Explicit formulas for the top Lyapunov exponents of planar linear stochastic differential equations

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Abstract

Our aim in this paper is to establish explicit formulas for the top Lyapunov exponents of planar linear stochastic differential equations. We use these formulas to examine the sample-path stability of a linear stochastic differential equations arising in fluid dynamics and of a model of stochastic Hopf bifurcation.

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1 Introduction

The characteristic Lyapunov exponent of a non-zero solution of a linear stochastic differential equation measures the asymptotic exponential growth rate of the norm of this solution. Thank to the Multiplicative Ergodic Theorem (see, [Ose68, Ar98]), the set of all possible Lyapunov exponents, called Lyapunov spectrum, of a linear stochastic differential equation consists of finite non-random real numbers.

It is well known that the Lyapunov spectrum indicates not only the stability of the corresponding linear stochastic differential equations but also some other important dynamical properties of the nonlinear perturbed stochastic systems such as the transience/recurrence, see [Bax90], the normal form theory, see [AI98] and the bifurcation theory, see [Ar98, Chapter 8]. Therefore, computing the Lyapunov exponents of linear stochastic differential equations is an extremely important task in the qualitative theory of stochastic

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differential equations. In this paper, our aim is to establish an explicit formula for the top Lyapunov exponent of planar linear stochastic differential equations of the form

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix} dt + B \begin{pmatrix} x_t \\ y_t \end{pmatrix} \circ dW_t, \tag{1}$$

where $A, B \in \mathbb{R}^{2\times 2}$. For this purpose, we consider two separated cases: the coefficients A and B of (1) do not (case (i)) or do (case (ii)) satisfy the Hörmander hypoellipticity condition.

Concerning the case (i), we are able to write explicitly the solutions of (1) and to use the Strong Law of Large Numbers for Martingales to compute explicitly the top Lyapunov exponent of (1). Meanwhile, for the case (ii), we first use the Furstenberg-Khasminskii formula to represent the top Lyapunov exponent as an integral of a function involving coefficients A and B over the stationary distribution of the induced flow of (1) on the unit circle. Finally, we compute explicitly the stationary distribution by solving the Fokker-Planck equation associated with the induced flow on the unit circle. To do this, depending on the Jordan normal form of the diffusion coefficient B, we can partition the unit circle on some open intervals such that on each open interval the associated Fokker-Planck equation is a solvable ordinary differential equation (Note that in general Fokker-Planck equations are implicit ordinary differential equations). Note that this procedure is also used in [IL99, IL01] to establish explicit formulas for the top Lyapunov exponent and the rotation number of (1) in the case that the Jordan normal form of B is $\begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{R}$.

The paper is organized as follows: In Section 2, we recall some fundamental aspects of Lyapunov exponents of linear stochastic differential equations. Section 3 is devoted to present our main results in this paper about explicit formulas for the top Lyapunov exponents of planar linear stochastic differential equations. These formulas are later used to detect the area of parameters for which a linear stochastic differential equation arising from fluid dynamics is sample-path asymptotically stable (Subsection 4.1) and the bifurcation value of a model of stochastic Hopf bifurcation (Subsection 4.2).

To conclude this introductory section, we introduce notations which are used throughout this paper. For a matrix M, let $\sigma(M)$ denote the set of all complex eigenvalues of M and let

$$\rho(M) := \{\max \mathrm{Re}\lambda : \lambda \in \sigma(M)\}.$$

Let $\langle\cdot\rangle$ denote the standard Euclidean inner product in $\mathbb{R}^2$ and $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2$, i.e. $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers.

2 Preliminaries

Consider a planar linear stochastic differential equation of the form

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix} dt + B \begin{pmatrix} x_t \\ y_t \end{pmatrix} \circ dW_t, \tag{2}$$

where $\begin{pmatrix} x_t \\ y_t \end{pmatrix} \in \mathbb{R}^2$ and $A, B \in \mathbb{R}^{2\times 2}$. Let $\Phi_{A,B}(t,\xi)$ denote the solution of (2) with $(x_0, y_0)^{\mathrm{T}} = \xi \in \mathbb{R}^2 \setminus \{0\}$. Then, the *top sample path Lyapunov exponent* $\lambda_{A,B}$ of (2) is defined by

$$\lambda_{A,B} = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi_{A,B}(t,\cdot)\| \qquad \text{a.s.},$$

see e.g. [Ar98]. To gain a formula to compute $\lambda_{A,B}$, we rewrite equation (2) in its polar coordinates by defining $r_t := \sqrt{x_t^2 + y_t^2}$ and $s_t := \left(\frac{x_t}{r_t}, \frac{y_t}{r_t}\right)^{\mathrm{T}}$. Using Ito's formula, see e.g. [KP92], we obtain

$$dr_t = \overline{f}_A(s_t) r_t \, dt + \overline{f}_B(s_t) r_t \circ dW_t, \quad ds_t = \overline{g}_A(s_t)\, dt + \overline{g}_B(s_t) \circ dW_t,$$

where for a matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, we define

$$\overline{f}_M(s) := \langle s, Ms\rangle \quad \text{and} \quad \overline{g}_M(s) := Ms - \overline{f}_M(s)s \qquad \text{for } s \in \mathbb{S}^1.$$

By identifying φ_t and $-\varphi_t$, the angular motion is in fact a motion on one-dimensional projective space. Writing $s_t = \begin{pmatrix} \cos\varphi_t \\ \sin\varphi_t \end{pmatrix}$, where $\varphi_t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, leads to

$$dr_t = f_A(\varphi_t) r_t \, dt + f_B(\varphi_t) r_t \circ dW_t, \quad d\varphi_t = g_A(\varphi_t)\, dt + g_B(\varphi_t) \circ dW_t, \tag{3}$$

where for a matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$$\begin{aligned} f_M(\varphi) &:= m_{11}\cos^2\varphi + m_{22}\sin^2\varphi + (m_{12}+m_{21})\cos\varphi\sin\varphi, \\ g_M(\varphi) &:= (m_{22}-m_{11})\cos\varphi\sin\varphi + m_{21}\cos^2\varphi - m_{12}\sin^2\varphi. \end{aligned}$$

Now we recall the well-known Furstenberg-Khasminskii formula for the top Lyapunov exponent of (2).

