Omega-limit sets and bounded solutions

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April 8, 2016

Abstract. We prove among other things that the omega-limit set of a bounded solution of a Hamilton system

$$\begin{cases} \dot{\mathbf{p}} = \dfrac{\partial H}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} = -\dfrac{\partial H}{\partial \mathbf{p}} \end{cases}$$

is containing a full-time solution so there are the limits of $\frac{1}{t}\int_0^t \mathbf{p}(s)ds$ and $\frac{1}{t}\int_0^t \mathbf{q}(s)ds$ as $t \to \infty$ for any bounded solution $(\mathbf{p}, \mathbf{q})$ of the Hamilton system. These limits are stationary points of the Hamilton system so if a Hamilton system has no stationary point then every solution of this system is unbounded.

2000 AMS Subject Classification: 34-02 (47D06)

Key Words: Beurling spectrum of a bounded full-time solution, precompact orbit, unique ergodic

1. INTRODUCTION

In this paper $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ denotes a complex Banach space. Let $A : \mathbb{X} \to \mathbb{X}$ be a bounded linear operator with compact spectrum $\sigma(A)$ and positive spectral radius $r(A)$. In [1] we proved that if $\sigma(A) \cap i\mathbb{R} = \{i\xi_1, i\xi_2, \cdots, i\xi_n\}$ then every bounded full-time solution of differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ has the form $\mathbf{u}(t) = \sum_{k=1}^{n} e^{i\xi_k t}\mathbf{v}_k$, where $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are fixed vectors of $\mathbb{X}$. Recall that full-time solution is the solution satisfying the differential equation for all $t \in \mathbb{R}$. For example, periodic solutions are full-time and bounded solutions. Moreover, if $\sigma(A) \cap i\mathbb{R} = \emptyset$ then 0 is the only bounded full-time solution. We used Beurling spectrum [1] and Fourier coefficients of a bounded function (on the real line) in the proof. More exactly, we proved that the Beurling spectrum of any bounded full-time solution is a subset of $\{\xi_1, \xi_2, \cdots, \xi_n\}$. For the delay equation $\dot{\mathbf{u}}(t) = -\mathbf{u}(t-\tau)$we proved that every almost periodic solution is periodic, so if there exists an almost periodic solution then the delay τ must be $\pi/2$. Generally, the spectrum of any bounded full-time solution of the delay equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t-\tau)$ is a compact subset of the interval $[-r(A), r(A)]$. Now consider a bounded solution $\mathbf{x}$ of

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) \text{ for } t > 0 \\ \quad \mathbf{x}(0) \text{ given in } \mathbb{X}. \end{cases}$$

Assume that the orbit $\{\mathbf{x}(t) : t \geq 0\}$ is relatively compact. Then the omega-limit set ω of $\mathbf{x}$ is a compact connected subset of $\mathbb{X}$ [4]. Moreover, ω is invariant under the group $T(t) = e^{At}$. Let $\mathbf{v}$ be a point in this omega-limit set and $\mathbf{u}(t) = T(t)\mathbf{v}$. Then $\mathbf{u}$ is a bounded full-time solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. On the other hand, $\Omega = \omega \cup \{\mathbf{x}(t) : t \geq 0\}$ is a compact subset of $\mathbb{X}$. Therefore, the semi-group $\{T(t)\}_{t\geq 0}$ acts injectively on Ω. By an ergodic theorem [6] we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{x}(s)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{u}(s)ds$. This limit is lying in the kernel of A. Specially, if $\sigma(A) \cap i\mathbb{R} = \emptyset$ then 0 is the only bounded full time solution. Thus, every bounded solution tends to 0 as $t \to \infty$. Now let $(\mathbf{p}, \mathbf{q})$ be a bounded solution of the Hamilton system

$$\begin{cases} \dot{\mathbf{p}} = \dfrac{\partial H}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} = -\dfrac{\partial H}{\partial \mathbf{p}}. \end{cases}$$

Then there is an injective continuous semi-flow $T(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $(\mathbf{p}(t), \mathbf{q}(t)) = T(t)(\mathbf{p}(0), \mathbf{q}(0))$ Then the omega-limit set ω of $(\mathbf{p}, \mathbf{q})$ is a compact connected subset of $\mathbb{R}^{2n}$ [4]. Moreover, ω is invariant under the group $T(t)$. The dynamical system $\left\langle \omega, \{T(t)\}_{t\in\mathbb{R}} \right\rangle$ is uniquely ergodic, since the only invariant (continuous) function on $\left\langle \omega, \{T(t)\}_{t\in\mathbb{R}} \right\rangle$ is the constant function. Let $\mathbf{v}$ be a point in this omega-limit set and $\mathbf{u}(t) = T(t)\mathbf{v}$. Then $\mathbf{u}$ is a bounded full-time solution of the differential equation

$$\begin{cases} \dot{\mathbf{p}} = \dfrac{\partial H}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} = -\dfrac{\partial H}{\partial \mathbf{p}}. \end{cases}$$

By an ergodic theorem [6] there are the limits of $\frac{1}{t}\int_0^t \mathbf{p}(s)ds$ and $\frac{1}{t}\int_0^t \mathbf{q}(s)ds$ as $t \to \infty$ for any bounded solution $(\mathbf{p}, \mathbf{q})$ of the Hamilton system. These limits are stationary points of the Hamilton system. Therefore, we have

Theorem A. If the gradient ∇H of a smooth hamiltonian H is nowhere 0 then every solution of the Hamilton system

$$\begin{cases} \dot{\mathbf{p}} = \dfrac{\partial H}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} = -\dfrac{\partial H}{\partial \mathbf{p}} \end{cases}$$

is unbounded.

