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Shortest Paths along a Sequence of Line Segments in Euclidean Spaces

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(Dedicated to Prof. Dr. sc. Hoang Xuan Phu on the occasion of his 60th birthday)

Abstract In this paper, some analytical and geometric properties of shortest ordered paths joining two given points with respect to a sequence of line segments in a Euclidean space, especially their existence, uniqueness, characteristics, and conditions for concatenation of two shortest ordered paths to be a shortest ordered path are presented. We then focus on straightest paths lying on a sequence of adjacent convex polygons in 2 or 3 dimensional spaces.

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1 Introduction

Computing a shortest path in a geometric domain (see Mitchell [1]) is a fundamental problem in computational geometry with many applications in different areas such as robotics, geographic information systems and navigation (see Agarwal et al. [2] and Sethian [3]).

To date, some authors propose many exact algorithms (see Chen and Han [4], Mitchell et al. [5], Sharir and Schorr [6],. . .) to find the shortest path joining two fixed points on a polyhedral surface. Some others authors consider the problem of finding the shortest path joining two points on a sequence of adjacent faces and use the results to find the solution of the above problem (see Pham-Trong et al. [7] and Xin and Wang [8]).

Several properties of shortest paths joining two points on a geometric domain were considered in some papers (see Hai and An [9], Sharir and Schorr [6], Mitchell et al. [5] and Hoai, An and Hai [10]). Sharir and Schorr [6] proved that the shortest path joining two points on a convex polyhedral surface cannot pass through any vertex except at its end-points. Mitchell et al. [5] proved that the general form of the shortest path on a sequence of edge-adjacent faces is a path which goes through an alternating sequence of vertices (these faces here are triangles). The unfolded image of the path passes through a sequence of line segments is a line segment, and the curve angle at any vertex which the path passes through is greater than or equal to π.

The problem of finding shortest path joining two points along a sequence of adjacent convex polygons can be simplified to the following: Given two points and a sequence of line segments in real three dimensional space, find a shortest path that joins these points and passes these line segments in a given order. Akman ([11]) presented an algorithm for finding such shortest paths. Liu and Wong [12] and Sautter [13] considered the problem of finding shortest gentle paths between two points in a sequence of line segments as a local problem of finding shortest gentle paths between two points on a polytope, where a path is gentle if its slope is less than or equal to a given positive number. When this number equal to π, the shortest gentle paths are the shortest paths.

Our problem covers the one of finding the shortest path lying in a simple polygon in the planar space that joins two given points and meets a given sequence of sides and vertices of the polygon in a required order. Our problem is studied under the analytical point of view.We present the existence and uniqueness of shortest ordered path (Theorem 3.2 and Corollary 3.4). We then investigate some characteristics of angles between shortest ordered path and line segments, especially when the path goes through common points some line segments (Theorem 4.1, Corollaries 4.2–4.3). Sufficient conditions under which concatenation of two shortest ordered paths be shortest are also given (Theorem 4.2 and Corollary 4.4). In the last section we introduce the concept of straightest path on a sequence of adjacent convex polygons, establish a relation between shortest paths and straightest paths joining two points on the sequence, and consider a discrete initial value problem of straightest paths. Theorem 5.1 can be used to find straightest path on a sequence of convex polygons from a given point and a given direction. Moreover, using Proposition 5.1 one can construct shortest paths joining two points and passing through a sequence of convex polygons from its finite straightest paths (see An, Giang, Phu and Polthier [14]).

2 Preliminaries

Given a metric space (X, d). A *path* in X is a continuous mapping γ from an interval $[t_0, t_1] \subset \mathbb{R}$ to X. We say that γ *joins* the point $\gamma(t_0)$ to the point $\gamma(t_1)$. The *length* of $\gamma : [t_0, t_1] \to X$ is the quantity $l(\gamma) = \sup_\sigma \sum_{i=1}^{k} d\big(\gamma(\tau_{i-1}), \gamma(\tau_i)\big)$, where the supremum is taken over the set of partitions $t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = t_1$ of $[t_0, t_1]$. The length of a path is additive, i.e., for any path γ :

$[t_0, t_1] \to X$ and $t_* \in [t_0, t_1]$, $l(\gamma) = l\big(\gamma_{|_{[t_0,t_*]}}\big) + l\big(\gamma_{|_{[t_*,t_1]}}\big)$, where $\gamma_{|_{[t_0,t_*]}}$ and $\gamma_{|_{[t_*,t_1]}}$ are restrictions of γ on $[t_0, t_*]$ and $[t_*, t_1]$, respectively (see [15]). For instance, let $(X, \|\cdot\|)$ be a normed space. A mapping $\gamma : [t_0, t_1] \to X$ is said to be an *affine path* if for any $\lambda \in [0, 1]$, $\gamma((1-\lambda)t_0 + \lambda t_1) = (1-\lambda)\gamma(t_0) + \lambda\gamma(t_1)$. For $x, y \in X$, $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, a path $\gamma : [t_0, t_1] \to X$ joining x to y is affine iff $\gamma(t) = (t_1 - t_0)^{-1}\big[(t_1 - t)x + (t - t_0)y\big]$. In this case, γ has length $\|x - y\|$ and the image $\gamma([t_0, t_1])$ is the *line segment* $[x, y] := \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$. We assume that all paths in this paper have finite length.

Let t_0, t_1 and t_2 be real numbers satisfying $t_0 \le t_1 \le t_2$. If $\gamma_1 : [t_0, t_1] \to X$ and $\gamma_2 : [t_1, t_2] \to X$ are two paths satisfying $\gamma_1(t_1) = \gamma_2(t_1)$, then we can define the path $\gamma : [t_0, t_2] \to X$ by setting $\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t_0 \le t \le t_1, \\ \gamma_2(t) & \text{if } t_1 \le t \le t_2. \end{cases}$ γ is called the *concatenation* of γ_1 and γ_2, denoted by $\gamma_1 * \gamma_2$, and we have $l(\gamma) = l(\gamma_1) + l(\gamma_2)$.

Let $\gamma : [t_0, t_1] \to X$ and $\eta : [\tau_0, \tau_1] \to X$ be two paths in X. We say that γ *is obtained from* η *by a change of parameter* if there exists a function $\psi : [t_0, t_1] \to [\tau_0, \tau_1]$ that is monotonic (in the weak sense), surjective and that satisfies $\gamma = \eta \circ \psi$. The function ψ is called *the change of parameter*. It is proved that the equality $l(\gamma) = l(\eta)$ always holds. We say that $\gamma : [t_0, t_1] \to X$ is *parametrized by arclength* if for all τ and τ' satisfying $t_0 \le \tau \le \tau' \le t_1$, we have $l\big(\gamma_{|_{[\tau,\tau']}}\big) = \tau' - \tau$. If $\gamma : [t_0, t_1] \to X$ is any path, then there always exists a path $\lambda : [0, l(\gamma)] \to X$ such that λ is parametrized by arclength and γ is obtained from λ by the change of the parameter $\psi : [t_0, t_1] \to [0, l(\gamma)]$ defined by $\psi(\tau) = l\big(\gamma_{|_{[t_0,\tau]}}\big)$. Most definitions and results above can be seen in [15].

In this paper, $\mathbb{E}$ denotes $\mathbb{R}^n$, equipped with the Euclidian norm $\|\cdot\|$, and $d(x, y)$ is $\|x - y\|$. For $x, y \in \mathbb{E}$, denote $]x, y] = [x, y] \setminus \{x\}$, $[x, y[= [x, y] \setminus \{y\}$, and $]x, y[= [x, y] \setminus \{x, y\}$. Note that when $x = y$, $[x, y] = \{x\}$ and $[x, y[=]x, y] =]x, y[= \emptyset$. If $x \ne y$, each point $z \in]x, y[$ is called an *interior point* of $[x, y]$. By abuse of notation, sometimes we also call the image $\gamma([t_0, t_1])$ the path $\gamma : [t_0, t_1] \to \mathbb{E}$.

3 Shortest Ordered Paths

Definition 3.1 Let a, b be points in $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singleton). A path $\gamma : [t_0, t_1] \to \mathbb{E}$ that joins a to b is called an *ordered path* with respect to the sequence $e_1, \ldots, e_k$ if there is a sequence of numbers $t_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_k \leq t_1$ such that $\gamma(\bar{t}_i) \in e_i$ for $i = 1, \ldots, k$ (see Fig. 1). γ is called a *shortest ordered path* if its length does not exceed any ordered path.

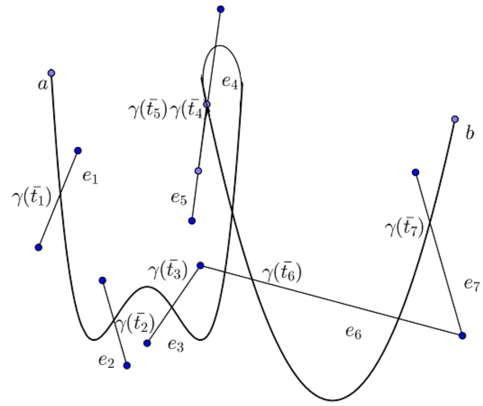

Fig. 1: An ordered path that joins a to b and passes through $e_1, e_2, e_3, e_4, e_5, e_6$ and e_7.

If γ joins a to b and is an ordered path with respect to the sequence $e_1, \ldots, e_k$, we call it an $OP(a,b)_{(e_1,\ldots,e_k)}$. If, in addition, it is a shortest path, then we say that γ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$.

In this paper we always assume that $e_i \neq e_{i+1}$ for all $i = 1, \ldots, k-1$. We also use the notation $SOP(a,b)_\emptyset$ or $[a,b]$ to denote any path that joins a to b, one-to-one, and has length $\|a - b\|$ and image $[a,b]$. Usually $SOP(a,b)_\emptyset$ is assumed to be an affine path joining a to b. The aim of this section is to prove that shortest ordered path joining two given points and with respect to a given sequence of line segments exists and in some sense is unique.

We start with an intuitive property of paths.

Lemma 3.1 *Let* $\gamma : [t_0, t_1] \to \mathbb{E}$ *be any path.*

(a) $l(\gamma) \geq \|\gamma(t_0) - \gamma(t_1)\|$ *and equality holds only if* $\gamma([t_0, t_1]) = [\gamma(t_0), \gamma(t_1)]$.

(b) If γ *is parametrized by arclength and* $l(\gamma) = \|\gamma(t_0) - \gamma(t_1)\| > 0$*, then* γ *is affine, that is,*

$$\gamma(t) = \frac{t_1 - t}{t_1 - t_0}\gamma(t_0) + \frac{t - t_0}{t_1 - t_0}\gamma(t_1), \quad t \in [t_0, t_1].$$

Proof. Set $a = \gamma(t_0)$ and $b = \gamma(t_1)$.

(a) Choosing the partition $\{t_0, t_1\}$ of $[t_0, t_1]$ gives $l(\gamma) \geq \|\gamma(t_0) - \gamma(t_1)\| = \|a - b\|$. Suppose that $l(\gamma) = \|a - b\|$. If there were $x = \gamma(\tau) \in \gamma([t_0, t_1])$ such that $x \notin [a, b]$, then we would have $l(\gamma) \geq \|\gamma(t_0) - \gamma(\tau)\| + \|\gamma(\tau) - \gamma(t_1)\| = \|a - x\| + \|x - b\| > \|a - b\|$, a contradiction. Thus $\gamma([t_0, t_1]) \subset [a, b]$. Moreover, since $\gamma([t_0, t_1])$ is connected, $a = \gamma(t_0)$ and $b = \gamma(t_1)$, it follows that $\gamma([t_0, t_1]) = [a, b]$.

(b) First observe that $t_1 - t_0 = l(\gamma) = \|a - b\| > 0$ and by (a), $\gamma([t_0, t_1]) = [a, b]$. Fix $t_0 < t < t_1$ and suppose $t = (1 - \lambda)t_0 + \lambda t_1$ and $\gamma(t) = (1 - \mu)a + \mu b$, $\lambda, \mu \in [0, 1]$. Since γ is parametrized by arclength, $l\big(\gamma_{|[t_0,t]}\big) = t - t_0 \geq \|\gamma(t) - a\|$ and $l\big(\gamma_{|[t,t_1]}\big) = t_1 - t \geq \|b - \gamma(t)\|$. Adding these inequalities gives $t_1 - t_0 \geq \|\gamma(t) - a\| + \|b - \gamma(t)\| \geq \|a - b\| = t_1 - t_0$. Hence $t - t_0 = \|\gamma(t) - a\|$, that is $\lambda(t_1 - t_0) = \mu\|a - b\| = \mu(t_1 - t_0)$, whence $\mu = \lambda = (t - t_0)/(t_1 - t_0)$. □

We now turn to properties of shortest ordered paths. Although the following result is simple, it is essential for others.