Theorem 1 (Furstenberg-Khasminskii formula)**.** Suppose that the following non-degeneracy condition holds:

(H) There is no $s \in \mathbb{S}^1$ such that

$$As = \langle As, s\rangle s \quad \text{and} \quad Bs = \langle Bs, s\rangle s.$$

Then, the top Lyapunov exponent $\lambda_{A,B}$ of (2) is given by

$$\lambda_{A,B} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(f_A(\varphi) + \frac{1}{2} f_B'(\varphi) g_B(\varphi) \right) p(\varphi)\, d\varphi,$$

where $p : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}_{\geq 0}$ is a smooth density function satisfying that $p(-\frac{\pi}{2}) = p(\frac{\pi}{2})$, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\varphi)\, d\varphi = 1$ and the following differential equation

$$\frac{1}{2} g_B(\varphi)^2 p'(\varphi) = \left(g_A(\varphi) - \frac{1}{2} g_B'(\varphi) g_B(\varphi) \right) p(\varphi) + C, \tag{4}$$

where C is a constant.

Proof. See e.g. [IL01, page 34-37]. □

3 Explicit formulas for top Lyapunov exponents

Our aim in this section is to establish explicit formulas for the top Lyapunov exponents of planar linear stochastic differential equations. For this purpose, we divide this section into two subsections. In Subsection 3.1, we consider degenerated linear stochastic differential equations, i.e. equations in which the drift and the diffusion coefficients do not fulfill condition (H) of Theorem 1. For systems satisfying condition (H), we give explicit formulas for their top Lyapunov exponents in Subsection 3.2.

3.1 Degenerated linear stochastic differential equations

Suppose that $A, B \in \mathbb{R}^{2\times 2}$ do not satisfy condition (H). Therefore, there exists $s \in \mathbb{S}^1$ such that

$$As = \langle As, s\rangle s \quad \text{and} \quad Bs = \langle Bs, s\rangle s,$$

which implies that $s \in \mathbb{S}^1$ is a common real eigenvector of A and B. Thus, for the orthogonal matrix $T \in \mathbb{R}^{2\times 2}$ defined by $Te_1 = s$ and $Te_2 := s^\perp$, we have $T^{-1}AT$ and $T^{-1}BT$ are lower triangular matrices. Hence, any degenerated linear stochastic differential equation can be transformed to a lower triangular linear stochastic differential equation and our aim in this subsection is to give an explicit formula for Lyapunov exponent of this class of linear stochastic differential equations. Before going to the main result in this subsection, we need the following preparatory lemma.

Lemma 2. Let $\alpha, \beta \in \mathbb{R}$ be arbitrary and $(W_t)_{t\in\mathbb{R}}$ be a Brownian motion defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the following statements hold almost surely:

(i) $\limsup_{t\to\infty} \frac{1}{t} \log \left| \int_0^t \exp(\alpha s) W_s(\omega)\, ds \right| \leq \alpha$.

(ii) $\lim_{t\to\infty} \frac{1}{t} \log \int_0^t \exp(\alpha s + \beta W_s(\omega))\, ds = \alpha$.

Proof. Let ε be an arbitrary positive number. Using Strong Law of Large Numbers for Martingales, there exists a measurable set $\widehat{\Omega}$ with $\mathbb{P}(\widehat{\Omega}) = 1$ and $\lim_{t\to\infty} \frac{W_t(\omega)}{t} = 0$ for all $\omega \in \widehat{\Omega}$, see e.g. [BGV10, Appendix A]. Choose and fix an arbitrary $\omega \in \widehat{\Omega}$. Thus, for $\omega \in \widehat{\Omega}$ there exists $T(\varepsilon, \omega) > 0$ such that

$$-\varepsilon s \leq W_s(\omega) \leq \varepsilon s \qquad \text{for all } s \geq T(\varepsilon, \omega).$$

Hence, for all $t \geq T(\varepsilon, \omega)$ we have

$$\int_{T(\varepsilon,\omega)}^t \exp(\alpha s) W_s(\omega)\, ds \leq \varepsilon \int_0^t \exp(\alpha s) s\, ds,$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left| \int_{T(\varepsilon,\omega)}^t \exp(\alpha s) W_s(\omega)\, ds \right| \leq \limsup_{t\to\infty} \frac{1}{t} \log \left(\varepsilon \int_0^t \exp(\alpha s) s\, ds \right) \leq \alpha \ .$$

Hence, (i) is proved. For all $t \geq T(\varepsilon, \omega)$ we also have

$$\begin{aligned}\int_{T(\varepsilon,\omega)}^{t} \exp\left(\alpha - |\beta|\varepsilon\right)s\, ds &\leq \int_{T(\varepsilon,\omega)}^{t} \exp(\alpha s + \beta W_s(\omega))\, ds \\ &\leq \int_{T(\varepsilon,\omega)}^{t} \exp\left(\alpha + |\beta|\varepsilon\right)s\, ds,\end{aligned}$$

which implies that

$$\alpha - |\beta|\varepsilon \leq \liminf_{t\to\infty} \frac{1}{t} \log \int_0^t \exp(\alpha s + \beta W_s(\omega))\, ds$$

and

$$\alpha + |\beta|\varepsilon \geq \limsup_{t\to\infty} \frac{1}{t} \log \int_0^t \exp(\alpha s + \beta W_s(\omega))\, ds.$$

Letting $\varepsilon \to 0$ yields (ii) and the proof is complete. □

Theorem 3 (Explicit formula for the top Lyapunov exponents of degenerated linear stochastic differential equations)**.** Consider system (2) where A, B are of the following form

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}.$$

Then,

$$\lambda_{A,B} = \rho(A) = \max\big\{a_{11}, a_{22}\big\}.$$

Proof. The stochastic differential equation for the first component x_t is

$$dx_t = a_{11} x_t\, dt + b_{11} x_t \circ dW_t,$$

which implies that

$$x_t = \exp\left(a_{11} t + b_{11} W_t\right) x_0.$$

The equation for the second component y_t is

$$dy_t = \left(a_{22} + \frac{b_{22}^2}{2}\right) y_t\, dt + \left(a_{21} + b_{21} \frac{b_{11} + b_{22}}{2}\right) x_t\, dt + \left(b_{21} x_t + b_{22} y_t\right) dW_t.$$

