Spectrum of a bounded sequence and inhomogeneous delay linear difference equations in a Banach space

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Abstract. We study the asymptotic behavior of a bounded solution of an inhomogeneous delay linear difference equation in a Banach space by using the spectrum of bounded sequences. We get a significant extension of excellent results in [1]. A new simple proof is also found for the famous Gelfand spectral radius theorem. Moreover, among other things we prove that if the spectrum of a bounded sequence $\{x_n\}_n$ is finite then $x_n = c_1\vartheta_1^n + c_2\vartheta_2^n + \cdots + c_k\vartheta_k^n + o(1)$ as $n \to \infty$ where $|\vartheta_1| = |\vartheta_2| = \cdots = |\vartheta_k| = 1$.

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1. Introduction

Let $\mathbb{X} = (\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ denote a Banach space and $T : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator. If the sequence of norms $\{\|T\|, \|T^2\|, \cdots\}$ is bounded then T is called a power bounded operator. Katznelson and Tzafriri [2] proved the following famous result.

Theorem A. *Let $T : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator and $\partial\mathbb{D}$ the unit circle. If T is power bounded and $\partial\mathbb{D} \cap \sigma(T) \subseteq \{1\}$ then*

$$\lim_{n\to\infty} \left(T^{n+1} - T^n\right) = 0.$$

Vu Quoc Phong [4] reproved this theorem. In this paper, we use a new method (the spectrum of a bounded sequence in a Banach space) to prove the existence of limit of T^n as $n \to \infty$.

2. Spectrum of a bounded sequence

Let $\mathbb{X} = (\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ denote a Banach space. Let $\mathbf{x} = \{x_0, x_1, \cdots\}$ denote a sequence with elements in $\mathbb{X}$ and $\ell^\infty(\mathbb{X})$ the Banach space of bounded sequences in $\mathbb{X}$ with the norm $\|\mathbf{x}\| = \sup\{\|x_0\|_{\mathbb{X}}, \|x_1\|_{\mathbb{X}}, \cdots\}$. Moreover, let $c_0(\mathbb{X})$ be the subpace of $\ell^\infty(\mathbb{X})$ consisting of vanishing sequences $\mathbf{x} = \{x_0, x_1, \cdots\}$ in $\mathbb{X}$ that is $\lim\limits_{n\to\infty} x_n = 0$. Let

$$S : \ell^\infty(\mathbb{X}) \to \ell^\infty(\mathbb{X})$$

denote the shift operator, that is $(S\mathbf{x})_n = x_{n+1}$. Let

$$\mathbb{Y} = \ell^\infty(\mathbb{X}) / c_0(\mathbb{X})$$

be the quotient space. The equivalent class containing $\mathbf{x} = \{x_0, x_1, \cdots\}$ is denoted by $\bar{\mathbf{x}} = \{\bar{x}_0, \bar{x}_1, \cdots\}$ The norm of an $\bar{\mathbf{x}} = \overline{\{x_0, x_1, \cdots\}} \in \mathbb{Y}$ is defined by $\|\bar{\mathbf{x}}\|_{\mathbb{Y}} = \limsup_{n\to\infty} \|x_n\|_{\mathbb{X}}$ and the reduced shift operator of S is denoted by $\overline{S} : \mathbb{Y} \to \mathbb{Y}$. Then $\overline{S}$ is an isometry operator so the spectrum of $\overline{S}$ is contained in the unit circle so the resolvent operator $R\left(\lambda, \overline{S}\right)$ of $\overline{S}$ is analytic and injective for every $|\lambda| \neq 1$. Hence, if $R\left(\lambda, \overline{S}\right)\bar{\mathbf{x}} = 0$ for some $|\lambda| \neq 1$

then $\bar{\mathbf{x}} = 0$ which means $\lim_{n\to\infty} x_n = 0$. Moreover, the norm of $R\left(\lambda, \overline{S}\right)$ is bounded by $\left|\left|\lambda\right| - 1\right|^{-1}$. Hence, $\lim_{\lambda\to\infty} R\left(\lambda, \overline{S}\right) = 0$. These conditions hold for resolvent of any isometry operator. The spectrum of a bounded sequence $\mathbf{x} = \{x_0, x_1, \cdots\}$ denoted by $\sigma(\mathbf{x})$ is the set of all essential (non-removable) singular points of $g(\lambda) = R\left(\lambda, \overline{S}\right)\bar{\mathbf{x}}$ (holomorphic function taking values in $\mathbb{Y}$). Then $\sigma(\mathbf{x})$ is contained in the unit circle $\partial\mathbb{D}$. Moreover, we have

Theorem 1. *$\sigma(\mathbf{x})$ is empty iff $\lim_{n\to\infty} x_n = 0$.*

Note 1. Theorem 1 is presented in [1] without strict proof. We refer [3] for readers intersted in complex function and spectral theory .