Theorem 3.1 *Let* $\gamma\big([t_0, t_1]\big)$ *be an* $SOP(a, b)_{(e_1,\ldots,e_k)}$ *and let* $t_0 =: \bar{t}_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_k \leq \bar{t}_{k+1} := t_1$ *satisfy* $\gamma(\bar{t}_i) \in e_i$ *for* $i = 1, \ldots, k$.

(a) If $\bar{t}_{j-1} \leq \alpha < \beta \leq \bar{t}_j$*, then* $l\big(\gamma_{|[\alpha,\beta]}\big) = \|\gamma(\alpha) - \gamma(\beta)\|$ *and* $\gamma([\alpha, \beta]) = [\gamma(\alpha), \gamma(\beta)]$*. If, in addition,* γ *is parametrized by arclength, then* γ *is affine on* $[\alpha, \beta]$.

(b) $\gamma\big([t_0, t_1]\big) = \cup_{i=0}^{k}[\gamma(\bar{t}_i), \gamma(\bar{t}_{i+1})]$*. Thus* $\gamma\big([t_0, t_1]\big)$ *is a polyline.*

Proof. On the contrary, suppose that $l\big(\gamma_{|[\alpha,\beta]}\big) > \|\gamma(\alpha) - \gamma(\beta)\|$. Define $\eta : [t_0, t_1] \to \mathbb{E}$ by $\eta(\tau) = \gamma(\tau)$ for $\tau \in [t_0, \alpha] \cup [\beta, t_1]$ and η is affine on $[\alpha, \beta]$. η is then an $OP(a, b)_{(e_1,\ldots,e_k)}$. However, since $l\big(\eta_{|[\alpha,\beta]}\big) =$

$\|\gamma(\alpha)-\gamma(\beta)\| < l\big(\gamma_{|[\alpha,\beta]}\big)$, $l(\eta) = l\big(\eta_{|[t_0,\alpha]}\big) + l\big(\eta_{|[\alpha,\beta]}\big) + l\big(\eta_{|[\beta,t_1]}\big) < l\big(\gamma_{|[t_0,\alpha]}\big) + l\big(\gamma_{|[\alpha,\beta]}\big) + l\big(\gamma_{|[\beta,t_1]}\big) = l(\gamma)$. This contradicts the fact that γ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$. Thus $l\big(\gamma_{|[\alpha,\beta]}\big) = \|\gamma(\alpha) - \gamma(\beta)\|$ and hence, by Lemma 3.1, $\gamma\big([\alpha,\beta]\big) = [\gamma(\alpha),\gamma(\beta)]$. In particular, $\gamma\big([\bar{t}_i,\bar{t}_{i+1}]\big) = [\gamma(\bar{t}_i),\gamma(\bar{t}_{i+1})]$ for $i = 0,\ldots,k$. Therefore, $\gamma\big([t_0,t_1]\big) = \cup_{i=0}^{k}\gamma([\bar{t}_i,\bar{t}_{i+1}]) = \cup_{i=0}^{k}[\gamma(\bar{t}_i),\gamma(\bar{t}_{i+1})]$, which is a polyline. Finally, if γ is parametrized by arclength, Lemma 3.1 says that γ is affine on $[\alpha,\beta]$. □

Remark 3.1 Let $f_1,\ldots,f_{k+1}$ be a sequence of convex polygons (these polygons are not necessarily distinct and f_i and f_{i+1} may be identical), let e_i be a common edge of f_i and f_{i+1} and let $a \in f_1$, $b \in f_{k+1}$ ($k \geq 1$). If $\gamma([t_0,t_1])$ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$, then by Theorem 3.1, $\gamma([t_0,t_1]) = \cup_{i=0}^{k}[\gamma(\bar{t}_i),\gamma(\bar{t}_{i+1})] \subset \cup_{i=1}^{k+1} f_i$, i.e., this shortest ordered path lies on the polygons. Conversely, if $\gamma : [t_0,t_1] \to \cup_{i=1}^{k+1} f_i$ is an $OP(a,b)_{(e_1,\ldots,e_k)}$ and has the minimum length in the family of all paths that join a to b, lie on the polygons f_i, and pass through the edges $e_1,\ldots,e_k$ in that order, then $\gamma([t_0,t_1])$ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$.

Thus determining a shortest path on the sequence of faces $f_1,\ldots,f_{k+1}$ is equivalent to determining an $SOP(a,b)_{(e_1,\ldots,e_k)}$. This is a reason why we investigate in this paper shortest ordered paths with respect to a sequence of line segments.

The following lemma says that restriction of a shortest ordered path on each subinterval of its domain is again a shortest ordered path. It can be seen as a "local property" of shortest ordered paths.

Lemma 3.2 *Suppose* $\gamma\big([t_0,t_1]\big)$ *is an* $SOP(a,b)_{(e_1,\ldots,e_k)}$, $\gamma(\bar{t}_i) \in e_i$ *for* $i = 1,\ldots,k$ *and* $t_0 =: \bar{t}_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_k \leq \bar{t}_{k+1} := t_1$. *If* $1 \leq m \leq n \leq k$ *and* $\bar{t}_{m-1} \leq \alpha \leq \bar{t}_m \leq \bar{t}_n \leq \beta \leq \bar{t}_{n+1}$, *then* $\gamma_{|[\alpha,\beta]}$ *is an* $SOP(\gamma(\alpha),\gamma(\beta))_{(e_m,\ldots,e_n)}$.

Proof. Let $x := \gamma(\alpha)$ and $y := \gamma(\beta)$. Clearly $\gamma_{|[\alpha,\beta]}$ is an $OP(x,y)_{(e_m,\ldots,e_n)}$. If $\gamma_{|[\alpha,\beta]}$ is not shortest, there exists a path $\eta : [\alpha,\beta] \to \mathbb{E}$ that is an $OP(x,y)_{(e_m,\ldots,e_n)}$ with $l(\eta) < l\big(\gamma_{|[\alpha,\beta]}\big)$. Then $\xi := \gamma_{|[t_0,\alpha]} * \eta * \gamma_{|[\beta,t_1]}$ is an $OP(a,b)_{(e_1,\ldots,e_k)}$ but $l(\xi) = l\big(\gamma_{|[t_0,\alpha]}\big) + l(\eta) + l\big(\gamma_{|[\beta,t_1]}\big) < l\big(\gamma_{|[t_0,\alpha]}\big) + l\big(\gamma_{|[\alpha,\beta]}\big) + l\big(\gamma_{|[\beta,t_1]}\big) = l(\gamma)$, a contradiction. Thus $\gamma_{|[\alpha,\beta]}$ must be an $SOP(x,y)_{(e_m,\ldots,e_n)}$. □

Lemma 3.3 *Suppose $\gamma\big([t_0,t_1]\big)$ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$ and $t_0 =: \bar{t}_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le \bar{t}_{k+1} := t_1$, $\gamma(\bar{t}_i) \in e_i$ for $i = 1,\ldots,k$. If $l\big(\gamma_{|[t,t']}\big) > 0$ for all $t,t' \in [t_0,t_1]$ and $t < t'$, then γ is one-to-one on each subinterval $[\bar{t}_i,\bar{t}_{i+1}]$, $0 \le i \le k$.*

Proof. Suppose conversely that γ is not one-to-one on $[\bar{t}_j,\bar{t}_{j+1}]$, i.e., there exist $\bar{t}_j \le \alpha < \beta \le \bar{t}_{j+1}$ with $\gamma(\alpha) = \gamma(\beta)$. Set $\delta := \beta - \alpha > 0$. Consider the mapping $\eta : [t_0, t_1 - \delta] \to \mathbb{E}$ defined by $\eta(t) = \gamma(t)$ if $t_0 \le t \le \alpha$ and $\eta(t) = \gamma(t+\delta)$ if $\alpha < t \le t_1 - \delta$. It is easy to see that η is an $OP(a,b)_{(e_1,\ldots,e_k)}$ and since $l\big(\gamma_{|[\alpha,\beta]}\big) > 0$, $l(\eta) = l\big(\eta_{|[t_0,\alpha]}\big) + l\big(\eta_{|[\alpha,t_1-\delta]}\big) = l\big(\gamma_{|[t_0,\alpha]}\big) + l\big(\gamma_{|[\beta,t_1]}\big) = l(\gamma) - l\big(\gamma_{|[\alpha,\beta]}\big) < l(\gamma)$. This contradicts the fact that γ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$. Thus γ is one-to-one on $[\bar{t}_i,\bar{t}_{i+1}]$. □

In this paper, sometimes we use the following assumption.

(A) The restriction of the path on each non-singleton interval of its domain has positive length.

Assumption (A) means that the path is not constant on any interval with positive length in its domain. It is satisfied, for instance, if the path is parametrized by arclength or if it is one-to-one.

Next we turn to the problem of existence and uniqueness of a shortest ordered path.

Lemma 3.4 *Suppose that $\gamma([t_0,t_1])$ and $\eta([\tau_0,\tau_1])$ are $OP(a,b)_{(e_1,\ldots,e_k)}$, $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$, $\tau_0 \le \bar{\tau}_1 \le \cdots \le \bar{\tau}_k \le \tau_1$ such that $x_i = \gamma(\bar{t}_i) \in e_i$ and $y_i = \eta(\bar{\tau}_i) \in e_i$ for $1 \le i \le k$.*

(a) If $a \ne x_1$, $a \ne y_1$ and the angle between $x_1 - a$ and $y_1 - a$ is not zero, then γ and η are not both shortest.

(b) If $b \ne x_k$, $b \ne y_k$ and the angle between $x_k - b$ and $y_k - b$ is not zero, then γ and η are not both shortest.

Proof. We prove for case (a), similar arguments apply to case (b). Suppose that γ and η are both $SOP(a,b)_{(e_1,\ldots,e_k)}$ and $l(\gamma) = l(\eta) = \sigma$. Let $x_0 = y_0 := a$, $x_{k+1} = y_{k+1} := b$ and $z_i := (x_i + y_i)/2$, $i = 0,1,\ldots,k+1$, so $z_0 = a$ and $z_{k+1} = b$. Let $\bar{\alpha}_0 := 0$, $\bar{\alpha}_1 := \|z_1 - a\|$, $\bar{\alpha}_2 := \bar{\alpha}_1 + \|z_2 - z_1\|$,..., $\bar{\alpha}_{k+1} := \bar{\alpha}_k + \|z_{k+1} - z_k\| = \bar{\alpha}_k + \|b - z_k\|$. Define $\varphi : [0, \bar{\alpha}_{k+1}] \to \mathbb{E}$ by $\varphi(\bar{\alpha}_i) = z_i$ for $i = 0,\ldots,k+1$, and φ is affine on each subinterval $[\bar{\alpha}_i, \bar{\alpha}_{i+1}]$. Then φ is continuous, $\varphi(\bar{\alpha}_i) \in e_i$ for $i = 1, \ldots, k$, i.e., φ is an $OP(a, b)_{(e_1,\ldots,e_k)}$. We have

$$l(\varphi) = \sum_{i=0}^{k} l\big(\varphi_{|[\bar{\alpha}_i,\bar{\alpha}_{i+1}]}\big) = \sum_{i=0}^{k} \|z_i - z_{i+1}\|. \tag{1}$$

Likewise, according to linearity and Theorem 3.1,

$$l(\gamma) = \sum_{i=0}^{k} \|x_i - x_{i+1}\| \quad \text{and} \quad l(\eta) = \sum_{i=0}^{k} \|y_i - y_{i+1}\|. \tag{2}$$

Since the angle between $x_1 - a$ and $y_1 - a$ is not zero,

$$2\|a - z_1\| < \|a - x_1\| + \|a - y_1\|. \tag{3}$$

Moreover, for $1 \le i \le k$,

$$2\|z_i - z_{i+1}\| = \|(x_i + y_i) - (x_{i+1} + y_{i+1})\| \le \|x_i - x_{i+1}\| + \|y_i - y_{i+1}\|. \tag{4}$$

Using (1)–(4) we get $2l(\varphi) = 2\sum_{i=0}^{k} \|z_i - z_{i+1}\| < \|a - x_1\| + \|a - y_1\| + 2\sum_{i=1}^{k} \|z_i - z_{i+1}\| \le \|a-x_1\|+\|a-y_1\|+\sum_{i=1}^{k} \|x_i-x_{i+1}\|+\sum_{i=1}^{k} \|y_i-y_{i+1}\| = \sum_{i=0}^{k} \|x_i-x_{i+1}\|+\sum_{i=0}^{k} \|y_i-y_{i+1}\| = l(\gamma) + l(\eta) = 2\sigma$, implying $l(\varphi) < \sigma$. This is impossible. Therefore γ and η are not both shortest. □

Suppose that $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $OP(a, b)_{(e_1,\ldots,e_k)}$. We say that γ *equals* η if at least one path can be obtained from the other by a strictly increasing change of parameter. Clearly, if γ equals η, then $l(\gamma) = l(\eta)$. If $t_0 < t_1$, $\tau_0 < \tau_1$, and $\gamma([t_0, t_1])$ is an $OP(a, b)_{(e_1,\ldots,e_k)}$, then there always exists an $OP(a, b)_{(e_1,\ldots,e_k)}$ defined on $[\tau_0, \tau_1]$ that is equal to γ, say $\eta = \gamma \circ \varphi$, where $\varphi : [\tau_0, \tau_1] \to [t_0, t_1]$ defined by $\varphi(\tau) = (\tau_1 - \tau_0)^{-1}(\tau_1 - \tau)t_0 + (\tau_1 - \tau_0)^{-1}(\tau - \tau_0)t_1$. This equality relation is in fact an equivalence relation on the family of $OP(a, b)_{(e_1,\ldots,e_k)}$. Whenever we identify ordered paths this way, shortest ordered path joining two points with respect to a given sequence of line segments is unique. This is stated in the following theorem which is also the main result in this section.