Using the variation of constants formula, see e.g. [KP92, pp. 120], we obtain that

$$y_t = \exp\left(a_{22}t + b_{22}W_t\right)\Big(y_0 + \Big(a_{21} + b_{21}\frac{b_{11}-b_{22}}{2}\Big)\, x_0 \int_0^t \exp\left(\alpha s + \beta W_s\right) ds + b_{21}x_0 \int_0^t \exp(\alpha s + \beta W_s)\, dW_s\Big),$$

where $\alpha := a_{11} - a_{22}$ and $\beta := b_{11} - b_{22}$. Since $\lim_{t\to\infty} \frac{W_t}{t} = 0$ it follows that

$$\lim_{t\to\infty} \frac{1}{t} \log|x_t| = a_{11} \quad \text{for } x_0 \neq 0$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log|y_t| = a_{22} \quad \text{for } x_0 = 0, y_0 \neq 0.$$

Consequently, $\lambda_{A,B} \geq \rho(A)$. To conclude the proof, it is sufficient to show that $\lambda_{A,B} \leq \rho(A)$. Equivalently, we prove that $\lim_{t\to\infty} \frac{1}{t} \log|y_t| \leq \rho(A)$. To do this, we consider the following separated cases:

Case 1: If $\beta = 0$, then by using Ito's formula, we obtain that

$$\int_0^t \exp(\alpha s)\, dW_s = \exp(\alpha t) W_t - \alpha \int_0^t \exp(\alpha s) W_s\, ds.$$

Therefore,

$$\begin{aligned} y_t = {} & \exp\left(a_{22}t + b_{22}W_t\right)\Big(y_0 + \Big(a_{21} + b_{21}\frac{b_{11}-b_{22}}{2}\Big)\, x_0 \int_0^t \exp\left(\alpha s\right)\, ds\Big) - \\ & \alpha b_{21} x_0 \exp\left(a_{22}t + b_{22}W_t\right) \int_0^t \exp(\alpha s) W_s\, ds + \\ & b_{21}x_0 \exp\left(a_{11}t + b_{11}W_t\right) W_t, \end{aligned}$$

which together with Lemma 2(i) implies that $\limsup_{t\to\infty} \frac{1}{t} \log|y_t| \leq \rho(A)$.

Case 2: If $\beta \neq 0$, then by using Ito's formula, we obtain that

$$\begin{aligned} \int_0^t \exp(\alpha s + \beta W_s)\, dW_s = {} & \frac{1}{\beta}\big(\exp(\alpha t + \beta W_t) - 1\big) \\ & - \Big(\frac{\alpha}{\beta} + \frac{\beta}{2}\Big) \int_0^t \exp(\alpha s + \beta W_s)\, ds. \end{aligned}$$

Thus,

$$\begin{aligned} y_t &= \Big(y_0 - \frac{b_{21}}{\beta}x_0\Big)\exp\left(a_{22}t + b_{22}W_t\right) + \frac{b_{21}}{\beta}x_0\exp(a_{11}t + b_{11}W_t) \\ &\quad + \big(a_{21} - b_{21}\frac{\alpha}{\beta}\big)x_0\exp\left(a_{22}t + b_{22}W_t\right)\int_0^t \exp\left(\alpha s + \beta W_s\right) ds. \end{aligned}$$

By virtue of Lemma 2(ii), $\lim_{t\to\infty}\frac{1}{t}\log|y_t| \le \rho(A)$ and the proof is complete. □

3.2 Non-degenerated linear stochastic differential equations

According to Theorem 1, the top Lyapunov exponent of a non-degenerated linear stochastic differential equation is given explicitly in terms of the stationary distribution of the induced flow on the unit circle. Hence, to obtain an explicit form of the top Lyapunov exponent, our attempt in this subsection is to solve explicitly non-zero solutions of (4). Note that the equation (4) can be singular in the sense that $g_B(\varphi)$ might be equal to zero for some values of φ. So, to solve (4) we distinguish the following types of the form of matrix B (up to a transformation generated by a non-singular matrix):

$$\begin{array}{llll} \text{Type I} & B = \alpha\,\mathrm{id}, & \text{Type II} & B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \\ \text{Type III} & B = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, & \text{Type IV} & B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \end{array}$$

where $\alpha, \beta \in \mathbb{R}$ and additionally, in Type II we assume that $\alpha \neq \beta$ and in Type IV we assume that $\beta \neq 0$.

Type I

Let B be of Type I. Then, any matrix $A \in \mathbb{R}^{2\times 2}$ commutes with B. In the following lemma, we compute the top Lyapunov exponent of (2) when A and B commute. Consequently, we obtain a formula for the top Lyapunov exponent of (2) when B is of Type I.

Lemma 4. Suppose that A and B are commutative, i.e. $AB = BA$. Then, for any non-zero initial value $(x_0, y_0)^{\mathrm{T}} \in \mathbb{R}^2\setminus\{0\}$ the characteristic Lyapunov exponent of the solution $\Phi_{A,B}(t,(x_0,y_0)^{\mathrm{T}})$ of (2) is

$$\limsup_{t\to\infty}\frac{1}{t}\log\left\|\Phi_{A,B}(t,(x_0,y_0)^{\mathrm{T}})\right\| = \lim_{t\to\infty}\frac{1}{t}\log\left\|\exp(tA)(x_0,y_0)^{\mathrm{T}}\right\|. \qquad (5)$$

In particular, $\lambda_{A,B} = \rho(A)$

Proof. Since the matrices A and B are commute it follows that the explicit solution of (2) is

$$\Phi_{A,B}(t,(x_0,y_0)^{\mathrm{T}}) = \exp(tA + BW_t)(x_0,y_0)^{\mathrm{T}} = \exp(W_tB)\exp(tA)(x_0,y_0)^{\mathrm{T}},$$

which implies that

$$\frac{\left\|\exp(tA)(x_0,y_0)^{\mathrm{T}}\right\|}{\|\exp(-W_tB)\|} \le \left\|\Phi_{A,B}(t)(x_0,y_0)^{\mathrm{T}}\right\| \le \|\exp(W_tB)\|\left\|\exp(tA)(x_0,y_0)^{\mathrm{T}}\right\|. \qquad (6)$$