Theorem 2. *$\sigma(\mathbf{x}) = \{1\}$ iff $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$.*

Note 2. Theorem 2 is also presented in [1] without strict proof. Similar to Theorem 2 we consider the case where $\sigma(\mathbf{x})$ has only one point. We have

Theorem 3. *$\sigma(\mathbf{x}) = \{\vartheta\}$ iff $\lim_{n\to\infty} (x_{n+1} - \vartheta x_n) = 0$.*

Proof: See Theorem 5 below.

Note 3. Theorems 1 and 3 give the following theorem which is invented by Katznelson and Tzafriri [2] and reproved by Vu Quoc Phong [4] in the case $\vartheta = 1$.

Theorem 4. *Let $T : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator and $\partial\mathbb{D}$ the unit circle. If T is power bounded (that is the sequence of norms $\{\|T\|, \|T^2\|, \cdots\}$ is bounded) and $\partial\mathbb{D} \cap \sigma(T) \subseteq \{\vartheta\}$ then*

$$\lim_{n\to\infty} \left(T^{n+1} - \vartheta T^n\right) = 0.$$

Proof: Let $x_n = T^n$ and consider the spectrum of $\mathbf{x} = \{x_0, x_1, \cdots\}$. We have at once that $\sigma(\mathbf{x}) \subseteq \partial\mathbb{D} \cap \sigma(T) \subseteq \{\vartheta\}$ so by Theorem 3, $\lim_{n\to\infty} (x_{n+1} - \vartheta x_n) = 0$. The proof is now complete.

Theorem 5. *Assume that $\sigma(\mathbf{x}) = \{\vartheta_1, \vartheta_2, \cdots, \vartheta_k\}$ is of k distinct points. Then there exist vectors $v_1, v_2, \cdots, v_k \in \mathbb{X}$ such that $x_n = v_1\vartheta_1^n + v_2\vartheta_2^n + \cdots +$*

$v_k\vartheta_k^n + o(1)$ *as* $n \to \infty$. *Specially, if* $\sigma(\mathbf{x}) \subseteq \{1\}$ *then there exists* $\lim x_n$ *as* $n \to \infty$.

Proof: Let

$$g(\lambda) = R\left(\lambda, \overline{S}\right)\bar{\mathbf{x}} = \sum_{j=1}^{k} \frac{\bar{\mathbf{c}}_j}{\lambda - \vartheta_j}.$$

This formula holds because the (resolvent) function $R\left(\lambda, \overline{S}\right)\bar{\mathbf{x}}$ has simple poles at isolated points $\{\vartheta_1, \vartheta_2, \cdots, \vartheta_k\}$ and $\lim_{\lambda\to\infty} g(\lambda) = \mathbf{0}$. Then

$$\bar{\mathbf{x}} = \sum_{j=1}^{k} \frac{\left(\lambda - \overline{S}\right)\bar{\mathbf{c}}_j}{\lambda - \vartheta_j} \quad \text{for every } \lambda \in \mathbb{C}\setminus\{\vartheta_1, \vartheta_2, \cdots, \vartheta_k\}. \qquad (*)$$

Let $\lambda \to \infty$ we get $\bar{\mathbf{x}} = \sum_{j=1}^{k} \bar{\mathbf{c}}_j$. Replace this back to $(*)$ we get

$$\sum_{j=1}^{k} \frac{\overline{S}\bar{\mathbf{c}}_j}{\lambda - \vartheta_j} = \sum_{j=1}^{k} \frac{\vartheta_j\bar{\mathbf{c}}_j}{\lambda - \vartheta_j} \quad \text{for every } \lambda \in \mathbb{C}\setminus\{\vartheta_1, \vartheta_2, \cdots, \vartheta_k\}$$

and consequently, $\overline{S}\bar{\mathbf{c}}_j = \vartheta_j\bar{\mathbf{c}}_j$ for $j = 1, 2, \cdots, k$. In the other words, $\bar{\mathbf{x}}$ is the sum of k eigen-sequences of the shift operator with respect to k eigenvalues $\vartheta_1, \vartheta_2, \cdots, \vartheta_k$. More exactly, we have $x_n = v_1\vartheta_1^n + v_2\vartheta_2^n + \cdots + v_k\vartheta_k^n + o(1)$ as $n \to \infty$ where $v_1, v_2, \cdots, v_k \in \mathbb{X}$ are fixed. The proof is now complete.