Theorem 3.2 *Let $a, b \in \mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments (these line segments are not necessarily distinct and some of them may be singleton). There exists a shortest ordered path joining a to* *b with respect to the sequence* $e_1, \dots, e_k$. *Moreover, this shortest ordered path is unique in the family of ordered paths satisfying assumption (A).*

Proof. *Existence* Denote $\mathcal{E} := e_1 \times \cdots \times e_k \subset \mathbb{E}^k$, $x_0 := a$, and $x_{k+1} := b$. Consider the function $\Phi : \mathbb{E}^k \to \mathbb{R}$, $\Phi(x_1, \dots, x_k) = \sum_{i=0}^{k} \|x_i - x_{i+1}\|$. For each $i = 0, 1, \dots, k$, the function $(x_1, \dots, x_k) \mapsto \|x_i - x_{i+1}\|$ is continuous on $\mathbb{E}^k$ and so Φ is continuous, too. As $\mathcal{E}$ is compact, there exists $(x_1^0, \dots, x_k^0) \in \mathcal{E}$ such that $\Phi(x_1^0, \dots, x_k^0) = \sigma := \min_{\mathcal{E}} \Phi$. Let $x_0^0 := a, \quad x_{k+1}^0 := b, \quad \bar{t}_0 := 0, \quad \bar{t}_i := \bar{t}_{i-1} + \|x_{i-1}^0 - x_i^0\|, \quad 1 \le i \le k+1$. Clearly, $\bar{t}_{k+1} = \sum_{i=1}^{k+1} \|x_{i-1}^0 - x_i^0\| = \sigma$. Consider the mapping $\gamma_0 : [0, \sigma] \to \mathbb{E}$ defined by $\gamma_0(\bar{t}_i) = x_i^0$, $i = 0, \dots, k+1$, and γ_0 is affine on each subinterval $[\bar{t}_i, \bar{t}_{i+1}]$. Then γ_0 is a path with $\gamma_0(\bar{t}_i) \in e_i$ for $i = 1, \dots, k$, so it is an $OP(a, b)_{(e_1, \dots, e_k)}$. Its length is $l(\gamma_0) = \sum_{i=0}^{k} l\big(\gamma_{0|[\bar{t}_i, \bar{t}_{i+1}]}\big) = \sum_{i=0}^{k} \|x_i^0 - x_{i+1}^0\| = \Phi(x_1^0, \dots, x_k^0) = \sigma$.

Suppose now that $\tilde{\gamma} : [t_0, t_1] \to \mathbb{E}$ is an $OP(a, b)_{(e_1, \dots, e_k)}$ and $\tilde{x}_i := \tilde{\gamma}(\tilde{t}_i) \in e_i$, $1 \le i \le k$, where $t_0 \le \tilde{t}_1 \le \cdots \le \tilde{t}_k \le t_1$. Set $\tilde{x}_0 := a, \tilde{x}_{k+1} := b$, $\tilde{t}_0 := t_0$, and $\tilde{t}_{k+1} := t_1$. Then, by Lemma 3.1, $l(\tilde{\gamma}) = \sum\limits_{i=0}^{k} l\big(\tilde{\gamma}_{|[\tilde{t}_i, \tilde{t}_{i+1}]}\big) \ge \sum\limits_{i=0}^{k} \|\tilde{x}_i - \tilde{x}_{i+1}\| = \Phi(\tilde{x}_1, \dots, \tilde{x}_k) \ge \Phi(x_1^0, \dots, x_k^0) = l(\gamma_0)$. Therefore γ_0 is an $SOP(a, b)_{(e_1, \dots, e_k)}$. Observe further that γ_0 is parametrized by arclength.

Uniqueness Suppose that $\eta([\tau_0, \tau_1])$ is an $SOP(a, b)_{(e_1, \dots, e_k)}$, $\tau_0 =: \bar{\tau}_0 \le \bar{\tau}_1 \le \cdots \le \bar{\tau}_k \le \bar{\tau}_{k+1} := \tau_1$ and $y_i = \eta(\bar{\tau}_i) \in e_i$, $1 \le i \le k$. Set $y_0 := \eta(\bar{\tau}_0) = a$, $y_{k+1} := \eta(\bar{\tau}_{k+1}) = b$.

We first consider the case when η is parametrized by arclength and $\tau_0 = 0$. Then $\tau_1 = \tau_1 - \tau_0 = l(\eta) = \sigma$, so $[\tau_0, \tau_1] = [0, \sigma]$.

Suppose now that the set $I = \{i : y_i \ne x_i^0\}$ is nonempty and $m := \min I$. As $y_0 = x_0^0 = a$ and $y_{k+1} = x_{k+1}^0 = b$, $1 \le m \le k$. Furthermore, by definition, $y_{m-1} = x_{m-1}^0$. If $x_m^0 \notin [x_{m-1}^0, y_m]$ and $y_m \notin [x_{m-1}^0, x_m^0]$, then Lemma 3.4 says that $\gamma_{0|[\bar{t}_{m-1}, t_1]}$ and $\eta_{|[\bar{\tau}_{m-1}, \tau_1]}$ are not both $SOP(x_{m-1}^0, b)_{(e_m, \dots, e_k)}$. This and Lemma 3.2 imply that γ_0 and η are not both $SOP(a, b)_{(e_1, \dots, e_k)}$, a contradiction. Therefore $x_m^0 \in [y_{m-1}, y_m[$ or $y_m \in [x_{m-1}^0, x_m^0[$ (since $y_{m-1} = x_{m-1}^0$).

If $y_m \in [x_{m-1}^0, x_m^0[$, by Theorem 3.1, there exists $\bar{t}'_m \in [\bar{t}_{m-1}, \bar{t}_m[$ for which $\gamma_0(\bar{t}'_m) = y_m$. Since γ_0 is parametrized by arclength, Theorem 3.1 says that γ_0 is still affine on each subintervals $[\bar{t}_{m-1}, \bar{t}'_m]$

and $[\bar{t}'_m, \bar{t}_{m+1}]$, and $\|x^0_{m-1} - y_m\| = l(\gamma_{0|[\bar{t}_{m-1},\bar{t}'_m]}) = \bar{t}'_m - \bar{t}_{m-1}$, $\|y_m - x^0_{m+1}\| = \bar{t}_{m+1} - \bar{t}'_m$. Thus we can relabel $\bar{t}_m := \bar{t}'_m$ and $x^0_m := y_m$. Likewise, if $x^0_m \in [y_{m-1}, y_m[$, we can relabel $y_m := x^0_m$. If $J = I \setminus \{m\}$ is nonempty, set $n = \min J$. It follows that $x^0_n \in [y_{n-1}, y_n[$ or $y_n \in [x^0_{n-1}, x^0_n[$ and we continue to reduce the number of elements of J. After finite steps we get $y_i = x^0_i$ for all i. Thus we can assume $y_i = x^0_i$ for all $i = 0, 1, \ldots, k+1$ and will prove that $\eta = \gamma_0$ on $[0, \sigma]$.

Observe that $\bar{\tau}_{i+1} - \bar{\tau}_i = l(\eta_{|[\bar{\tau}_i,\bar{\tau}_{i+1}]}) = \|x^0_i - x^0_{i+1}\| = \bar{t}_{i+1} - \bar{t}_i$ for $i = 0, \ldots, k$. Condition $\bar{\tau}_0 = \bar{t}_0 = 0$ implies that $\bar{\tau}_1 = \bar{t}_1$, $\bar{\tau}_2 = \bar{t}_2, \ldots, \bar{\tau}_{k+1} = \bar{t}_{k+1} = \sigma$. Suppose $\bar{\tau}_{j+1} > \bar{\tau}_j$. Since η is parametrized by arclength and $l(\eta_{|[\bar{\tau}_j,\bar{\tau}_{j+1}]}) = \|\eta(\bar{\tau}_j) - \eta(\bar{\tau}_{j+1})\|$, Lemma 3.1 says that η is affine on $[\bar{\tau}_j, \bar{\tau}_{j+1}]$. Hence $\eta(t) = \gamma_0(t)$ on $[\bar{\tau}_j, \bar{\tau}_{j+1}]$ and therefore $\eta(t) = \gamma_0(t)$ on $[0, \sigma]$.

For the general case when $[\tau_0, \tau_1]$ is arbitrary and η is not necessarily parametrized by arclength but satisfies assumption (A), we express $\eta = \zeta \circ \psi$, where $\psi : [\tau_0, \tau_1] \to [0, \sigma]$ is defined by $\psi(\tau) = l(\eta_{|[\tau_0,\tau]})$ and $\zeta : [0, \sigma] \to \mathbb{E}$ is parametrized by arclength (see [15]). Let $\bar{t}'_i = \psi(\bar{\tau}_i)$. Since ψ is strictly increasing, we have $0 = \psi(\tau_0) \leq \psi(\bar{\tau}_1) = \bar{t}'_1 \leq \cdots \leq \psi(\bar{\tau}_{k+1}) = \bar{t}'_{k+1} = \sigma$. Hence $\zeta(0) = a$, $\zeta(\sigma) = b$, $\zeta(\bar{t}'_i) = \zeta \circ \psi(\bar{\tau}_i) = \eta(\bar{\tau}_i) = y_i$ for $i = 1, \ldots, n$, i.e., ζ is an $OP(a, b)_{(e_1,\ldots,e_n)}$. As ζ is parametrized by arclength, $l(\zeta) = \sigma - 0 = \sigma$, so that ζ is an $SOP(a, b)_{(e_1,\ldots,e_n)}$. By the proof above, $\zeta = \gamma_0$. Therefore, $\eta = \gamma_0 \circ \psi$, i.e., the two paths η and γ_0 are equal. The proof is complete. □

We can apply the arguments in the proof of the first part of Theorem 3.2 to prove existence of solutions of problems with variable endpoints.

Corollary 3.1 *Let A, B be nonempty compact subsets of $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments.*

(a) Let $b \in \mathbb{E}$ be a fixed point. In the family of $OP(a, b)_{(e_1,\ldots,e_k)}$, where $a \in A$, there exists a shortest path.

(b) Let $a \in \mathbb{E}$ be a fixed point. In the family of $OP(a, b)_{(e_1,\ldots,e_k)}$, where $b \in B$, there exists a shortest path.

Proof. (a) The proof is similar to the first part of that of Theorem 3.2, where Φ is replaced with $\tilde{\Phi}$: $\mathbb{E}^{k+1} \to \mathbb{R}$, $\tilde{\Phi}(x_0, x_1, \ldots, x_k) = \left(\sum_{i=0}^{k-1} \|x_i - x_{i+1}\|\right) + \|x_k - b\|$, on the compact set $A \times e_1 \times \cdots \times e_k$. The proof for part (b) is similar. □

Corollary 3.2 *Let A, B be nonempty compact subsets of $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments. In the family of $OP(a, b)_{(e_1,\ldots,e_k)}$, where $a \in A, b \in B$, there exists a shortest path.*

Corollary 3.3 *Let A be a nonempty compact subset of $\mathbb{E}$ and let $e_1, \ldots, e_k$ be a sequence of line segments. In the family of $OP(a, a)_{(e_1,\ldots,e_k)}$, where $a \in A$, there exists a shortest path.*

Observe that shortest ordered paths in Corollaries 3.1–3.3 may not be unique. Moreover, by Theorem 3.1, all shortest paths in Corollaries 3.1–3.3 are always polylines.