Since $\lim_{t\to\infty}\frac{W_t}{t} = 0$ it follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\|\exp(W_tB)\| = \lim_{t\to\infty}\frac{1}{t}\log\|\exp(-W_tB)\| = 0,$$

which together with (6) proves (5). Since

$$\max_{(x_0,y_0)^{\mathrm{T}}\in\mathbb{R}^2\setminus\{0\}}\lim_{t\to\infty}\frac{1}{t}\log\left\|\exp(tA)(x_0,y_0)^{\mathrm{T}}\right\| = \rho(A),$$

it follows that $\lambda_{A,B} = \rho(A)$. The proof is complete. □

Theorem 5 (Explicit formula for the top Lyapunov exponents of linear SDE of Type I)**.** Consider system (2) with the drift part $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and with the diffusion part $B = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{R}$. Then, the top Lyapunov exponent of (2) is given by $\lambda_{A,B} = \rho(A)$.

Type II

In this part, we consider the case that B is of Type II. Note that for all diagonal singular matrices $F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ we have $FBF^{-1} = B$. In the following remark, a suitable diagonal matrix F which enables to simplify the form of the drift term A is found:

Remark 6. Let $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$. Then, the pair (A, B) satisfy condition (H) of Theorem 1 iff $a_{12}a_{21} \neq 0$. Depending on the sign of $a_{12}a_{21}$, we can simplify A as follows:

(a) If $a_{12}a_{21} > 0$, then for $F = \begin{pmatrix} \operatorname{sign}(a_{12})\sqrt{\frac{a_{21}}{a_{12}}} & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$\widehat{A} := FAF^{-1} = \begin{pmatrix} a_{11} & \sqrt{a_{12}a_{21}} \\ \sqrt{a_{12}a_{21}} & a_{22} \end{pmatrix}. \tag{7}$$

(b) If $a_{12}a_{21} < 0$, then for $F = \begin{pmatrix} \operatorname{sign}(a_{12})\sqrt{-\frac{a_{21}}{a_{12}}} & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$\widehat{A} := FAF^{-1} = \begin{pmatrix} a_{11} & \sqrt{-a_{12}a_{21}} \\ -\sqrt{-a_{12}a_{21}} & a_{22} \end{pmatrix}. \tag{8}$$

Theorem 7 (Explicit formula for the top Lyapunov exponents of linear SDE of Type II)**.** Consider system (2) with the drift part $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and with the diffusion part $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Then, the following statements hold:

(a) If $a_{12}a_{21} > 0$, then

$$\lambda_{A,B} = \frac{a_{11} + a_{22}}{2} + \frac{\int_0^\infty P_{A,B}(u) \exp\left(-\frac{2\sqrt{a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1+u^2}{u}\right) du}{2\int_0^\infty Q_{A,B}(u) \exp\left(-\frac{2\sqrt{a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1+u^2}{u}\right) du},$$

where

$$\begin{aligned} P_{A,B}(u) &:= u^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}-2}\Big((a_{11} - a_{22})(1 - u^2) + 4\sqrt{a_{12}a_{21}}u + \\ &\qquad +2(\alpha-\beta)^2\frac{u^2}{1+u^2}\Big), \\ Q_{A,B}(u) &:= u^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}-2}(1 + u^2). \end{aligned}$$

(b) If $a_{12}a_{21} < 0$, then

$$\begin{aligned}
\lambda_{A,B} &= \frac{a_{11}+a_{22}}{2} \\
&+\frac{1}{C}\int_0^\infty R_{A,B}(u)e^{\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-u^2}{u}}\int_0^u \frac{v^{-\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{1+v^2}e^{-\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-v^2}{v}}\,dv\,du \\
&+\frac{1}{C}\int_{-\infty}^0 R_{A,B}(u)e^{\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-u^2}{u}}\int_{-\infty}^u \frac{|v|^{-\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{1+v^2}e^{-\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-v^2}{v}}\,dv\,du,
\end{aligned}$$

where

$$R_{A,B}(u) := |u|^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}-2}\left(\frac{a_{11}-a_{22}}{2}(1-u^2)+(\alpha-\beta)^2\frac{u^2}{1+u^2}\right)$$

and

$$\begin{aligned}
C &:= \int_0^\infty S_{A,B}(u)e^{\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-u^2}{u}}\int_0^u \frac{v^{-\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{1+v^2}e^{-\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-v^2}{v}}\,dv\,du \\
&+\int_{-\infty}^0 S_{A,B}(u)e^{\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-u^2}{u}}\int_{-\infty}^u \frac{|v|^{-\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{1+v^2}e^{-\frac{2\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1-v^2}{v}}\,dv\,du,
\end{aligned}$$

here $S_{A,B}(u) := (1+u^2)|u|^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}-2}$.

Proof. Let $\lambda_{\widehat{A},B}$ denote the top Lyapunov exponent of the following linear stochastic differential equation

$$dX_t = \widehat{A}X_t + BX_t \circ dW_t, \tag{9}$$

where $\widehat{A}$ is defined as in Remark 6. Obviously, $\lambda_{A,B} = \lambda_{\widehat{A},B}$ and our aim is to compute $\lambda_{\widehat{A},B}$. For this purpose, by definition of B, we have $g_B(\varphi) = \frac{\beta-\alpha}{2}\sin 2\varphi$. Then, the equation for the stationary distribution of the induced flow on $\mathbb{S}^1$ is

$$\sin^2 2\varphi\, p'(\varphi) = \left(\frac{8g_{\widehat{A}}(\varphi)}{(\beta-\alpha)^2} - 2\sin 4\varphi\right)p(\varphi) + C, \tag{10}$$

where C is a constant. Using the explicit formula of $g_{\widehat{A}}(\varphi)$, the differential equation corresponding to the linear part of (10) is given by