For example, consider the system $\ddot{x} = -\sin x$ with $x(0) = 0$. If $\dot{x}(0) > 2$ then $x(t)$ is unbounded. If $\dot{x}(0) = 2$ then

$$x(t) = 2\text{arc sin}\,\frac{e^{2t} - 1}{e^{2t} + 1}$$

which is increasingly tending to π as $t \to \infty$. If $\dot{x}(0) \in (0, 2)$ then $x(t)$ is periodic and bounded by π in the time and both $\frac{1}{t}\int\limits_0^t x(s)ds$ and $\frac{1}{t}\int\limits_0^t \dot{x}(s)ds$ tend to 0 as $t \to \infty$. Moreover, the period of this solution is

$$2\int_0^A \frac{dx}{\sqrt{2\cos x - 2 + \dot{x}(0)^2}},$$

where $A = \arccos\left(1 - \frac{\dot{x}(0)^2}{2}\right)$ is the maximal value of $x(t)$.

2. MAIN RESULTS

Let $T(t) : \mathbb{X} \to \mathbb{X}$ for $t \geq 0$ denote a semi-group with (unbounded and close) generator A. Let $\mathbf{x}(t) = T(t)\mathbf{x}(0)$ denote a bounded solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Assume that the orbit $\{\mathbf{x}(t) : t \geq 0\}$ is relatively compact. Then the omega-limit set ω of $\mathbf{x}$ is a compact connected subset of $\mathbb{X}$ [4]. Moreover, ω is invariant under the semi-group $\{T(t)\}_{t\geq 0}$. Clearly, $T(t) : \omega \to \omega$ is bijective. It is easy to prove that the dynamical system $\left\langle \omega, \{T(t)\}_{t\in\mathbb{R}} \right\rangle$ is uniquely ergodic [6]. In fact, the only invariant (continuous) function on $\left\langle \omega, \{T(t)\}_{t\in\mathbb{R}} \right\rangle$ is the constant function. Hence, there is a unique Borel probability measure μ on ω [6] such that

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \varphi(\mathbf{u}(s))\, ds = \int_{\omega} \varphi(\mathbf{v})\, d\mu(\mathbf{v}).$$

Here, φ denotes a continuous function on ω and $\mathbf{u}(s) = T(s)\mathbf{v}$ for some $\mathbf{v} \in \omega$. Therefore, there is the limit of $\frac{1}{t}\int_0^t \mathbf{u}(s)ds$ as $t \to \infty$. Similarly, the limit of $\frac{1}{t}\int_0^t \mathbf{x}(s)ds$ exists as $t \to \infty$.

Theorem B. Let A denote the generator of a linear semigroup $T(t) : \mathbb{X} \to \mathbb{X}$ for $t \geq 0$. Let $\mathbf{x}(t) = T(t)\mathbf{x}(0)$ denote a bounded solution of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Assume that the orbit $\{\mathbf{x}(t) : t \geq 0\}$ is pre-compact. Then the limit of $\frac{1}{t}\int_0^t \mathbf{x}(s)ds$ exists as $t \to \infty$. This limit is a vector in the kernel of the operator A. If $\sigma(A) \cap i\mathbb{R} = \{i\xi_1, i\xi_2, \cdots, i\xi_n\}$ then every bounded full-time solution of differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ has the form $\mathbf{u}(t) = \sum\limits_{k=1}^{n} e^{i\xi_k t}\mathbf{v}_k$, where $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are fixed vectors of $\mathbb{X}$. Specially, if $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ then every bounded solution of pre-compact orbit tends to a vector in the kernel of A as $t \to \infty$.

Proof: As we have mentioned before, the dynamics on the omega limit set of $\mathbf{x}$ is uniquely ergodic. Moreover, this limit set contains a full time bounded solution. Let $\mathbf{u}$ denote a bounded full-time solution of $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. Then $(\lambda - D)^{-1}\mathbf{u}(t) = (\lambda - A)^{-1}\mathbf{u}(t)$ for any $t \in \mathbb{R}$ and $\lambda \notin i\mathbb{R} \cup \sigma(A)$.

Here D denotes the differential operator with spectrum $i\mathbb{R}$. Therefore, for any point ξ in the Beurling spectrum of u we have $i\xi \in \sigma(A)$. Hence, if $\sigma(A) \cap i\mathbb{R} = \{i\xi_1, i\xi_2, \cdots, i\xi_n\}$ then the Beurling spectrum of any bounded full-time solution is a subset of $\{\xi_1, \xi_2, \cdots, \xi_n\}$. Thus, $\mathbf{u}(t) = \sum_{k=1}^{n} e^{i\xi_k t}\mathbf{v}_k$, where $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are fixed vectors of $\mathbb{X}$ [1], [5]. Now consider a bounded solution $\mathbf{x}$ of pre-compact orbit. Then the omega-limit set of $\mathbf{x}$ should contain a bounded full time solution $\mathbf{u}(t) = \sum_{k=1}^{n} e^{i\xi_k t}\mathbf{v}_k$. Specially, if $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$ then the omega limit set of any bounded solution with pre-compact orbit has only one element. This element is a vector of the kernel of A. The proof is now complete.

Remark. The last statement in our Theorem makes a significant extension of results in [2], [3]. Indeed, the authors have proved the existence of the $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{x}(s)ds$ only.

Acknowledgement. Deepest appreciation is extended towards the NAFOSTED (the National Foundation for Science and Techology Development in Vietnam) for the financial support.

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