Applying Corollary 3.3 to the problem of finding an inscribed polygon in a given convex polygon $\mathcal{P} \subset \mathbb{R}^2$ with a minimum perimeter, we find that this problem has a solution. Some properties of angles of this inscribed polygon will be derived in the next section.

Corollary 3.4 *If $\gamma([t_0, t_1])$ and $\eta([\tau_0, \tau_1])$ are $SOP(a, b)_{(e_1,\ldots,e_k)}$ and parametrized by arclength, then $\eta(\tau) = \gamma(\tau - \tau_0 + t_0)$ and $\gamma(t) = \eta(t - t_0 + \tau_0)$. In addition, if $\tau_0 = t_0$ (hence $\tau_1 = t_1$), then $\eta(t) = \gamma(t)$ for all $t \in [t_0, t_1]$.*

Proof. By Theorem 3.2, there is a strictly increasing and surjective function $\psi : [\tau_0, \tau_1] \to [t_0, t_1]$ such that $\eta = \gamma \circ \psi$. We have $\psi(\tau_0) = t_0$ and $\psi(\tau_1) = t_1$. As γ and η are parametrized by arclength, $\tau - \tau_0 = l\big(\eta_{|[\tau_0,\tau]}\big) = l\big(\gamma_{|[\psi(\tau_0),\psi(\tau)]}\big) = \psi(\tau) - \psi(\tau_0) = \psi(\tau) - t_0$ for every $\tau \in [\tau_0, \tau_1]$. It follows that $\psi(\tau) = \tau - \tau_0 + t_0$, giving $\eta(\tau) = \gamma\big(\psi(\tau)\big) = \gamma(\tau - \tau_0 + t_0)$. Conversely, for each $t \in [t_0, t_1]$, $\tau := t - t_0 + \tau_0 \in [\tau_0, \tau_1]$ and so $\eta(t - t_0 + \tau_0) = \eta(\tau) = \gamma(\tau - \tau_0 + t_0) = \gamma(t)$. □

Corollary 3.5 *Let $\mathcal{D}$ be a polytope in $\mathbb{R}^3$ whose faces are convex. Let $f_1, \ldots, f_k$ be a sequence of faces of $\mathcal{D}$ such that $f_i \cap f_{i+1}$ is an edge for $i = 1, \ldots, k-1$, $a \in f_1$, and $b \in f_k$. In the family of ordered paths satisfying assumption (A), there exists uniquely a shortest ordered path lying on the surface of $\mathcal{D}$, joining a to b, and going orderly through $f_1, \ldots, f_k$.*

As in Remark 3.1, this shortest path is an $SOP(a,b)_{(e_1,\ldots,e_k)}$, where $e_i = f_i \cap f_{i+1}$, $i = 1,\ldots,k-1$.

Applying Theorem 3.2 to polytopes in $\mathbb{R}^3$ and simple polygons on the plane $\mathbb{R}^2$ we get the following well-known results that are shown in [5], [6].

Corollary 3.6 *(a) Let a and b be any two points on a surface $\mathcal{S}$ in $\mathbb{R}^3$ consisting of finite adjacent convex polygons. In the family of paths joining a to b and lying totally on $\mathcal{S}$, there exists a shortest path. This shortest path is a polyline.*

(b) Let a and b be any two points in a simple polygon $\mathcal{P} \subset \mathbb{R}^2$. There exists a shortest path joining a to b that lies totally in $\mathcal{P}$ and furthermore, it is a polyline.

Proof. (a) The case both a and b lie on the same polygon is trivial, so we assume that they belong different polygons. Each path joining a to b and lying on $\mathcal{S}$ is an ordered path with respect to some sequence $\mathcal{E}$ consisting of common edges of adjacent polygons in $\mathcal{S}$. Let $\gamma_{\mathcal{E}}$ be an $SOP(a,b)_{(\mathcal{E})}$. Since the family of such sequences $\mathcal{E}$ is finite, the shortest path γ in the family $\{\gamma_{\mathcal{E}}\}$ is the required path. By Theorem 3.1, each $\gamma_{\mathcal{E}}$ is a polyline, so is γ.

(b) Partition $\mathcal{P}$ into nonoverlap convex polygons and apply part (a) to obtain the required result.

□

4 Conditions for Concatenation of Two Shortest Ordered Paths to be a Shortest Ordered Path

We have known in Theorem 3.1 that every shortest ordered path is a polyline. In this section we consider some geometric characteristics of shortest ordered paths and represent conditions under which concatenation of two shortest ordered paths is a shortest ordered path. If $j < i$, then $SOP(x,y)_{(e_i,\ldots,e_j)}$ is understood to be $SOP(x,y)_{\emptyset}$.

Lemma 4.1 *Suppose γ is an $OP(a,b)_{(e_1,\ldots,e_k)}$ and $b \in e_n \cap e_{n+1} \cap \cdots \cap e_k$. γ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$ iff it is an $SOP(a,b)_{(e_1,\ldots,e_{n-1})}$. Similarly, if $a \in e_1 \cap e_2 \cap \cdots \cap e_m$, then γ is an $SOP(a,b)_{(e_1,\ldots,e_k)}$ iff it is an $SOP(a,b)_{(e_{m+1},\ldots,e_k)}$.*

This is derived from the fact that under the condition $b \in e_n \cap e_{n+1} \cap \cdots \cap e_k$, a path is an $OP(a,b)_{(e_1,\dots,e_{n-1})}$ iff it is an $OP(a,b)_{(e_1,\dots,e_k)}$. The other case is similar.

The following generalizes a part of Lemma 4.1.

Lemma 4.2 *Let $\gamma\big([t_0,t_1]\big)$ be an $SOP(a,b)_{(e_1,\dots,e_k)}$, $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_k \le t_1$, $x_i := \gamma(\bar{t}_i) \in e_i$ for $i = 1,\dots,k$.*

(a) If e_* is a line segment such that $e_* \cap [x_{j-1},x_j] \neq \emptyset$, $2 \le j \le k$, and $e_* \neq e_{j-1}, e_* \neq e_j$, then $\gamma\big([t_0,t_1]\big)$ is an $SOP(a,b)_{(e_1,\dots,e_{j-1},e_*,e_j,\dots,e_k)}$. If $e_* \cap [a,x_1] \neq \emptyset$, and $e_* \neq e_1$, then $\gamma\big([t_0,t_1]\big)$ is an $SOP(a,b)_{(e_*,e_1,\dots,e_k)}$. The case $e_* \cap [x_k,b] \neq \emptyset$ is similar.

(b) If $e_{*1},\dots,e_{*m}$ are line segments that contain the same point belonging to $[x_{j-1},x_j]$ for some $j \in \{2,\dots,k\}$, then γ is also an $SOP(a,b)_{(e_1,\dots,e_{j-1},e_{*1},\dots,e_{*m},e_j,\dots,e_k)}$. The cases $e_{*1},\dots,e_{*m}$ contain the same point of $[a,x_1]$ or $[x_k,b]$ are similar.

Proof. (a) We prove for the case $2 \le j \le k$. Other cases are proved similarly. Suppose $\eta\big([\tau_0,\tau_1]\big)$ is an $OP(a,b)_{(e_1,\dots,e_{j-1},e_*,e_j,\dots,e_k)}$, $\tau_0 \le \cdots \le \bar{\tau}_{j-1} \le \bar{\tau}_* \le \bar{\tau}_j \le \cdots \le \tau_1$ such that $y_i := \eta(\bar{\tau}_i) \in e_i$ for $1 \le i \le k$ and $y_* = \eta(\bar{\tau}_*) \in e_*$. Clearly, η is also an $OP(a,b)_{(e_1,\dots,e_k)}$, so $l(\gamma) \le l(\eta)$. Now take an $x_* \in e_* \cap [x_{j-1},x_j]$. Since $\gamma([\bar{t}_{j-1},\bar{t}_j]) = [x_{j-1},x_j]$, there is $\bar{t}_* \in [\bar{t}_{j-1},\bar{t}_j]$ with $\gamma(\bar{t}_*) = x_*$. Thus γ is also an $OP(a,b)_{(e_1,\dots,e_{j-1},e_*,e_j,\dots,e_n)}$ and therefore γ is an $SOP(a,b)_{(e_1,\dots,e_{j-1},e_*,e_j,\dots,e_n)}$. (b) follows directly from (a). □

We now turn to our main problem of this section, the one of concatenation of two shortest paths. We first consider the simplest case: the concatenation of a path and a line segment.

If $\gamma\big([t_0,t_1]\big)$ is any path which joins $a = \gamma(t_0)$ to $b = \gamma(t_1)$, and c is any point in $\mathbb{E}$, for abbreviation, we denote by $\gamma * [b,c]$ the concatenation of γ and any one-to-one shortest path $\xi : [t_1,t_2] \to \mathbb{E}$ joining b and c. (If $b = c$, $\gamma * [b,c] = \gamma$.) The notation $[c,a] * \gamma$ is defined similarly. Clearly $l\big(\gamma * [b,c]\big) = l(\gamma) + \|c-b\|$ and $l\big([c,a] * \gamma\big) = l(\gamma) + \|c-a\|$. Likewise, if $\gamma_1\big([t_0,t_1]\big)$, $\gamma_2\big([t_1',t_2]\big)$ are paths which join a to b and c to d, respectively, and $t_1 < t_1'$, $b \neq c$, the notation $\gamma_1 * [b,c] * \gamma_2$ denotes the concatenation

$\gamma_1 * \xi * \gamma_2$, where ξ is defined on $[t_1, t_1']$, one-to-one, and is a shortest path joining b to c. In these notations, ξ is usually chosen to be affine.

The following lemma says that if we elongate the first or last line segment of a shortest ordered path, we get a new shortest ordered path.

Lemma 4.3 *Suppose* $\gamma\big([t_0, t_1]\big)$ *is an* $SOP(a, b)_{(e_1,\dots,e_n)}$ *and* $t_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_n \leq t_1$ *such that* $x_i := \gamma(\bar{t}_i) \in e_i$ *for* $1 \leq i \leq n$.

(a) If $x_n \neq b$ *and* $q \in \mathbb{E}$ *such that* $b \in]x_n, q[$*, then* $\gamma * [b, q]$ *is an* $SOP(a, q)_{(e_1,\dots,e_n)}$.

(b) If $x_1 \neq a$ *and* $p \in \mathbb{E}$ *such that* $a \in]p, x_1[$*, then* $[p, a] * \gamma$ *is an* $SOP(p, b)_{(e_1,\dots,e_n)}$.

(c) Set $x_0 = a$*. If* $x_m \neq x_{m+1} = \cdots = x_n = b$*,* $(0 \leq m \leq n-1)$*, and* $b \in]x_m, q[$ *then* $\gamma * [b, q]$ *is an* $SOP(a, q)_{(e_1,\dots,e_n)}$.

Proof. (a) Let $\eta([\tau_0, \tau_1])$ be any $OP(a, q)_{(e_1,\dots,e_n)}$, $\tau_0 \leq \bar{\tau}_1 \leq \cdots \leq \bar{\tau}_n \leq \tau_1$, and $y_i := \eta(\bar{\tau}_i) \in e_i$ for $i = 1, \dots, n$. Suppose $b = (1-\lambda)x_n + \lambda q$ for some $\lambda \in]0, 1[$. Set $z_i := (1-\lambda)x_i + \lambda y_i,\ 1 \leq i \leq n, \quad z_0 := (1-\lambda)x_1 + \lambda a, \quad z_{n+1} := (1-\lambda)x_n + \lambda q = b$. Then for $i = 1, \dots, n-1$,

$$\|z_i - z_{i+1}\| = \|(1-\lambda)(x_i - x_{i+1}) + \lambda(y_i - y_{i+1})\| \leq (1-\lambda)\|x_i - x_{i+1}\| + \lambda\|y_i - y_{i+1}\|. \quad (5)$$

Let ξ be the path going through $z_0, z_1, \dots, z_{n+1}$ such that ξ is an affine path joining z_i and z_{i+1}, $0 \leq i \leq n$. ξ is an $OP(z_0, b)_{(e_1,\dots,e_n)}$. By Theorem 3.1, there exits a $t^* \in [t_0, \bar{t}_1]$ satisfying $\gamma(t^*) = z_0$. Lemma 3.2 states that $\gamma_{|[t^*, t_1]}$ is an $SOP(z_0, b)_{(e_1,\dots,e_n)}$. According to Theorem 3.1 and (5) we have

$$\begin{aligned} \|z_0 - x_1\| + \sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \|x_n - b\| &= l\big(\gamma_{|[t^*, t_1]}\big) \leq l(\xi) = \sum_{i=0}^{n} \|z_i - z_{i+1}\| \\ &\leq \|z_0 - z_1\| + \left[(1-\lambda)\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda \sum_{i=1}^{n-1} \|y_i - y_{i+1}\|\right] + \|z_n - b\|. \end{aligned} \quad (6)$$

Noting that $\|z_0 - x_1\| = \lambda\|a - x_1\|, \quad \|x_n - b\| = \lambda\|x_n - q\|, \|z_0 - z_1\| = \lambda\|a - y_1\|, \quad \|z_n - b\| = \lambda\|y_n - q\|$, we deduce from (6) that $\lambda\|a - x_1\| + \lambda\sum_{i=1}^{n-1} \|x_i - x_{i+1}\| + \lambda\|x_n - q\| \leq \lambda\|a - y_1\| + \lambda\sum_{i=1}^{n-1} \|y_i - y_{i+1}\| + \lambda\|y_n - q\|$, Since $\|x_n - q\| = \|x_n - b\| + \|b - q\|$, the above inequality yields

$\lambda\big(l(\gamma) + \|b - q\|\big) \leq \lambda l(\eta)$, which implies $l(\gamma * [b, q]) = l(\gamma) + \|b - q\| \leq l(\eta)$. Thus, $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\dots,e_n)}$.