$$\frac{p'(\varphi)}{p(\varphi)} = \frac{4}{(\alpha-\beta)^2}\left(\frac{\widehat{a}_{22}-\widehat{a}_{11}}{\sin 2\varphi} + \frac{(\widehat{a}_{12}+\widehat{a}_{21})\cos 2\varphi}{\sin^2 2\varphi} + \frac{\widehat{a}_{21}-\widehat{a}_{12}}{\sin^2 2\varphi}\right) - 4\cot 2\varphi,$$

where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{\pm\frac{\pi}{2}, 0\}$. A solution of the preceding equation on $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{\pm\frac{\pi}{2}, 0\}$ is

$$\Pi(\varphi) := \frac{|\tan\varphi|^{\frac{2(\widehat{a}_{22}-\widehat{a}_{11})}{(\alpha-\beta)^2}}}{\sin^2 2\varphi \exp\left(\frac{2(\widehat{a}_{12}+\widehat{a}_{21})}{(\alpha-\beta)^2 \sin 2\varphi} + \frac{2(\widehat{a}_{21}-\widehat{a}_{12})\cot 2\varphi}{(\alpha-\beta)^2}\right)}. \tag{11}$$

By variation of constants formula, we have

$$p(\varphi) = \begin{cases} \Pi(\varphi)\left(\frac{p(\frac{\pi}{4})}{\Pi(\frac{\pi}{4})} + C\int_{\frac{\pi}{4}}^{\varphi} \frac{1}{\sin^2 2s\Pi(s)}\,ds\right) & \text{for } \varphi \in (0, \frac{\pi}{2}), \\ \Pi(\varphi)\left(\frac{p(-\frac{\pi}{4})}{\Pi(-\frac{\pi}{4})} + C\int_{-\frac{\pi}{4}}^{\varphi} \frac{1}{\sin^2 2s\Pi(s)}\,ds\right) & \text{for } \varphi \in (-\frac{\pi}{2}, 0). \end{cases} \tag{12}$$

Now, we consider two separated cases:

(a) Let $\widehat{A}$ be of the form (7). Then, by (11) the function $\Pi(\varphi)$ is given by

$$\Pi(\varphi) = \frac{|\tan\varphi|^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{\sin^2 2\varphi \exp\left(\frac{4\sqrt{a_{12}a_{21}}}{(\alpha-\beta)^2}\frac{1}{\sin 2\varphi}\right)},$$

which implies that $\lim_{\varphi\to 0^-}\Pi(\varphi) = \lim_{\varphi\to -\frac{\pi}{2}^+}\Pi(\varphi) = \infty$. Therefore, by boundedness of p and (12), we have

$$\frac{p(-\frac{\pi}{4})}{\Pi(-\frac{\pi}{4})} + C\int_{-\frac{\pi}{4}}^{0} \frac{1}{\Pi(s)\sin^2 2s}\,ds = \frac{p(-\frac{\pi}{4})}{\Pi(-\frac{\pi}{4})} + C\int_{-\frac{\pi}{4}}^{-\frac{\pi}{2}} \frac{1}{\Pi(s)\sin^2 2s}\,ds = 0.$$

Consequently, $p(-\frac{\pi}{4}) = C = 0$ and using the fact that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\varphi)\,d\varphi = 1$, we obtain that

$$p(\varphi) = \begin{cases} \frac{\Pi(\varphi)}{\int_0^{\frac{\pi}{2}}\Pi(s)\,ds}, & \text{if } \varphi \in [0, \frac{\pi}{2}], \\ 0, & \text{if } \varphi \in [-\frac{\pi}{2}, 0). \end{cases}$$

In light of Theorem 1 and from the fact that

$$f_{\widehat{A}}(\varphi) + \frac{1}{2}f_B'(\varphi)g_B(\varphi) = \frac{a_{11}+a_{22}}{2} + \frac{a_{11}-a_{22}}{2}\cos 2\varphi + \sqrt{a_{12}a_{21}}\sin 2\varphi + \frac{(\alpha-\beta)^2}{4}\sin^2 2\varphi,$$

we have

$$\lambda_{A,B} = \frac{a_{11}+a_{22}}{2} + \frac{\int_0^{\frac{\pi}{2}}\left(\frac{a_{11}-a_{22}}{2}\cos 2\varphi + \sqrt{a_{12}a_{21}}\sin 2\varphi + \frac{(\alpha-\beta)^2}{4}\sin^2 2\varphi\right)\Pi(\varphi)\,d\varphi}{\int_0^{\frac{\pi}{2}}\Pi(\varphi)\,d\varphi}.$$

Changing variable $\varphi = \arctan u$ in the preceding integral completes the proof of this part.

(b) Let $\widehat{A}$ be of the form (8). Then, by (11) the function $\Pi(\varphi)$ is of the following form

$$\Pi(\varphi) = \frac{|\tan\varphi|^{\frac{2(a_{22}-a_{11})}{(\alpha-\beta)^2}}}{\sin^2 2\varphi \exp\left(-\frac{4\sqrt{-a_{12}a_{21}}}{(\alpha-\beta)^2}\cot 2\varphi\right)},$$

which implies that $\lim_{\varphi\to 0^+}\Pi(\varphi) = \lim_{\varphi\to -\frac{\pi}{2}^+}\Pi(\varphi) = \infty$. Consequently, from boundedness of $p(\varphi)$ and (12) we have

$$\frac{p(\frac{\pi}{4})}{\Pi(\frac{\pi}{4})} + C\int_{\frac{\pi}{4}}^{0} \frac{1}{\sin^2 2s\, \Pi(s)}\, ds = \frac{p(-\frac{\pi}{4})}{\Pi(-\frac{\pi}{4})} + C\int_{-\frac{\pi}{4}}^{-\frac{\pi}{2}} \frac{1}{\sin^2 2s\, \Pi(s)}\, ds = 0.$$

Thus,

$$p(\varphi) = \begin{cases} C\int_0^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds & \text{for } \varphi\in(0,\frac{\pi}{2}), \\ C\int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds & \text{for } \varphi\in(-\frac{\pi}{2},0). \end{cases}$$

