii) If, in addition, f is convenient, then

$$\lim_{(x,y)\to\infty,(x,y)\in V_1} |f(x,y)| = \infty.$$

Proof. The proof of Theorem 7.6 follows directly from lemma 7.5. $\square$

Corollary 7.7. *(i) If f is non-degenerate at infinity, then f has no sequences of the first type, or equivalently, the following Łojasiewicz inequality near to the set $f^{-1}(0)$ exists:*

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\rho}$$

for all $(x,y) \in \{(x,y) \in \mathbb{R}^2 || f(x,y)| < \delta\}$, where c, ρ and δ are positive numbers.

(ii) If f is convenient and non-degenerate at infinity, then there are no sequences of the first and the second types of f, or equivalently, there exist $c > 0, \alpha > 0$ and $\beta > 0$ such that

$$|f(x,y)| \geq \min\{dist((x,y), f^{-1}(0))^{\alpha}, dist((x,y), f^{-1}(0))^{\beta}\}$$

for all $(x,y) \in \mathbb{R}^2$.

Remark 7.8. Assertion (ii) of Corollary 7.7 was proved in [H1] for the case of abritrary dimension by a different method. Nevertheless, assertion (ii) of Theorem 7.6 is stronger than the fact that there are no sequences of the first and the second types of f.

Theorem 7.9. *If f is convenient and non-degenerate at infinity, then the exponient $\mathcal{L}_{0,\infty}(V_1), \mathcal{L}_{\infty}(V_1)$, $\mathcal{L}_{0,\infty}(f)$ and $\mathcal{L}_{\infty}(f)$ can be expressed in terms of the Newton part at infinity of f.*

Proof. It follows from Lemma 7.5 and Section 6.1 that $\mathcal{L}_{0,\infty}(V_1)$ and $\mathcal{L}_{\infty}(V_1)$ can be expressed in terms of $d(\sigma_i), v(\sigma_i), i = 1, \cdots, k$.
The fact that the exponents $\mathcal{L}_{0,\infty}(f)$ and $\mathcal{L}_{\infty}(f)$ are determined in terms of $d(\sigma_i), v(\sigma_i), i = 1, \cdots, k$ follows from Sections 6.1 and 6.2, and Lemma 7.4 (v). $\square$

8 A version of Hörmander's inequality

In [Hor], L. Hörmander proved the following version of the global Łojasiewicz inequality

Theorem 8.1. ([Hor], Lemma 1 and Lemma 2).
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Then there exist $c > 0, \mu > 0, \mu' > 0$ and $\mu'' > 0$ such that

(i)

$$|f(x)| \geq c.dist(x, f^{-1}(0))^{\mu}$$

for all $x \in \{x \in \mathbb{R}^n | \parallel x \parallel < 1\}$.

(ii)

$$(1+\parallel x \parallel)^{\mu'}|f(x)| \geq c.dist(x, f^{-1}(0))^{\mu''} \tag{22}$$

for all $x \in \{x \in \mathbb{R}^n | \parallel x \parallel \geq 1\}$.

Clearly, ones need the factor $(1+\parallel x \parallel)^{\mu'}$ for controlling the bad behavior of f at infinity, which was caused by the existence of the first and the second types of f. Note that, the concrete values of μ' and μ'' are not given in [Hor].

In this section, in the spirit of the Hörmander inequality, we propose, for the case $n = 2$, and another version of the Hörmander inequality. In our formula,the exponent μ'' is equal to 1, the factor $(1+\parallel x \parallel)^{\mu'}$ also appears and the exponent μ' will be given with a concrete value. This value can be easily computed via the Puiseux's roots at infinities of f and of $\dfrac{\partial f}{\partial y}$.

Since the factor $(1+\parallel x \parallel)^{\mu'}$ is needed only in the case when there are sequences of the first or the second types of f, in this section that we will consider only this case. In other words, we assume that the following set $\mathcal{P}^*$ is not empty

$$\mathcal{P}^*: \ = \mathcal{P}_0^*(\frac{\partial f}{\partial y}) \cup \mathcal{P}_\infty^*(\frac{\partial f}{\partial y}) \cup \mathcal{P}_0^*(\frac{\partial \overline{f}}{\partial y}) \cup \mathcal{P}_\infty^*(\frac{\partial \overline{f}}{\partial y})$$

where

$$\mathcal{P}_0^*(\frac{\partial f}{\partial y}): \ = \{\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})| \ \ v(f(x, \overline{\lambda}(x)) < 0 \quad \text{and} \quad \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) \geq 0\}$$

$$\mathcal{P}_\infty^*(\frac{\partial f}{\partial y}): \ = \{\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})| \ \ v(f(x, \overline{\lambda}(x)) \leq 0 \quad \text{and} \quad \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} v(\overline{\lambda}(x) - \lambda(x)) > 0\}$$