Part (b) is proved similarly. To prove (c) we observe that, by Lemma 4.1, γ is an $SOP(a, b)_{(e_1,\dots,e_m)}$. Applying part (a) we find that $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\dots,e_m)}$. That $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\dots,e_n)}$ follows from Lemma 4.2. □

We now study some characteristics of a shortest ordered path basing on properties of angles between the path and line segments e_is. If u and v are nonzero vectors in $\mathbb{E}$, we denote by $\angle(u, v)$ the angle between u and v, which does not excess π.

Theorem 4.1 *Let $e_1, \dots, e_k$ be a sequence of line segments. Let $\gamma([t_0, t_1])$ be an $SOP(a, b)_{(e_1,\dots,e_{n-1})}$ and $t_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_{n-1} \leq t_1$ such that $x_i = \gamma(\bar{t}_i) \in e_i$ and $n \leq k$. Set $x_0 = a$. Suppose $x_{n-1} \neq b$, $b \in e_n \cap \cdots \cap e_k$, and each e_j is nonsingleton for $j = n, \dots, k$. Let $q \in \mathbb{E}$, $q \neq b$. Then $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\dots,e_k)}$ iff for any $y_j \in e_j$ and $y_j \neq b$, $j = n, \dots, k$, we have $\theta \geq \pi$, where $\theta := \angle(x_{n-1} - b, y_n - b) + \sum_{j=n}^{k-1} \angle(y_j - b, y_{j+1} - b) + \angle(y_k - b, q - b)$.*

Proof. Suppose that $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\dots,e_k)}$ and that $y_j \in e_j$, $y_j \neq b$, $j = n, \dots, k$. We show that $\theta \geq \pi$. By Lemma 3.2, $\xi := \gamma_{|[\bar{t}_{n-1}, t_1]} * [b, q]$ is an $SOP(x_{n-1}, q)_{(e_n,\dots,e_k)}$. Setting $y_{k+1} := q$, $v_j := y_j - b$, $j = n, \dots, k + 1$, we have $\theta = \angle(x_{n-1} - b, v_n) + \sum_{j=n}^{k} \angle(v_j, v_{j+1})$. Let L_j be the line containing b and y_j, $n \leq j \leq k + 1$. Let P be the plane containing L_n and x_{n-1} (if $x_{n-1} \in L_n$, P is any plane containing L_n).

Set $y'_n := y_n$. If $L_n \neq L_{n+1}$, denote by $y'_{n+1} \in P$ the point such that $\triangle x_{n-1} b y'_n$ and $\triangle y'_n b y'_{n+1}$ do not overlap (i.e., x_{n-1} and y'_{n+1} are on opposite sides of L_n) and $\triangle y'_n b y'_{n+1} = \triangle y_n b y_{n+1}$; if $x_{n-1} \in L_n$, y'_{n+1} lies on any side of L_n. If $L_n = L_{n+1}$, i.e., $y_{n+1} - b = \lambda(y_n - b)$ for some λ, then we choose y'_{n+1} satisfying $y'_{n+1} - b = \lambda(y'_n - b)$. Similarly, $y'_{n+2} \in P$ is the point such that $\triangle y'_{n+1} b y'_{n+2} = \triangle y_{n+1} b y_{n+2}$ and the triangles $\triangle y'_n b y'_{n+1}$ and $\triangle y'_{n+1} b y'_{n+2}$ do not overlap, and so on. We then obtain a sequence $y'_n, \dots, y'_{k+1}$ in P such that $\|y'_j - y'_{j+1}\| = \|y_j - y_{j+1}\|$, $\|y'_j - b\| = \|y_j - b\|$, and $\angle(v'_j, v'_{j+1}) =$

$\angle(v_j, v_{j+1})$, where $v'_j = y'_j - b$ for $j \geq n$. Then $\theta = \angle(x_{n-1} - b, v'_n) + \sum_{j=n}^{k} \angle(v'_j, v'_{j+1})$. If $\theta < \pi$, there exist $z_{n-1} \in]x_{n-1}, b[$, $z'_n \in]y'_n, b[$,..., $z'_{k+1} \in]y'_{k+1}, b[$ that are near b and such that they are collinear. Let $z_j \in]y_j, b[$ satisfy $\|z_j - b\| = \|z'_j - b\|$, $j \geq n$. We have $\|z_{n-1} - z_n\| + \cdots + \|z_k - z_{k+1}\| = \|z_{n-1} - z'_n\| + \cdots + \|z'_k - z'_{k+1}\| = \|z_{n-1} - z'_{k+1}\| < \|z_{n-1} - b\| + \|b - z'_{k+1}\| = \|z_{n-1} - b\| + \|b - z_{k+1}\|$ and hence $[x_{n-1}, z_{n-1}] * [z_{n-1}, z_n] * \cdots * [z_k, z_{k+1}] * [z_{k+1}, q]$ is an $OP(x_{n-1}, q)_{(e_n, \ldots, e_k)}$ whose length is less than that of ξ. This is impossible. Therefore $\theta \geq \pi$.

Next suppose conversely that for any $y_j \in e_j$ and $y_j \neq b$, $j = n, \ldots, k$, we have $\theta \geq \pi$. We first observe that $\gamma * [b, q]$ is an $OP(a, q)_{(e_1, \ldots, e_k)}$. Let $\eta([\tau_0, \tau_1])$ be any $SOP(a, q)_{(e_1, \ldots, e_k)}$, $\tau_0 \leq \bar{\tau}_1 \leq \cdots \leq \bar{\tau}_k \leq \bar{\tau}_{k+1} := \tau_1$, and $y_i := \eta(\bar{\tau}_i) \in e_i$ for $i = 1, \ldots, k$, $y_{k+1} := \eta(\bar{\tau}_{k+1}) = q$. If $b \in [y_j, y_{j+1}]$ for some $j \geq n$, then there is $\tau_b \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$ with $\eta(\tau_b) = b$ and so $l(\eta) = l\big(\eta_{|[\tau_0, \tau_b]}\big) + l\big(\eta_{|[\tau_b, \tau_1]}\big) \geq l(\gamma) + \|b - q\| = l(\gamma * [b, q])$, because $\eta_{|[\tau_0, \tau_b]}$ is also an $OP(a, b)_{(e_1, \ldots, e_{n-1})}$ and $l\big(\eta_{|[\tau_b, \tau_1]}\big) \geq \|b - q\|$.

Consider the case $b \notin [y_j, y_{j+1}]$ (and so $y_j \neq b$) for all $j = n, \ldots, k$. Let $y'_n, \ldots, y'_{k+1}$ be points defined as in the first part of the proof. Set $\theta_{n-1} := \angle(x_{n-1} - b, y'_n - b)$ and for $n \leq j \leq k$, $\theta_j := \angle(x_{n-1} - b, y'_n - b) + \angle(y'_n - b, y'_{n+1} - b) + \cdots + \angle(y'_j - b, y'_{j+1} - b)$. We have $\theta_k = \theta \geq \pi$. Let $r = \min\{j \geq n : \theta_j \geq \pi\}$. If $\theta_r = \pi$, set $y^{*\prime} = y'_{r+1}$ and $y^* = y_{r+1}$, $\tau^* = \bar{\tau}_{r+1}$. Suppose $\theta_r > \pi$. Observe that $\angle(y'_r - b, y'_{r+1} - b) = \angle(y_r - b, y_{r+1} - b) < \pi$ since if this angle is π, $b \in]y_r, y_{r+1}[$, a contradiction. As $\theta_{r-1} < \pi$ and $\theta_r = \theta_{r-1} + \angle(y'_r - b, y'_{r+1} - b) > \pi$, there exists $y^{*\prime} \in]y'_r, y'_{r+1}[$ such that $\theta_{r-1} + \angle(y'_r - b, y^{*\prime} - b) = \pi$. Let $y^* \in]y_r, y_{r+1}[$ satisfy $\|y^* - y_r\| = \|y^{*\prime} - y'_r\|$ and $\tau^* \in]\bar{\tau}_r, \bar{\tau}_{r+1}[$ satisfy $\eta(\tau^*) = y^*$. Since $b \in]x_{n-1}, y^{*\prime}[$, by Lemma 4.3(c), $\gamma_1 * [b, y^{*\prime}]$ is an $SOP(a, y^{*\prime})_{(e_1, \ldots, e_n)}$. Thus

$$\begin{aligned} l\big(\eta_{|[\tau_0, \tau^*]}\big) &= l\big(\eta_{|[\tau_0, \bar{\tau}_n]}\big) + \|y_n - y_{n+1}\| + \cdots + \|y_{r-1} - y_r\| + \|y_r - y^*\| \\ &= l\big(\eta_{|[\tau_0, \bar{\tau}_n]}\big) + \|y'_n - y'_{n+1}\| + \cdots + \|y'_{r-1} - y'_r\| + \|y'_r - y^{*\prime}\| \\ &\geq l\big(\gamma_1 * [b, y^{*\prime}]\big) = l(\gamma_1) + \|b - y^{*\prime}\| = l(\gamma_1) + \|b - y^*\|. \end{aligned}$$

We deduce that $l(\eta) \geq l\big(\eta_{|[\tau_0, \tau^*]}\big) + \|y^* - q\| \geq l(\gamma) + \|b - y^*\| + \|y^* - q\| \geq l(\gamma) + \|b - q\| = l(\gamma * [b, q])$. In all cases, $l(\eta) \geq l(\gamma * [b, q])$, showing that $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1, \ldots, e_k)}$. □

Theorem 4.1 gives a criterion to recognize an ordered path to be shortest: we just measure the angles between line segments of the path and e_is at points of intersection. To illustrate the theorem let us consider a special case. Suppose that $e_i = [b, b_i]$, $i = 1, \ldots, r$, are non-singleton line segments with the same endpoint b. Let $a, q \in \mathbb{E} \setminus \{b\}$. If the sum $\angle(a-b, b_1-b) + \sum_{i=1}^{r-1} \angle(b_i - b, b_{i+1} - b) + \angle(b_r - b, q - b)$ is not less than π, then $[a, b] * [b, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_r)}$. For instance, let $a, q \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Then $\gamma = [a, \mathbf{0}] * [\mathbf{0}, q]$ is the shortest path that joins a and q and meets the x-, y-, and z-axes. Indeed, suppose η is any path joining a and q and meets x-, y-, and z-axes at $b_1 = (\lambda, 0, 0)$, $b_2 = (0, \mu, 0)$, and $b_3 = (0, 0, \nu)$. If one of the points b_1, b_2, b_3 is coincident with $b := \mathbf{0} = (0, 0, 0)$, then clearly, $l(\gamma) \le l(\eta)$. Assume $b_i \neq b$ for all $i = 1, 2, 3$ and η is an $OP(a, q)_{([b,b_2],[b,b_1],[b,b_3])}$. Since $\angle(a-b, b_2-b)+\angle(b_2-b, b_1-b)+\angle(b_1-b, b_3-b)+\angle(b_3-b, q-b) \ge \pi$, Theorem 4.1 says that, in the family $OP(a, q)_{([b,b_2],[b,b_1],[b,b_3])}$, $l(\gamma) \le l(\eta)$.

We now consider several consequences of Theorem 4.1.