Since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\varphi)\, d\varphi = 1$ it follows that

$$C = \frac{1}{\int_0^{\frac{\pi}{2}}\int_0^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi + \int_{-\frac{\pi}{2}}^{0}\int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi}.$$

A direct computation yields that

$$f_{\widehat{A}}(\varphi) + \frac{1}{2}f_B'(\varphi)g_B(\varphi) = \frac{a_{11}+a_{22}}{2} + \frac{a_{11}-a_{22}}{2}\cos 2\varphi + \frac{(\alpha-\beta)^2}{4}\sin^2 2\varphi,$$

which together with Theorem 1 implies that

$$\begin{aligned}\lambda_{A,B} &= \frac{a_{11}+a_{22}}{2} + \frac{\int_0^{\frac{\pi}{2}}\left(\frac{a_{11}-a_{22}}{2}\cos 2\varphi + \frac{(\alpha-\beta)^2}{4}\sin^2 2\varphi\right)\int_0^{\varphi}\frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi}{\int_0^{\frac{\pi}{2}}\int_0^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi + \int_{-\frac{\pi}{2}}^{0}\int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi} \\ &\quad + \frac{\int_{-\frac{\pi}{2}}^{0}\left(\frac{a_{11}-a_{22}}{2}\cos 2\varphi + \frac{(\alpha-\beta)^2}{4}\sin^2 2\varphi\right)\int_{-\frac{\pi}{2}}^{\varphi}\frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi}{\int_0^{\frac{\pi}{2}}\int_0^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi + \int_{-\frac{\pi}{2}}^{0}\int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi(\varphi)}{\sin^2 2s\,\Pi(s)}\, ds\, d\varphi}.\end{aligned}$$

Changing variables $s = \arctan v$ and $\varphi = \arctan u$ in the preceding integral completes the proof. □

Type III

In this subsection, we recall the result in [IL99, IL01] about explicit formula of Lyapunov exponent of linear stochastic differential equations whose diffusion parts are of Type III. Note that in this case a pair (A, B) do not satisfy the degeneracy condition (H) iff the matrix A is not of lower triangular form, i.e. $a_{12} \neq 0$.

Theorem 8 (Explicit formula for the Lyapunov exponents of linear SDE of Type III)**.** Consider system (2) with the drift part $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and with the diffusion part $B = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{R}$. Then,

$$\lambda_{A,B} = \frac{a_{11} + a_{22}}{2} + \frac{1}{2} |a_{12}| \frac{\int\limits_0^\infty \sqrt{v} \exp\left(-\frac{1}{6} |a_{12}| v^3 + \frac{v}{2|a_{12}|} (\mu_1 - \mu_2)^2\right) dv}{\int\limits_0^\infty \frac{1}{\sqrt{v}} \exp\left(-\frac{1}{6} |a_{12}| v^3 + \frac{v}{2|a_{12}|} (\mu_1 - \mu_2)^2\right) dv}.$$

Type IV

In this subsection, we consider the case that the matrix B has a pair of conjugated complex eigenvalues, i.e.

$$B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad \text{where } \alpha, \beta \in \mathbb{R}. \tag{13}$$

Theorem 9 (Explicit formula for the top Lyapunov exponents of linear SDE of Type IV)**.** Consider system (2) with the drift part $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and with the diffusion part $B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. Define a function $\Pi_{A,B} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$\Pi_{A,B}(\varphi) := \exp\left(\frac{a_{11} - a_{22}}{2\beta^2} \cos 2\varphi + \frac{a_{12} + a_{21}}{2\beta^2} \sin 2\varphi - \frac{a_{12} - a_{21}}{\beta^2} \varphi\right).$$

Then, the top Lyapunov exponents of (2) is given by:

If $a_{12} = a_{21}$, then

$$\lambda_{A,B} = \frac{a_{11} + a_{22}}{2} + \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{a_{11}-a_{22}}{2} \cos 2\varphi + \frac{a_{12}+a_{21}}{2} \sin 2\varphi\right) \Pi_{A,B}(\varphi)\, d\varphi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\, d\varphi}.$$

If $a_{12} \neq a_{21}$ then

$$\lambda_{A,B} = \frac{1}{\Gamma_{A,B}(\exp(\frac{\pi(a_{12}-a_{21})}{\beta^2})-1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Pi_{A,B}(\varphi)}\, d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_A(\varphi)\Pi_{A,B}(\varphi)\, d\varphi$$

$$+\frac{1}{\Gamma_{A,B}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\varphi} f_A(\varphi)\frac{\Pi_{A,B}(\varphi)}{\Pi_{A,B}(u)}\, du\, d\varphi.$$

where

$$\Gamma_{A,B} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi_{A,B}(\varphi)}{\Pi_{A,B}(u)}\, du\, d\varphi + \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\, d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Pi_{A,B}(\varphi)}\, d\varphi}{\exp(\frac{\pi(a_{12}-a_{21})}{\beta^2})-1}. \quad (14)$$

Proof. By definition of $g_B(\varphi)$, we have $g_B(\varphi) = \beta$ for all $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, the condition (H) holds for all $A \in \mathbb{R}^{2\times 2}$ and the differential equation of $p(\varphi)$ is given by

$$p'(\varphi) = \frac{2g_A(\varphi)}{\beta^2} p(\varphi) + C, \quad (15)$$

where C is a constant. From the definition of $\Pi_{A,B}$, we get that

$$\Pi_{A,B}(\varphi) = \exp\left(\int_{-\frac{\pi}{2}}^{\varphi} \frac{2g_A(u)}{\beta^2}\, du\right) \exp\left(\frac{(a_{12}-a_{21})\pi - (a_{11}-a_{22})}{2\beta^2}\right).$$

Therefore, $\Pi_{A,B}(\varphi)$ is a nontrivial solution of the corresponding linear equation of (15) given by $p'(\varphi) = \frac{2g_A(\varphi)}{\beta^2} p(\varphi)$. By variation of constants formula, we arrive at

$$p(\varphi) = \Pi_{A,B}(\varphi) \left(\kappa + C \int_{-\frac{\pi}{2}}^{\varphi} \frac{1}{\Pi_{A,B}(u)}\, du\right), \quad (16)$$

where κ is a constant. In the remaining part, we solve κ and C by using the properties that $p(-\frac{\pi}{2}) = p(\frac{\pi}{2})$ and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(u)\, du = 1$. From definition of $\Pi_{A,B}$, we have

$$\frac{\Pi_{A,B}(-\frac{\pi}{2})}{\Pi_{A,B}(\frac{\pi}{2})} = \exp\left(\frac{\pi(a_{12}-a_{21})}{\beta^2}\right).$$