3. Inhomogeneous delay linear difference equations

Now let $B : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator. Then B can be extended to the space $\ell^\infty(\mathbb{X})$ by letting $(B\mathbf{x})_n = Bx_n$ and also to the space $\mathbb{Y} = \ell^\infty(\mathbb{X})/c_0(\mathbb{X})$. The spectrums of B in the spaces $\mathbb{X}$ and $\mathbb{Y}$ are the same. We are interested in the bounded solutions of the linear difference equation

$$x_{n+1} = Bx_n + y_n$$

where $\mathbf{y} = \{y_0, y_1, \cdots\}$ is a vanishing sequence in $\mathbb{X}$. Clearly, for any solution $\mathbf{x} = \{x_0, x_1, \cdots\}$ we have $S\bar{\mathbf{x}} = B\bar{\mathbf{x}}$. Therefore, the spectrum of any solution

$\mathbf{x} = \{x_0, x_1, \cdots\}$ is contained in the spectrum of the operator B (and in the unit circle). Theorem 5 gives the following theorem which was proved in [1] for the case $k = 1$ and $\vartheta_1 = 1$.

Theorem 6. *Let $B : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator and $\partial\mathbb{D}$ the unit circle. If $\partial\mathbb{D} \cap \sigma(B) = \{\vartheta_1, \vartheta_2, \cdots, \vartheta_k\}$ then for every bounded solution $\mathbf{x} = \{x_0, x_1, \cdots\}$ of the linear difference equation*

$$x_{n+1} = Bx_n + y_n \quad \text{for } n = 0, 1, \cdots,$$

where $\mathbf{y} = \{y_0, y_1, \cdots\}$ is a vanishing sequence in $\mathbb{X}$, we have $x_n = v_1\vartheta_1^n + v_2\vartheta_2^n + \cdots + v_k\vartheta_k^n + o(1)$ as $n \to \infty$ where $v_1, v_2, \cdots, v_k \in \mathbb{X}$ are fixed. Specially, if $\partial\mathbb{D} \cap \sigma(B) \subseteq \{1\}$ then there exists $\lim x_n$ as $n \to \infty$.

For the delay equation

$$x_{n+p} = Bx_n + y_n \quad \text{for } n = 0, 1, \cdots,$$

we have the following result.

Theorem 7. *Let $B : \mathbb{X} \to \mathbb{X}$ denote a bounded linear operator and $\partial\mathbb{D}$ the unit circle. If $\partial\mathbb{D} \cap \sigma(B) \subseteq \{\vartheta\}$ then for every bounded solution $\mathbf{x} = \{x_0, x_1, \cdots\}$ of the delay linear difference equation*

$$x_{n+p} = Bx_n + y_n \quad \text{for } n = 0, 1, \cdots,$$

where $\mathbf{y} = \{y_0, y_1, \cdots\}$ is a vanishing sequence in $\mathbb{X}$, we have

$$\lim_{n\to\infty} (x_{n+1} - \vartheta x_n) = 0.$$

(Here p denotes a fixed positive integer.)

Proof: Clearly, for any bounded solution $\mathbf{x} = \{x_0, x_1, \cdots\}$ we have $S^p\bar{\mathbf{x}} = B\bar{\mathbf{x}}$. Therefore, the spectrum of $S^{p-1}\mathbf{x}$ is contained in the spectrum of the operator B (and in the unit circle). Consequently, this spectrum is empty or of only one point and our Theorem follows.

4. Appendix: Holomorphic functions in a Banach space and Resolvent of an isometry operator

We are interested in those functions $f:\mathbb{C}\to\mathbb{X}$ which can be expressed as power series $f(z)=\sum\limits_{k=0}^{\infty}z^{k}x_{k}$ where $\{x_{k}\}_{k=0}^{\infty}\subseteq\mathbb{X}$. This series is convergent in the norm topology of $\mathbb{X}$. This means that $\sum\limits_{k=0}^{\infty}\|x_{k}\|_{\mathbb{X}}|z|^{k}<\infty$ for every $z\in\mathbb{C}$. These functions are called entire functions in the Banach space $\mathbb{X}$. If this series is finite we say about polynomial function (operator). The complex derivative of f is $f'(z)=\sum\limits_{k=1}^{\infty}kz^{k-1}x_{k}$. Moreover, the complex integral of f gives

$$x_{k}=\frac{1}{2\pi i}\oint\limits_{|z|=R}\frac{f(z)\,dz}{z^{k+1}}\quad\text{for } k=0,1,2,\cdots.$$