Corollary 4.1 *Suppose $\gamma\big([t_0, t_1]\big)$ is an $SOP(a, b)_{(e_1,\ldots,e_n)}$ and $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_n \le t_1$ such that $x_i := \gamma(\bar{t}_i) \in e_i$ for $1 \le i \le n$, $x_n \neq b$. Let $y \in e_n$, $y \neq x_n$, and let q be any point in $\mathbb{E}$ such that $q \neq x_n$ and $\angle(y - x_n, q - x_n) = \angle(y - x_n, b - x_n)$. Then $\gamma_{|[t_0,\bar{t}_n]} * [x_n, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_n)}$. In particular, if we rotate the last line segment $[x_n, b]$ about the line containing e_n, we then get a new shortest ordered path with respect to $e_1, \ldots, e_n$.*

Proof. Set $x_0 := a$. If $x_0 = x_1 = \cdots = x_n$, then there is nothing to prove. So we assume that $m = \max\{i : x_i \neq b\}$ exists. By Theorem 4.1, for any $y_i \in e_i$, $y_i \neq x_n$, $m+1 \le i \le n$, $\angle(x_m - x_n, y_{m+1} - x_n) + \angle(y_{m+1} - x_n, y_{m+2} - x_n) + \cdots + \angle(y_n - x_n, b - x_n) \ge \pi$. Hence $\angle(x_m - x_n, y_{m+1} - x_n) + \angle(y_{m+1} - x_n, y_{m+2} - x_n) + \cdots + \angle(y_n - x_n, q - x_n) \ge \pi$ since $\angle(y_n - x_n, q - x_n) = \angle(y_n - x_n, b - x_n)$. Applying Theorem 4.1 once more, we find that $\gamma_{|[t_0,\bar{t}_n]} * [x_n, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_n)}$. □

Corollary 4.2 *Suppose $\gamma([t_0, t_1])$ is an $SOP(a, b)_{(e_1,\ldots,e_n)}$ and $t_0 \le \bar{t}_1 \le \cdots \le \bar{t}_n \le t_1$ such that $x_i := \gamma(\bar{t}_i) \in e_i$ for $1 \le i \le n$, $x_0 := a$, $x_{n+1} := b$. Suppose also that $x_{j-1} \neq x_j$ and $x_{j+1} \neq x_j$.*

(a) If $y_j \in e_j$, $y_j \neq x_j$, then $\theta := \angle(x_{j-1} - x_j, y_j - x_j) + \angle(y_j - x_j, x_{j+1} - x_j) \geq \pi$. In particular, if x_j is an interior point of e_j, then $\theta = \pi$.

(b) Let L be the line containing e_j. If $x_{j-1} \in L$ and $x_{j+1} \notin L$, then x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ and $\theta > \pi$. The case $x_{j+1} \in L$ and $x_{j-1} \notin L$ is similar.

(c) If $x_{j-1}, x_{j+1} \in L$, then either $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$ or $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, where $y_j \in e_j$, $y_j \neq x_j$. In the latter case x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.

Proof. (a) Applying Theorem 4.1 to $\gamma_{|[t_0, \bar{t}_{j+1}]}$ and sequence $e_1, \ldots, e_j$, we get $\theta \geq \pi$. If x_j is an interior point of e_j, take $y'_j \in e_j$ such that x_j is an interior point of $[y_j, y'_j]$. We then also have $\theta' := \angle(x_{j-1} - x_j, y'_j - x_j) + \angle(y'_j - x_j, x_{j+1} - x_j) \geq \pi$. Since $\theta + \theta' = 2\pi$, $\theta = \theta' = \pi$.

(b) Suppose $x_{j-1} \in L$ and $x_{j+1} \notin L$. Since $\theta \neq \pi$, $\theta > \pi$ and according to part (a), x_j must be an endpoint. If $\|x_{j-1} - x_j\| > \min_{x \in e_j} \|x_{j-1} - x\|$, there exists $\bar{x} \in e_j$ with $\|x_{j-1} - \bar{x}\| < \|x_{j-1} - x_j\|$. Since $\bar{x} \in [x_{j-1}, x_j[$ and $x_{j+1} \notin L$, $\|x_{j-1} - x_j\| + \|x_j - x_{j+1}\| = \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_j\| + \|x_j - x_{j+1}\| > \|x_{j-1} - \bar{x}\| + \|\bar{x} - x_{j+1}\|$, i.e, $l\big(\gamma_{|[\bar{t}_{j-1}, \bar{t}_{j+1}]}\big) > l\big([x_{j-1}, \bar{x}] * [\bar{x}, x_{j+1}]\big)$, a contradiction. Thus $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$.

(c) If $x_j \in [x_{j-1}, x_{j+1}]$, then $\angle(x_{j-1} - x_j, x_{j+1} - x_j) = \pi$. If otherwise, say $x_{j-1} \in [x_j, x_{j+1}]$, then an analysis which is similar to that in part (b) shows that x_j is an end point of e_j with $\|x_{j-1} - x_j\| = \min_{x \in e_j} \|x_{j-1} - x\|$ and $\angle(x_{j-1} - x_j, y_j - x_j) = \angle(x_{j+1} - x_j, y_j - x_j) = \pi$, for $y_j \in e_j, y_j \neq x_j$. $\square$

The following result is a converse of Corollary 4.2 and it gives sufficient conditions for an ordered path to be shortest.

Corollary 4.3 *Let $\gamma([t_0, t_1])$ be an $SOP(a, b)_{(e_1, \ldots, e_{n-1})}$, $b \in e_n$, and $t_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_{n-1} \leq t_1$ such that $x_i := \gamma(\bar{t}_i) \in e_i$. Let $q \in \mathbb{E}$, $q \neq b$. Set $x_0 = a$ and suppose that $m = \max\{i : x_i \neq b\}$ exits and that there is $y \in e_n, y \neq b$.*

(a) If $\theta := \angle(x_m - b, y - b) + \angle(y - b, q - b) = \pi$, then $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1, \ldots, e_n)}$.

(b) If b is an end point of e_n and $\theta > \pi$, then $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_n)}$.

Part (a) is a generalization of Lemma 4.3.

Proof. (a) For any $y' \in e_n$, $y' \neq b$, we always have $\angle(x_m - b, y' - b) + \angle(y' - b, q - b) = \pi$. Thus by Theorem 4.1, $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_m,e_n)}$. If $m < n - 1$, it follows from Lemma 4.2(b) that $\gamma * [b, q]$ is an $SOP(a, q)_{(e_1,\ldots,e_n)}$. (b) is proved similarly. □

To illustrate Corollaries 4.2 and 4.3 let us consider a triangle in $\mathbb{R}^2$ having acute angles (Fig. 2). For $u \in e_1$ fixed, Corollary 4.3 states that the triangle uvw has minimum perimeter because angles θ at v and w are π. Fig. 2 shows that $\angle(w - u, z_1 - u) + \angle(z_1 - u, v - u) \neq \pi$. Thus uvw is not the inscribed triangle with minimum perimeter since the angles at u do not satisfy Corollary 4.2(a).

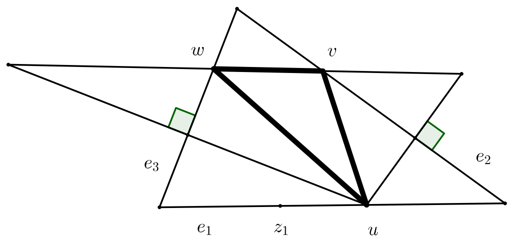

Fig. 2: $SOP(u, u)_{(e_1,e_2,e_3)} = [u, v] * [v, w] * [w, u]$

We are now in a position to consider the general case of concatenation of two shortest ordered paths. Roughly speaking, the following theorem says that if the last line segment of a shortest ordered path and the first of the other overlap, the two paths can be joined to become a shortest ordered path.

Theorem 4.2 *Let $e_1, \ldots, e_k$ be a sequence of line segments. Suppose that $\gamma_1([t_0, t_1])$ is an $SOP(a, b)_{(e_1,\ldots,e_{n-1})}$ and $\gamma_2([t^*, t_2])$ is an $SOP(c, d)_{(e_n,\ldots,e_k)}$, where $t^* < t_1 < t_2$, and $\gamma_1(t_1) = \gamma_2(t_1) = b$. Suppose also that γ_1, γ_2 satisfy assumption (A). If $t_1 \leq \bar{t}_n \leq \cdots \leq \bar{t}_k \leq t_2$, $x_i := \gamma_2(\bar{t}_i) \in e_i$ for $n \leq i \leq k$, and if there exists $\epsilon > 0$ such that $\gamma_1([t_1 - \epsilon, t_1]) \subset \gamma_2([t^*, t_1])$ then the concatenation γ of γ_1 and $\gamma_{2|[t_1,t_2]}$ is an $SOP(a, d)_{(e_1,\ldots,e_k)}$.*

Note that Lemma 4.3 is a particular case of this theorem.

Proof. Let $t_0 =: \bar{t}_0 \leq \bar{t}_1 \leq \cdots \leq \bar{t}_{n-1} \leq t_1$, $x_i := \gamma_1(\bar{t}_i) \in e_i$ for $i = 1, \ldots, n-1$, and $x_0 := \gamma_1(\bar{t}_0) = a$. We can also assume, by applying Lemma 4.3 if necessary, that $b = x_n \in e_n$, i.e., $\bar{t}_n = t_1$. Set $\bar{t}_{k+1} := t_2$ and note that γ is an $OP(a,d)_{(e_1,\ldots,e_n)}$. The proof is divided into three cases.

Case 1: $k = n$ and $e_n \neq \{b\}$. Since γ_2 is an $SOP(c,d)_{(e_n)}$, $\gamma_2([t^*, t_1]) = [c, b]$, $\gamma_2([t_1, t_2]) = [b, d]$, and $b \neq c$, $b \neq d$ (by assumption (A)). Set $m = \max\{i < n : x_i \neq b\}$. By virtue of Lemma 3.2, we can assume $t^* = t_1 - \epsilon$ and $c \in [x_m, b[$. Applying Theorem 4.1 to γ_2 we find that for every $y \in e_n$, $y \neq b$, $\angle(c-b, y-b) + \angle(y-b, d-b) \geq \pi$. Thus this theorem also says that γ is an $SOP(a,d)_{(e_1,\ldots,e_m,e_n)}$ and hence it is an $SOP(a,d)_{(e_1,\ldots,e_n)}$ by Lemma 4.2.

Case 2: $k \geq n+1$ and all e_j, $j = n, \ldots, k$, are not singleton. Set $x_{k+1} := d$, $m = \max\{i < n : x_i \neq b\}$, $r = \min\{j > n : x_j \neq b\}$, and assume that $c = \gamma_1(t_1 - \epsilon) \in [x_m, b[$. Then, by Lemma 3.2, $\gamma_{2|[t^*,\bar{t}_r]} = [c,b] * [b, x_r]$ is an $SOP(c, x_r)_{(e_n,\ldots,e_{r-1})}$. By Theorem 4.1, for $\gamma_{2|[t^*,\bar{t}_r]}$, all angles θ at b are not less than π and hence, $\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]}$ is an $SOP(a, x_r)_{(e_1,\ldots,e_m,e_n,\ldots,e_{r-1})}$. If $m < n-1$, Lemma 4.2 again shows that $\gamma_1 * \gamma_{2|[t_1,\bar{t}_r]}$ is an $SOP(a, x_r)_{(e_1,\ldots,e_{r-1})}$. We continue in this fashion to obtain, after a finite number of times, that γ is an $SOP(a,d)_{(e_1,\ldots,e_k)}$.

Case 3: e_j is singleton for some $j \geq n$. The particular case when e_n is singleton, i.e., $e_n = \{b\}$, is derived immediately from the following lemma.

Lemma 4.4 *Let $\zeta([\alpha_0,\alpha_1])$ be an $OP(p,u)_{(e_1,\ldots,e_l)}$ and $\xi([\alpha_1,\alpha_2])$ an $OP(u,q)_{(e_{l+1},\ldots,e_m)}$. Then $\psi = \zeta * \xi$ is an $SOP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$ iff ζ is an $SOP(p,u)_{(e_1,\ldots,e_l)}$ and ξ an $SOP(u,q)_{(e_{l+1},\ldots,e_m)}$.*

Indeed, if ψ is a shortest ordered path, then so are ζ and ξ (Lemma 3.2). Conversely, suppose ζ and ξ are shortest ordered paths. Let $\eta([\tau_0,\tau_1])$ be any $OP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$, $\tau_0 \leq \bar{\tau}_1 \leq \cdots \leq \bar{\tau}_l \leq \bar{\tau}_u \leq \bar{\tau}_{l+1} \leq \cdots \leq \bar{\tau}_m \leq \tau_1$, and $y_i := \eta(\bar{\tau}_i) \in e_i$ for $i = 1, \ldots, m$, $\eta(\bar{\tau}_u) = u$. Since $\eta_{|[\tau_0,\bar{\tau}_u]}$ is an $OP(p,u)_{(e_1,\ldots,e_l)}$ and $\eta_{|[\bar{\tau}_u,\tau_1]}$ is an $OP(u,q)_{(e_{l+1},\ldots,e_m)}$, $l(\eta) = l(\eta_{|[\tau_0,\bar{\tau}_u]}) + l(\eta_{|[\bar{\tau}_u,\tau_1]}) \geq l(\zeta) + l(\xi) = l(\psi)$. Therefore ψ is an $SOP(p,q)_{(e_1,\ldots,e_l,\{u\},e_{l+1},\ldots,e_m)}$.