and $\overline{f}(x, y) = f(x, y)$

$$\mathcal{P}_0^*(\frac{\partial \overline{f}}{\partial y}): \ = \{\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})| \ \ v(f(x, \overline{\lambda}(x)) < 0 \quad \text{and} \quad \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(\overline{f})} v(\overline{\lambda}(x) - \lambda(x)) \geq 0\}$$

$$\mathcal{P}_\infty^*(\frac{\partial \overline{f}}{\partial y}): \ = \{\overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})| \ \ v(f(x, \overline{\lambda}(x)) \leq 0 \quad \text{and} \quad \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(\overline{f})} v(\overline{\lambda}(x) - \lambda(x)) > 0\}$$

Let $\overline{\lambda} \in \mathcal{P}^*$, we put

$$\theta(\overline{\lambda}): \ = v(f(x, \overline{\lambda}(x)) \quad \text{if } \overline{\lambda} \in \mathcal{P}_0^*(\frac{\partial f}{\partial y}) \cup \mathcal{P}_\infty^*(\frac{\partial f}{\partial y})$$

and

$$\theta(\overline{\lambda}): \ = v(\overline{f}(x, \overline{\lambda}(x)) \quad \text{if } \overline{\lambda} \in \mathcal{P}_0^*(\frac{\partial \overline{f}}{\partial y}) \cup \mathcal{P}_\infty^*(\frac{\partial \overline{f}}{\partial y})$$

$$\rho(\overline{\lambda}): \ = \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(f)} \{v(\overline{\lambda}(x) - \lambda(x))\} \quad \text{if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y})$$

and

$$\rho(\overline{\lambda}):\ = \min_{\lambda \in \mathcal{P}_{\mathbb{R}}(\overline{f})} \{v(\overline{\lambda}(x) - \lambda(x))\} \quad \text{if } \overline{\lambda} \in \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$$

We also put

$$\nu(\overline{\lambda}) = \rho(\overline{\lambda}) - \theta(\overline{\lambda})$$

Finally, we put

$$\nu(f) = \max_{\overline{\lambda} \in \mathcal{P}^*} \nu(\overline{\lambda})$$

Clearly, $\nu(f) > 0$, since $\mathcal{P}^* \neq \emptyset$.

Theorem 8.2. *Let*

$$f(x,y) = a_0 y^d + a_1(x) y^{d-1} + \cdots + a_d(x),$$

where $d = degf$. Then there exist $\mu > 0$ and $c > 0$ such that

$$|f(x,y)|^{\frac{1}{\mu}} + |f(x,y)|^{\frac{1}{d}} + (1+|x|)^{\nu(f)} |f(x,y)| \geq c.dist((x,y), f^{-1}(0)) \tag{23}$$

for all $(x,y) \in \mathbb{R}^2$.

Proof. Firstly we show that there exist $\mu > 0$ and $c_1 > 0$ such that

$$|f(x,y)|^{\frac{1}{\mu}} + (1+|x|)^{\nu(f)} |f(x,y)| \geq c_1.dist((x,y), f^{-1}(0)) \quad \text{for all } (x,y) \in V_1. \tag{24}$$

Let $|x| \geq r > 0$, for r sufficiently large. Then $(x,y) \in V_1$ if only if there exists $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) \cup \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$ such that $(x,y) = (x, \overline{\lambda}(x))$.
We see then

$$|f(x,y)| = |f(x, \overline{\lambda}(x))| \asymp |x|^{\theta(\overline{\lambda})}$$

and

$$dist((x,y), f^{-1}(0)) = dist((x, \overline{\lambda}(x)), f^{-1}(0)) \asymp |x|^{\rho(\overline{\lambda})}$$

Hence, for any $\overline{\lambda}(x) \in \mathcal{P}_{\mathbb{R}}(\frac{\partial f}{\partial y}) \cup \mathcal{P}_{\mathbb{R}}(\frac{\partial \overline{f}}{\partial y})$ we have, for some $c_1 > 0$:

$$|f(x, \overline{\lambda}(x))|.(1+|x|)^{\nu(f)} \geq c_1.dist((x, \overline{\lambda}(x)), f^{-1}(0))$$

or equivalently

$$|f(x,y)|(1+|x|)^{\nu(f)} \geq c_1.dist((x,y), f^{-1}(0)) \tag{25}$$

for all $(x,y) \in V_1 \cap \{(x,y) \in \mathbb{R}^2 | |x| \geq r\}$.
The set $V_1 \cap \{(x,y) \in \mathbb{R}^2 | |x| \leq r\}$ is compact. Hence, by the classical Łojasiewicz inequality, there exist $\mu > 0$ and $c_2 > 0$ such that