So, from (16) the equality $p(-\frac{\pi}{2}) = p(\frac{\pi}{2})$ leads to

$$\exp\left(\frac{\pi(a_{12}-a_{21})}{\beta^2}\right)\kappa = \kappa + C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Pi_{A,B}(u)}\,du.$$

To solve the preceding equality, we consider the following separated cases:
Case 1: If $a_{12} = a_{21}$, then $C = 0$. Hence, from $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\varphi)\,d\varphi = 1$ we derive that $\kappa = \frac{1}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\,d\varphi}$. According to Theorem 1, we obtain

$$\begin{aligned}
\lambda(A,B) &= \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_A(\varphi)\,\Pi_{A,B}(\varphi)\,d\varphi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\,d\varphi} \\
&= \frac{a_{11}+a_{22}}{2} + \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{a_{11}-a_{22}}{2}\cos 2\varphi + \frac{a_{12}+a_{21}}{2}\sin 2\varphi\right)\Pi_{A,B}(\varphi)\,d\varphi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\,d\varphi}.
\end{aligned}$$

Case 2: If $a_{21} \neq a_{12}$, then from $p(-\frac{\pi}{2}) = p(\frac{\pi}{2})$ we derive that

$$\kappa = C\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Pi_{A,B}(u)}\,du}{\exp(\frac{\pi(a_{12}-a_{21})}{\beta^2}) - 1}.$$

Since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\varphi)\,d\varphi = 1$ it follows that

$$C\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\varphi} \frac{\Pi_{A,B}(\varphi)}{\Pi_{A,B}(u)}\,du\,d\varphi + \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Pi_{A,B}(\varphi)\,d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Pi_{A,B}(\varphi)}\,d\varphi}{\exp(\frac{\pi(a_{12}-a_{21})}{\beta^2}) - 1}\right) = 1.$$

Hence, $C = \frac{1}{\Gamma_{A,B}}$, where $\Gamma_{A,B}$ is defined as in (14) and using Theorem 1, the proof is complete. □

4 Applications

4.1 Sample-path stability of a linear stochastic differential equation arsing from fluid dynamics

In this subsection, we consider the following linear stochastic differential equation

$$\begin{pmatrix} du_t \\ dv_t \end{pmatrix} = \begin{pmatrix} -R^{-1} & 1 \\ 0 & -R^{-1} \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} \circ dW_t, \tag{17}$$

where $\sigma \neq 0$. This system was discussed in [BK14] and arises from a model in fluid dynamics. In [BK14], the authors compute explicitly the area of parameters for which system (17) is mean-square asymptotically stable. Instead of studying mean-square asymptotic stability, our aim is to find the area of parameters for which system (17) is sample-path asymptotically stable.

Theorem 10. System (17) is asymptotically stable if and only if two parameters R, σ satisfy

$$\begin{aligned} \frac{\Gamma_{A,B}}{R} \quad > \quad & \frac{1}{\exp(\frac{\pi}{\sigma^2}) - 1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2\varphi}{2} \frac{\exp\left(\frac{\sin 2\varphi}{2\sigma^2} - \frac{\varphi}{\sigma^2}\right)}{\exp\left(\frac{\sin 2u}{2\sigma^2} - \frac{u}{\sigma^2}\right)}\, du\, d\varphi \\ & + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\varphi} \frac{\sin 2\varphi}{2} \frac{\exp\left(\frac{\sin 2\varphi}{2\sigma^2} - \frac{\varphi}{\sigma^2}\right)}{\exp\left(\frac{\sin 2u}{2\sigma^2} - \frac{u}{\sigma^2}\right)}\, du\, d\varphi, \end{aligned}$$

where

$$\Gamma_{A,B} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\varphi} \frac{\exp\left(\frac{\sin 2\varphi}{2\sigma^2} - \frac{\varphi}{\sigma^2}\right)}{\exp\left(\frac{\sin 2u}{2\sigma^2} - \frac{u}{\sigma^2}\right)}\, du\, d\varphi + \frac{\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(\frac{\sin 2\varphi}{2\sigma^2} - \frac{\varphi}{\sigma^2}\right) d\varphi\right)^2}{\exp(\frac{\pi}{\sigma^2}) - 1} > 0. \tag{18}$$

Proof. The diffusion coefficient of this linear stochastic differential equation if of Type IV. Hence, we can use Theorem 9 to compute and to determine the sign of the top Lyapunov exponent of (17). A direct computation yields that

$$\Pi_{A,B}(\varphi) = \exp\left(\frac{\sin 2\varphi}{2\sigma^2} - \frac{\varphi}{\sigma^2}\right) \quad \text{and} \quad f_A(\varphi) = -\frac{1}{R} + \frac{\sin 2\varphi}{2}.$$

Therefore, the constant $\Gamma_{A,B}$ defined in (14) is computed explicitly as in (18). Hence, by Theorem 9, we have

$$\begin{aligned}\Gamma_{A,B}\lambda_{A,B} &= -\frac{1}{R}\Gamma_{A,B} + \frac{1}{\exp(\frac{\pi}{\sigma^2})-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2\varphi}{2}\frac{\exp\left(\frac{\sin 2\varphi}{2\sigma^2}-\frac{\varphi}{\sigma^2}\right)}{\exp\left(\frac{\sin 2u}{2\sigma^2}-\frac{u}{\sigma^2}\right)}\,du\,d\varphi \\ &\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\varphi} \frac{\sin 2\varphi}{2}\frac{\exp\left(\frac{\sin 2\varphi}{2\sigma^2}-\frac{\varphi}{\sigma^2}\right)}{\exp\left(\frac{\sin 2u}{2\sigma^2}-\frac{u}{\sigma^2}\right)}\,du\,d\varphi,\end{aligned}$$

which completes the proof. □

As is proved in [BK14, Theorem 4.2], system (17) is mean-square asymptotically stable iff R and σ satisfy

$$\begin{aligned}\frac{6}{R} &> -\sigma^2 + \sqrt[3]{8\sigma^6 + 27\sigma^2 + 3\sqrt{3\sigma^4(16\sigma^4+27)}} \\ &\quad + \frac{4\sigma^4}{\sqrt[3]{8\sigma^6 + 27\sigma^2 + 3\sqrt{3\sigma^4(16\sigma^4+27)}}}.\end{aligned}$$