We now prove the theorem for the case $s = \min\{j \geq n : e_j \text{ is singleton}\} > n$, say $e_s = \{u\}$. If $t_1 = \bar{t}_{n+1} = \cdots = \bar{t}_s$, then $b = u \in e_{n+1} \cap \cdots \cap e_s$ and by Lemma 4.1, γ_1 is also an $SOP(a,b)_{(e_1,\ldots,e_{s-1})}$ and $\gamma_{2|[t_1,t_2]}$ an $SOP(b,d)_{(e_{s+1},\ldots,e_k)}$. Lemma 4.4 states that γ is an $SOP(a,d)_{(e_1,\ldots,e_k)}$. If $\bar{t}_s > t_1$, by Cases 1 and 2 we find that $\xi = \gamma_1 * \gamma_{2|[t_1,\bar{t}_s]}$ is an $SOP(a,u)_{(e_1,\ldots,e_{s-1})}$, we then apply Lemma 4.4 again to ξ and $\gamma_{2|[\bar{t}_s,t_2]}$ to obtain the required result. The proof of the theorem is complete. □

Corollary 4.4 *Suppose $\gamma_1([t_0,t_1])$ is an $SOP(a,b)_{(e_1,\ldots,e_n)}$ and $\gamma_2([t_1,t_2])$ is an $SOP(b,c)_{(e_{n+1},\ldots,e_k)}$. Suppose also that γ_1, γ_2 satisfy assumption (A). Then $\gamma = \gamma_1 * \gamma_2$ is an $SOP(a,c)_{(e_1,\ldots,e_k)}$ iff there exists $\epsilon > 0$ such that $\gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]}$ is a shortest ordered path.*

Proof. If $\gamma = \gamma_1 * \gamma_2$ is an $SOP(a,c)_{(e_1,\ldots,e_k)}$ then $\zeta := \gamma_{1|[t_1-\epsilon,t_1]} * \gamma_{2|[t_1,t_1+\epsilon]} = \gamma_{|[t_1-\epsilon,t_1+\epsilon]}$ is a shortest ordered path for each $\epsilon \leq \min\{t_1 - t_0, t_2 - t_1\}$ (Lemma 3.2). Conversely suppose that ζ is a shortest ordered path and satisfies assumption (A). Since $\gamma_{1|[t_1-\epsilon,t_1]} = \zeta_{|[t_1-\epsilon,t_1]}$, applying Theorem 4.2 to γ_1 and ζ we find that $\xi = \gamma_1 * \zeta_{|[t_1,t_1+\epsilon]}$ is a shortest ordered path. Applying Theorem 4.2 again to ξ and γ_2 we obtain that $\gamma = \xi * \gamma_{2|[t_1+\epsilon,t_2]}$ is an $SOP(a,c)_{(e_1,\ldots,e_k)}$. □

Loosely speaking, the above result says that if the last segment of the first shortest ordered path and the first segment of the second form a shortest ordered path, then their concatenation is also a shortest ordered path.

5 Straightest Paths and a Discrete Initial Value Problem

Let $S = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of (not necessary distinct) adjacent convex polygons in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e., $e_i := f_i \cap f_{i+1}$ is an edge for $1 \leq i \leq k$. Polthier and Schimies [16] presented a new concept of geodesics: straightest geodesics are paths that have equal path angle on both sides at each point. In this section we consider "straightest paths" which are lightly differ from the original and in fact are particular shortest ordered paths.

Definition 5.1 A *straightest path* $\gamma : [t_0, t_1] \to \cup_{i=1}^{k+1} f_i$ on the sequence S is a path that satisfies assumption (A) and the following conditions:

(a) there exist integers $1 \leq m \leq n \leq k$ and a sequence of numbers $t_0 =: \bar{t}_{m-1} < \bar{t}_m \leq \cdots \leq \bar{t}_n < \bar{t}_{n+1} := t_1$ such that $x_i := \gamma(\bar{t}_i) \in e_i$ for $m \leq i \leq n$ and $x_{m-1} := \gamma(t_0) \in f_m$, $x_{n+1} := \gamma(t_1) \in f_{n+1}$;

(b) $l\big(\gamma_{|[\bar{t}_i, \bar{t}_{i+1}]}\big) = \|x_i - x_{i+1}\|$ for $m - 1 \leq i \leq n$;

(c) For $z_i \in e_i$, $z_i \neq x_i$, we have $\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi$ if $x_i \in e_i$ is not a common vertex, and $\angle(x_{i^*-1} - x_i, z_{i^*} - x_i) + \sum_{j=i^*}^{r-1} \angle(z_j - x_i, z_{j+1} - x_i) + \angle(z_r - x_i, x_{r+1} - x_i) = \pi$ if $x_{i^*-1} \neq x_{i^*} = x_{i^*+1} = \cdots = x_r \neq x_{r+1}$ and $i^* \leq i \leq r$.

It is conventional to define straightest path joining a to b on the same polygon (with respect to an empty sequence of common edges) to be any path that joins a to b, one-to-one, and has length $\|a - b\|$, i.e., an $SOP(a, b)_\emptyset$.

Condition (a) states that a straightest path is an ordered path. Condition (b) and Lemma 3.1 imply that $\gamma([\bar{t}_i, \bar{t}_{i+1}]) = [x_i, x_{i+1}] \subset f_{i+1}$ for $m - 1 \leq i \leq n$. We observe also that condition (c) does not depend on the choice of $z_i \in e_i$, $m \leq i \leq n$. Corollary 4.3 and Theorem 4.1 show that γ is an $SOP(x_{m-1}, x_{n+1})_{(e_m, \ldots, e_n)}$. Note also that assumption (A), the conditions $t_0 < \bar{t}_m$, $\bar{t}_n < t_1$, and (b) imply that $x_0 \neq x_1$ and $x_n \neq x_{n+1}$. These conditions do not restrict the definition of straightest paths since if x_0 belongs to a common edge then we choose $m = \max\{j : x_0 \in f_j\}$. Similarly, we can assume that x_{n+1} does not belong to e_n.

There are reasons why we study straightest paths: First the shortest path joining two given points on the surface of a polytope D whose angle at every vertex is strictly less than 2π does not go through any vertex of D (see [6]). Thus, by Corollary 4.2, the path is straightest. Second, every shortest ordered path joining two points on a surface S consisting of convex polygons is composed of straightest paths joining vertexes of S (Proposition 5.1).

Let $\gamma([t_0, t_1])$ be a straightest path on $S = (f_1, f_2, \ldots, f_{k+1})$, $\gamma(t_0) \in f_m$, and v a nonzero vector that is parallel to f_m. If there exists $t^* > t_0$ such that $\gamma([t_0, t^*]) \subset f_m$ and $\gamma(t^*) - \gamma(t_0) = \lambda v$ for some $\lambda > 0$ we say that γ *starts* at $\gamma(t_0)$ in the direction of v.

As usual, we first consider the problem of existence of straightest paths defined by a given starting point and a direction.

Theorem 5.1 *Let $S = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of adjacent convex polygons. Let $a, p \in f_m$, $a \neq p$, and $v = p - a$. Then there exists uniquely a longest straightest path $\gamma([t_0, t_1])$ on S starting at a and in the direction of v. Moreover, if $\eta([\tau_0, \tau_1])$ is any straightest path on S starting at a and in the direction of v, then η equals $\gamma_{|[t_0,t^*]}$ for some $t_0 \leq t^* \leq t_1$. Thus every straightest path on S can be extended to a longest straightest path.*

Proof. There is nothing to prove if $k = 0$, $S = \{f_1\}$. Thus we assume, without loss of the generality, that $k \geq 1$, $a, p \in f_1$, and $a \notin f_2$.

First we construct γ. Denote $e_i = f_i \cap f_{i+1}$, $i = 1, \ldots, k$. Set $t_0 = 0$ and $\bar{t}_1 = \max\{t : a + tv \in f_1\}$. Since $p = a + v \in f_1$, $\bar{t}_1 \geq 1$. Define $\gamma_1 : [0, \bar{t}_1] \to f_1$ by $\gamma_1(0) = a$, $\gamma_1(\bar{t}_1) = x_1 := a + \bar{t}_1 v \in f_1$, and γ_1 is affine on $[0, \bar{t}_1]$. If $x_1 \notin e_1$ we put $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$.

Suppose $x_1 \in e_1$. Choose any $z_1 \in e_1$, $z_1 \neq x_1$. If there exists $p_1 \in f_2$ such that

$$\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, p_1 - x_1) = \pi, \tag{7}$$

set $v_1 := p_1 - x_1$, $\sigma_1 := \max\{t : x_1 + tv_1 \in f_2\} \geq 1$, $\bar{t}_2 := \bar{t}_1 + \sigma_1$, and $x_2 := x_1 + \sigma_1 v_1 \in f_2$. We then define $\gamma_2 : [\bar{t}_1, \bar{t}_2] \to f_2$ by $\gamma_2(\bar{t}_1) = x_1$, $\gamma_2(\bar{t}_2) = x_2$, and γ_2 is affine on $[\bar{t}_1, \bar{t}_2]$. Clearly $\gamma_1 * \gamma_2$ is a straightest path starting at a, in the direction of v, and joining a to x_2. Observe that such a p_1 always exists if x_1 is an interior point of e_1.

Suppose x_1 is an endpoint of e_1 and there is no $p_1 \in f_2$ satisfying (7). If x_1 is not a common vertex of e_1 and e_2, set $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$. Assume that x_1 is a common vertex of e_1 and e_2. Let $r := \max\{i : x_1 \in e_j,\ 1 \leq j \leq i\}$. Choose $z_j \in e_j$, $z_j \neq x_1$, $1 \leq j \leq r$, and let z^*_{r+1} be a point on the second edge of f_{r+1} that has an endpoint x_1 and $z^*_{r+1} \neq x_1$. Let $\theta_1 := \angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots + \angle(z_{r-1} - x_1, z_r - x_1) + \angle(z_r - x_1, z^*_{r+1} - x_1)$. We call θ_1 the *angle of incidence at* x_1. If $\theta_1 < \pi$, set $t_1 := \bar{t}_1$ and $\gamma := \gamma_1$. Assume $\theta_1 \geq \pi$. Put $s := \max\{i : \angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots + \angle(z_{i-1} - x_1, z_i - x_1) < \pi\}$. Since there is no $p_1 \in f_2$ satisfying (7), $2 \leq s \leq r$ and there exists $p_s \in f_{s+1} \setminus f_s$, such that $\angle(a - x_1, z_1 - x_1) + \angle(z_1 - x_1, z_2 - x_1) + \cdots + \angle(z_s - x_1, p_s - x_1) = \pi$. Set $\bar{t}_2 = \cdots = \bar{t}_s := \bar{t}_1$, $x_2 = \cdots = x_s := x_1$, $v_s := p_s - x_1$, and $\sigma_s := \max\{t : x_s + tv_s \in f_{s+1}\}, \quad \bar{t}_{s+1} := \bar{t}_s + \sigma_s, \quad x_{s+1} := x_s + \sigma_s v_s$. As $\sigma_s \geq 1$, $\bar{t}_{s+1} > \bar{t}_s$. Define $\gamma_i : [\bar{t}_{i-1}, \bar{t}_i] \to f_i$, $\gamma_i(t) = x_1$, $2 \leq i \leq s$, $\gamma_{s+1} : [\bar{t}_s, \bar{t}_{s+1}] \to f_{s+1}$, $\gamma_{s+1}(\bar{t}_s) = x_s$, $\gamma_{s+1}(\bar{t}_{s+1}) = x_{s+1}$, and γ_{s+1} is affine. From the construction we find that $\gamma_1 * \gamma_2 * \cdots * \gamma_{s+1}$ is a straightest path joining a to x_{s+1} and starts at a in the direction of v.

We continue in this fashion and finally obtain a sequence of points $x_1, \ldots, x_{n+1}$ satisfying the following conditions: i) $x_n \in e_n$, $x_{n+1} \in f_{n+1} \setminus f_n$, and $1 = \max\{t > 0 : x_n + t(x_{n+1} - x_n) \in f_{n+1}\}$; ii) either $x_{n+1} \notin e_{n+1}$ (for instance, when $n = k$) or iii) $x_{n+1} \in e_{n+1}$ and the angle of incidence at x_{n+1} is less than π. We then set $t_1 := \bar{t}_{n+1} = \bar{t}_n + \sigma_n$ and define $\gamma_{n+1} : [\bar{t}_n, t_1] \to f_{n+1}$ by $\gamma_{n+1}(\bar{t}_n) = x_n$, $\gamma_{n+1}(t_1) = x_{n+1} = x_n + \sigma_n v_n$, and γ_{n+1} is affine. Let $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_{n+1}$. We find that γ satisfies assumption (A) and is an $OP(a, x_{n+1})_{(e_1,\ldots,e_n)}$. Moreover, γ also satisfies conditions (b) and (c) of Definition 5.1. Thus γ is a straightest path starting at a and in the direction of v.