Hence, if $\|f(z)\|_{\mathbb{X}}$ is bounded then f is constant (a vector of the Banach space $\mathbb{X}$). Now let Ω be an open (unbounded) region of complex plane. Consider those functions $g:\Omega\to\mathbb{X}$ such that for any $z_{0}\in\Omega$ there is $\delta>0$ such that $g(z)=\sum\limits_{k=0}^{\infty}(z-z_{0})^{k}x_{k}$ for some $\{x_{k}\}_{k=0}^{\infty}\subseteq\mathbb{X}$ and $|z-z_{0}|<\delta$. These functions of this condition are called holomorphic functions in the region Ω. If we can extend g to the whole complex plane without breaking this condition then g is also called an entire function. Otherwise, we say about the essential singularity of g. A point $z_{0}\in\mathbb{C}$ is called an essential singularity point of g if g cannot be extended to $\Omega\cup\{z_{0}\}$ without breaking the holomorphy condition. Most of time we are interested in the resolvent $(\lambda-A)^{-1}$ of a linear bounded operator $A:\mathbb{X}\to\mathbb{X}$. This is a holomorphic function defined on $\mathbb{C}\backslash\sigma(A)$ by Laurent series

$$(\lambda-A)^{-1}=\sum_{n=0}^{\infty}\frac{A^{n}}{\lambda^{n+1}}$$

which is convergent for all $|\lambda|>\rho(A)$ ($\rho(A)$ denotes the spectral radius of A). On the other hand, if $\mathbb{X}$ is finite dimensional then the resolvent $(\lambda-A)^{-1}$ has finite poles in $\sigma(A)$ (the spectrum of A). Let χ_{A} denote the characteristic polynomial of A. Then $\chi_{A}(z)=0$ for every $z\in\sigma(A)$ and the resolvent $(\lambda-A)^{-1}$ is holomorphic in $\mathbb{C}\setminus\sigma(A)$. Therefore, we can write

$$(\lambda-A)^{-1}=\frac{\phi(\lambda)}{\chi_{A}(\lambda)},$$

where $\phi:\mathbb{C}\to B(\mathbb{X})$ is a holomorphic function ($B(\mathbb{X})$ denotes the set of continuous linear operators on $\mathbb{X}$ and χ_{A} denotes the characteristic polynomial of A). Now multiply with $(\lambda-A)\chi_{A}(\lambda)$ side by side we have $\chi_{A}(\lambda)I=(\lambda-A)\phi(\lambda)$. Let $\lambda=A$ we have $\chi_{A}(A)=0$. The famous Caley-Hamilton theorem is proved. Moreover, we can prove the famous theorem of I. Gelfand on the spectral radius as follows. Let $a_{n}=\ln\|A^{n}\|$. Then $a_{n+m}=\ln\|A^{n+m}\|\leq\ln(\|A^{n}\|\,\|A^{m}\|)=\ln\|A^{n}\|+\ln\|A^{m}\|=a_{n}+a_{m}$ and consequently, there is $\lim a_{n}/n=:\ln r$, that is $\lim\|A^{n}\|^{1/n}=r$. On the other hand, the relsovent series

$$(\lambda-A)^{-1}=\sum_{n=0}^{\infty}\frac{A^{n}}{\lambda^{n+1}}$$

is absolutely convergent for all $|\lambda| > \rho(A)$ and

$$\sum_{n=0}^{\infty} \frac{\|A^n\|}{\rho(A)^n} = \infty.$$

If $\lim \|A^n\|^{1/n} = r < \rho(A)$ then $\|A^n\| < [\rho(A) - \varepsilon]^n$ for all $n > N$ and consequently,

$$\infty = \sum_{n>N} \frac{\|A^n\|}{\rho(A)^n} < \sum_{n>N} \left[\frac{\rho(A) - \varepsilon}{\rho(A)}\right]^n < \infty$$

which is a contradiction. If $\lim \|A^n\|^{1/n} = r > \rho(A)$ then $\|A^n\| > [\rho(A) + \varepsilon]^n$ for all $n > N$ and the series

$$\sum_{n=0}^{\infty} \frac{\|A^n\|}{|\lambda|^n}$$

divergent for $|\lambda| = \rho(A) + \varepsilon > \rho(A)$ which is a contradiction. In the next sections we are specially interested in the resolvent of isometry operators. More exactly, if $A : \mathbb{X} \to \mathbb{X}$ is an isometry linear operator then the spectrum of A is contained in the unit circle and the norm of $(\lambda - A)^{-1}$ is bounded by $\big||\lambda| - 1\big|^{-1}$. On the other hand, any isolated essential singular point of $(\lambda - A)^{-1}$ is a simple pole [1].

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