The following lemma is devoted to show that mean-square asymptotic stability implies sample-path asymptotic stability for an arbitrary planar linear stochastic differential equation

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix} dt + B \begin{pmatrix} x_t \\ y_t \end{pmatrix} \circ dW_t. \tag{19}$$

Recall that the *mean-square Lyapunov exponent* of (19) denoted by $\lambda^{\mathrm{ms}}_{A,B}$ is defined by

$$\lambda^{\mathrm{ms}}_{A,B} := \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\|\Phi_{A,B}(t,\cdot)\|^2,$$

see e.g. [AKO86].

Lemma 11. Consider (19) and suppose that $\lambda^{\mathrm{ms}}_{A,B} < 0$. Then, $\lambda_{A,B} < 0$.

Proof. Note that in the case that the coefficients A and B of (17) satisfy condition (H) of Theorem 1, then the assertion of this lemma was proved in [Ar84, Corollary 1]. Hence, we only need to deal with the case that the pair $\{A, B\}$ does not satisfy (H). Based on the arguments at the beginning of

Subsection 3.1, we can assume w.l.o.g. that A and B are upper triangular matrices, i.e.

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & 0 \\ b_{12} & b_{22} \end{pmatrix}.$$

Using Ito's formula, we obtain that

$$\begin{aligned}
\frac{d\,\mathbb{E}x_t^2}{dt} &= 2(a_{11}+b_{11}^2)\,d\mathbb{E}x_t^2, \\
\frac{d\,\mathbb{E}x_t y_t}{dt} &= \left(a_{11}+a_{22}+\frac{(b_{11}+b_{22})^2}{2}\right)\mathbb{E}x_t y_t, \\
&\qquad + \left(a_{21}+\frac{b_{21}(b_{11}+b_{22})}{2}+b_{11}b_{21}\right)\mathbb{E}x_t^2 \\
\frac{d\,\mathbb{E}y_t^2}{dt} &= 2\left(a_{22}+b_{22}^2\right)\mathbb{E}y_t^2 + b_{21}^2\mathbb{E}x_t^2 + 2\left(a_{21}+\frac{b_{21}(b_{11}+3b_{22})}{2}\right)\mathbb{E}x_t y_t.
\end{aligned}$$

Therefore, $\lambda_{A,B}^{\mathrm{ms}} < 0$ is equivalent to the fact that

$$a_{11}+b_{11}^2 < 0 \quad \text{and} \quad a_{22}+b_{22}^2 < 0,$$

which implies together with Theorem 3 that $\lambda_{A,B} < 0$. The proof is complete. □

4.2 A model of stochastic Hopf bifurcation

Consider a model of stochastic Hopf bifurcation of the following form

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = \begin{pmatrix} -y_t + (a - b(x_t^2+y_t^2))x_t \\ x_t + (a - b(x_t^2+y_t^2))y_t \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \circ dW_t, \tag{20}$$

where $a \in \mathbb{R}$ and $b, \sigma_1, \sigma_2 > 0$ are parameters. Note that the preceding model with an additional assumption that $\sigma_1 = \sigma_2$ is studied in [Bax94]. Our aim in this section is to find the bifurcation value of (20) in the remaining case that $\sigma_1 \neq \sigma_2$. For this purpose, the linearized equation along the trivial solution of (20) is

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \circ dW_t. \tag{21}$$

Let $\delta := \sigma_1 - \sigma_2$ and denote by $a_{\text{bif}}(\delta)$ the bifurcation value satisfying that when a changes increasingly and crosses $a_{\text{bif}}(\delta)$, the system (21) changes from stability to instability. For the case that $\delta > 0$, in light of Theorem 7 the bifurcation value $a_{\text{bif}}(\delta)$ is

$$\begin{aligned} a_{\text{bif}}(\delta) &= -\frac{\delta^2}{C}\int_0^\infty \frac{\exp\left(\frac{2(1-u^2)}{\delta^2 u}\right)}{1+u^2}\int_0^u \frac{\exp\left(-\frac{2(1-v^2)}{\delta^2 v}\right)}{1+v^2}\,dv\,du \\ &\quad - \frac{\delta^2}{C}\int_{-\infty}^0 \frac{\exp\left(\frac{2(1-u^2)}{\delta^2 u}\right)}{1+u^2}\int_{-\infty}^u \frac{\exp\left(-\frac{2(1-v^2)}{\delta^2 v}\right)}{1+v^2}\,dv\,du, \end{aligned}$$

where

$$\begin{aligned} C := &\int_0^\infty \frac{1+u^2}{u^2}\exp\left(\frac{2(1-u^2)}{\delta^2 u}\right)\int_0^u \frac{\exp\left(\frac{2(1-v^2)}{\delta^2 v}\right)}{1+v^2}\,dv\,du \\ + &\int_{-\infty}^0 \frac{1+u^2}{u^2}\exp\left(\frac{2(1-u^2)}{\delta^2 u}\right)\int_{-\infty}^u \frac{\exp\left(\frac{2(1-v^2)}{\delta^2 v}\right)}{1+v^2}\,dv\,du. \end{aligned}$$

By virtue of Theorem 5, $a_{\text{bif}}(0) = 0$, i.e. when $\delta = 0$ the bifurcation value of the stochastic Hopf bifurcation model (20) coincides with the Hopf bifurcation value of the deterministic part. Meanwhile, when $\delta > 0$, the bifurcation value of the stochastic Hopf bifurcation model (20) is strictly smaller than the Hopf bifurcation value of the deterministic part.

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