$$|f(x,y)| \geq c_2.dist((x,y), f^{-1}(0))^{\mu} \tag{26}$$

for all $(x,y) \in V_1 \cap \{(x,y) \in \mathbb{R}^2 || x| \leq r\}$. Thus, (24) follows from (25) and (26).
As consequence, the desired inequality (23) holds true if $(x,y) \in V_1$.
Now, let (x,y) be an abritrary point of $\mathbb{R}^2$ such that $(x,y) \notin f^{-1}(0) \cup V_1$. Then, by Lemma 3.2 there exists a point $(x,y_*) \in \mathbb{R}^2$ such that

$$(x,y_*) \in f^{-1}(0) \cup V_1$$

$$|f(x,y)| \geq |f(x,y_*)| \tag{27}$$

and

$$\| (x,y) - (x,y_*) \| = |y - y_*| \leq c_3.|f(x,y)|^{\frac{1}{d}} \tag{28}$$

with some $c_3 > 0$. We have then

$$dist((x,y), f^{-1}(0)) \leq dist((x,y),(x,y_*)) + dist((x,y_*), f^{-1}(0))$$

Hence, using (26), (27), (28), we get

$$\begin{aligned} dist((x,y), f^{-1}(0)) &\leq \frac{1}{c_3}|f(x,y_*)|^{\frac{1}{d}} + \frac{1}{c_2}|f(x,y_*)|^{\frac{1}{\mu}} + \frac{1}{c_1}|f(x,y_*)|(1+|x|)^{\nu(f)} \\ &\leq \frac{1}{c_3}|f(x,y)|^{\frac{1}{d}} + \frac{1}{c_2}|f(x,y)|^{\frac{1}{\mu}} + \frac{1}{c_1}|f(x,y)|(1+|x|)^{\nu(f)}. \end{aligned}$$

This implies that desired inequality (23) holds true. □

Clearly, the value $\nu(f)$, depending only of the behavior of f on V_1, is not the smallest one's satisfying the inequality (22). In what follows we will give an another form of Hörmander's inequality, in which the factor $(1+|x|)^{\nu}$ will appear with the optimal value.
We still use the notation of 6.2 - 6.3 of section 6.
We put

$$\widehat{\mathcal{L}}_{0,\infty}(f) \colon = \max\{t(f), L(\lambda_{ij}^{\mathbb{R}}) | D(\lambda_{ij}^{\mathbb{R}}) < 0\}$$

$$\mathcal{H}(f) \colon = \max\{D(\lambda_{ij}^{\mathbb{R}}) - v(f(x, \lambda_{ij}^{\mathbb{R}}(x))) | D(\lambda_{ij}^{\mathbb{R}}) \geq 0\}$$

Lemma 8.3. *The following statements hold*

(i) There are $c, R, \delta, \alpha > 0$ satisfying the inequality

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\alpha}$$

for all $\| (x,y) \| \geq R$, and $dist((x,y), f^{-1}(0)) < \delta$.
Moreover, $\widehat{\mathcal{L}}_{0,\infty}(f)$ is the infimum of the exponents α satisfying this inequality.

(ii) If f has sequences of the first and the second types, then there are $c, R, r > 0$ and $\nu > 0$ satisfying the inequality

$$(1+ \parallel (x,y) \parallel)^{\nu} |f(x,y)| \geq c.dist((x,y), f^{-1}(0))$$

for all $\parallel (x,y) \parallel \geq R$ and $dist((x,y), f^{-1}(0)) \geq r$.
Moreover, $\mathcal{H}(f) > 0$ and $\mathcal{H}(f)$ is the infimum of the exponents ν satisfying this inequality.

Proof. (i) is proved in [[VD], Proposition 2.6].

(ii) The proof of (ii) can done by the some method as in [[VD], Proposition 2.7].

□

Now, assume that c, R, δ be as in the Lemma 8.3. Let μ_0 denote the Łojasiewicz exponent of f in the domain $\parallel (x,y) \parallel \leq R$, i.e. μ_0 is the smallest exponents α such that

$$|f(x,y)| \geq c.dist((x,y), f^{-1}(0))^{\alpha},$$

for all $\parallel (x,y) \parallel \leq R$.

We have

Theorem 8.4. *If f has sequences of the first and the second types, then there exists $c > 0$ such that*

$$|f(x,y)|^{\frac{1}{\mu_0}} + |f(x,y)|^{\frac{1}{\mathcal{L}_{0,\infty}(f)}} + (1+ \parallel (x,y) \parallel)^{\mathcal{H}(f)} |f(x,y)| \geq c.dist((x,y), f^{-1}(0)),$$

for all $(x,y) \in \mathbb{R}^2$.

Proof. It follows from Lemma 8.3 □

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