Suppose now that $\eta([\tau_0, \tau_1])$ is a straightest path on S starting at a and in the direction of v. Let $c := \eta(\tau_1)$. The step-by-step construction of γ shows that there exists $t_c \in]t_0, t_1]$ such that $c = \gamma(t_c)$ and η and $\gamma_{|[t_0,t_c]}$ are shortest paths joining a to c with respect to the same sequence of line segments. Thus by the uniqueness of shortest ordered path (Theorem 3.2), η equals $\gamma_{|[t_1,t_c]}$. This implies also that η can be extended to γ and γ is the unique longest straightest path. □

Remark 5.1 Notice that shortest ordered path joining two given points on a sequence of adjacent convex polygons always exists but this may not be true for straightest path.

In the problem of computing shortest paths on a polyhedral surface, a key geometric concept is the notion of *planar unfolding* around a sequence of adjacent convex polygons $S = (f_1, f_2, \dots, f_{k+1})$ (see [5]). We unfold the sequence S as follow: Rotate f_1 around e_1 until its plane coincides with that of f_2, rotate f_1 and f_2 around e_2 until their plane coincides with that of f_3, continue this way until all faces $f_1, f_2, \dots, f_k$ lie on the plane of f_{k+1}. The idea of planar unfolding was used to prove Theorem 4.1. We now investigate the image of a straightest path under a planar unfolding.

Suppose that $\gamma([t_0, t_1])$ is a straightest path joining $a \in f_1$ and $b \in f_{k+1}$, $t_0 < \bar{t}_1 \leq \dots \leq \bar{t}_k < t_1$, $x_i := \gamma(\bar{t}_i) \in e_i$ for $1 \leq i \leq k$. For each i, choose $z_i \in e_i$, $z_i \neq x_i$. Let a_1 be the image of a under the planar unfolding around e_1. Since $a \neq x_1$, $a_1 \neq x_1$. If $x_1 \neq x_2$, we have $\angle(a_1 - x_1, z_1 - x_1) = \angle(a - x_1, z_1 - x_1)$ so that $\angle(a_1 - x_1, z_1 - x_1) + \angle(z_1 - x_1, x_2 - x_1) = \pi$. Hence $x_1 \in]a_1, x_2[$, i.e., a_1, x_1, x_2 are collinear. Similarly, let a_2 and $x_{1,2}$ be the images of a_1 and x_1 under the planar unfolding around e_2, respectively. If $x_2 \neq x_3$, then $\angle(x_{1,2} - x_2, z_2 - x_2) = \angle(x_1 - x_2, z_2 - x_2)$ whence $\angle(x_{1,2} - x_2, z_2 - x_2) + \angle(z_2 - x_2, x_3 - x_2) = \pi$, i.e., $a_2, x_{1,2}, x_2, x_3$ are collinear. If $x_1 = x_2 = \dots = x_r \neq x_{r+1}$, then by condition (c), a_r, x_1, x_{r+1} are collinear, where a_r is the image of a under the planar unfolding around e_1, e_2, $\dots$, e_r. Repeating this argument we finally find that the image of γ under the planar unfolding around $e_1, \dots, e_k$ is a line segment. Conversely, if the image of γ under the planar unfolding around $e_1, \dots, e_k$ is a line segment, then the angles of γ at edges are π, so γ is a straightest path. We thus arrive at the following result.

Lemma 5.1 *Let γ be an $OP(a, b)_{(e_1, \dots, e_k)}$ ($a \in f_1$, $b \in f_{k+1}$) on the sequence S in $\mathbb{R}^3$. Then γ is straightest iff its planar unfolding around $e_1, \dots, e_k$ is a line segment.*

Thus roughly speaking, straightest paths are ordered paths whose images under planar unfoldings are line segments.

A straightest path is a shortest ordered path. Hence if there exists a straightest path joining two given points $a \in f_m$ and $b \in f_{n+1}$ on the sequence S, $(m \leq n)$, then it is the shortest path joining these points that lies entirely in the polygons and passes through $e_m, \ldots, e_n$.

Conversely, we have the following result, which was presented by O'Rourke et al. (see [17]).

Proposition 5.1 *Every shortest ordered path joining two vertexes on $S = (f_1, f_2, \ldots, f_{k+1})$ is composed of straightest paths joining vertexes of S.*

Proof. Let $\gamma([t_0, t_1])$ be a shortest ordered path joining two vertexes $a \in f_1$ and $b \in f_{k+1}$ on S, $t_0 < \bar{t}_1 \leq \cdots \leq \bar{t}_k < t_1$, $x_i = \gamma(\bar{t}_i) \in e_i$, $1 \leq i \leq k$. If γ does not pass through any vertexes of S except a and b, then each x_i is an interior point of e_i. By Corollary 4.2, for $z_i \in e_i, z_i \neq x_i$, $i = 1, \ldots, k$, $\angle(x_{i-1} - x_i, z_i - x_i) + \angle(z_i - x_i, x_{i+1} - x_i) = \pi$, where $x_0 := a$ and $x_{k+1} := b$. Thus γ is straightest.

If γ passes through vertexes $v_0 := a, v_1 = \gamma(t_1^*), \ldots, v_l = \gamma(t_l^*), v_{l+1} := b$ (in that order) and there are no vertexes belonging $\gamma(\,]t_0, t_1^*[\,)$, $\gamma(\,]t_1^*, t_2^*[\,)$, ..., $\gamma(\,]t_l^*, t_1[\,)$, then by Lemma 3.2, restrictions of γ on $[t_0, t_1^*], \ldots, [t_l^*, t_1]$ are shortest ordered paths joining vertexes v_j and v_{j+1} of S. By the proof above, each restriction is a straightest path. Thus γ is composed of straightest paths joining vertexes of S. □

Theorem 5.2 *Let $S = (f_1, f_2, \ldots, f_{k+1})$ be a sequence of adjacent convex polygons and let a, q_1, q_2, q_3 be points in f_m such that $q_2 \in]q_1, q_3[$ and a, q_1, q_3 are not collinear. Let $v_i = q_i - a$, $i = 1, 2, 3$. Assume that γ_1, γ_2 and γ_3 are straightest paths starting at a and in the directions of v_1, v_2, v_3, respectively, and γ_1, γ_3 cut a line segment $e \subset f_n$ $(n \geq m)$ at y_1 and y_3, respectively. If γ_2 is the longest straightest path, then it meets e at some point $y_2 \in]y_1, y_3[$.*

Proof. We prove for the case S is in $\mathbb{R}^3$, the other case is similar. Without restriction of generality we assume that $m = 1$ and $n = k + 1$. Let $\mathcal{U}$ be the planar unfolding around the sequence $e_1, \ldots, e_k$. By Lemma 5.1, the images of $\gamma_1, \gamma_2, \gamma_3$ under $\mathcal{U}$ are line segments $[a', y_1^*]$, $[a', y_2^*]$, $[a', y_3^*]$, respectively and we have $y_1 \in [a', y_1^*]$ and $y_3 \in [a', y_3^*]$. Let q_1', q_2', q_3' be the images of q_1, q_2, q_3 under $\mathcal{U}$. Since q_2 is between q_1 and q_3, q_2' is between q_1' and q_3'. Assume $q_2' = \alpha q_1' + \beta q_3'$ where $\alpha, \beta > 0$, $\alpha + \beta = 1$.

Letting $v'_i := q'_i - a'$, $i = 1, 2, 3$, we get $v'_2 = \alpha v'_1 + \beta v'_3$. Since q_i lies on the path γ_i and $q_i \neq a$, $q'_2 \in]a', y^*_2]$, $q'_1 \in]a', y_1]$, and $q'_3 \in]a', y_3]$. Assume $y_1 - a' = \lambda v'_1$ and $y_3 - a' = \nu v'_3$, $\lambda, \nu > 0$. Set $\mu := \lambda\nu/(\alpha\nu + \beta\lambda) > 0$. We have $y_2 := a' + \mu v'_2 = a' + \mu\alpha v'_1 + \mu\beta v'_3 = \frac{\alpha\nu}{\alpha\nu+\beta\lambda} y_1 + \frac{\beta\lambda}{\alpha\nu+\beta\lambda} y_3 \in]y_1, y_3[$. Let S' be the sequence of images of $f_1, \ldots, f_{k+1}$ under the planar unfolding $\mathcal{U}$. S' is the sequence of convex polygons with adjacent edges being images of $e_1, \ldots, e_k$ under $\mathcal{U}$ and the triangle $a'y_1y_3$ is contained in the union of polygons in S'. As the image of γ_2 is $[a', y^*_2]$, the longest line segment starting at a' in the direction of v'_2, we have $[a', y_2] \subset [a', y^*_2]$. This means that the line segment $[a', y^*_2]$ meets $[y_1, y_3]$ at $y_2 \in]y_1, y_3[$, i.e., the longest straightest path γ_2 meets e at y_2. □

6 Conclusions

Recently, An et al. [14] presented a related work which is concerned to straightest paths. They proved that straightest paths solve the boundary value problem on an processed domain which is extended from a "funnel" and an alternative triangle face of that funnel.

We hope that some results in this paper could be used to study Steiner's problem (see [18]): Given a convex polygon in the plane, find an inscribed polygon of minimal perimeter.

References

1. Mitchell, J. S.: Geometric shortest paths and network optimizations, in The Handbook of Computational Geometry. Elsevier Science (Chapter 15), 633-701 (2000)
2. Agarwal, P. K., Har-Peled, S. and Karia, M.: Computing approximate shortest paths on convex polytopes. Algorithmica, 227-242 (2002)
3. Sethian, J. A.: Fast marching methods. SIAM Review 41 (2), 199-235 (1999)
4. Chen, J., Han, Y.: Shortest paths on a polyhedron; Part I: Computing shortest paths. International Journal of Computational Geometry and Applications 6 (2), 127-144 (1996)
5. Mitchell, J. S. B., Mount, D. M. and Papadimitriou, C. H.: The discrete geodesic problem. SIAM Journal on Computing 16 (4), 633-701 (1987)

6. Sharir, M. and Schorr, A.: On shortest paths in polyhedral spaces. Siam J. Comput. 15 (1), (1986)
7. Pham-Trong, V., Szafran, N. and Biard, L.: Pseudo-geodesics on three-dimensional surfaces and pseudo-geodesic meshes. Numerical Algorithms 26, 305-315 (2001)
8. Xin, S.-Q. and Wang, G.-J.: Efficiently determining a locally shortest path on polyhedral surfaces. Computer-Aided Design 39, 1081-1090 (2007)
9. Hai, N. N. and An, P. T.: Blaschke-type theorem and separation of disjoint closed geodesic convex sets. Journal of Optimization Theory and Applications 15 (3), 541-551 (2011)
10. Hoai, T. V., An, P. T. and Hai, N. N.: Multiple Shooting Approach for Computing Approximately Shortest Paths on Convex Polytopes. Journal of Computational and Applied Mathematics, accepted, November (2016)
11. Akman, V. : Unobstructed Shortest Paths in Polyhedral Enviroments, Springer (1987)
12. Liu, L. and Wong, R. C.-W.: Finding shortest path on land surface. Proceedings of the 2011 ACM SIGMOD International Conference on Management of data (2011)
13. Sautter, L.: Approximate shortest gentle paths on terrains. Diplom thesis, Karlsruhle Institute of Technology (2014)
14. An, P. T., Giang, D. T., Phu, H. X. and Polthier, K.: Straightest geodesics on a triangle sequence in 3D. Third ERC "SDModels" Workshop - Discrete Models in Geometry and Topology, Freie Universitaet Berlin, Germany, March 23-26 (2015)
15. Papadopoulos, A.: Metric Spaces, Convexity and Nonpositive Curvature, 2ed., European Mathematical Society (2014)
16. Polthier, K. and Schmies, M.: Straightest geodesics on polyhedral surfaces. Mathematical Visualization, 391, Springer (1998)
17. O'Rourke, J. , Suri, S. and Booth, H.: Shortest paths on polyhedral surfaces. Manuscript, The Johns Hopkins Univ., Baltimore, MD (1984)
18. J. Steiner.: Gesammelte Werke. Band 2, Berlin, p.45 (1882)