Dynamics of a real quadratic polynomial on its Julia set and a compact interval

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Abstract. We prove among other things that the Julia set of a real quadratic polynomial $P = 1 - az^2$ having an an absolutely continuous invariant measure in $[-1, 1]$ should be real.

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1. INTRODUCTION

Let $P = az^d + \cdots$ denote a polynomial of degree $d > 1$. Denote by $P^n = \underbrace{P \circ P \circ \cdots \circ P}_{n}$ the $n-$th iterate of P. The Julia set $J = J(P)$ of P is compact and nonempty. The Hausdorff dimension of J is positive ($\geq \ln d / \ln K_0$, with $K_0 = \max \{|P'(x)| : x \in J\} > 1$) [3]. Let $\mathbb{C}_\infty$ denote the Riemann sphere and $A(\infty) = \left\{z \in \mathbb{C}_\infty : \lim_{n\to\infty} P^n(z) = \infty\right\}$. Then $\partial A(\infty) = J$. Let μ denote the equilibrium measure of $J(P)$ and $\Sigma(P)$ the symmetry group of $J(P)$. It is proved in [4] that $\Sigma(P)$ is infinite if and only if $J(P)$ is a circle. For example, if $P = z^d$ then $J(P)$ is the unit circle, $\mathrm{cap}(J) = 1$ and $\mu = \frac{d\theta}{2\pi}$; if $P = T_d$ (Chebisev polynomial of the first kind) then $J(P) = [-1, 1]$, $\mathrm{cap}(J) = 1/2$ and $\mu = \frac{dx}{\pi\sqrt{1-x^2}}$. Brolin [6] proved that for monic polynomial $P = z^d + \cdots$, the sequence of zero counting measures of $P^n - a$ converges weakly to the equilibrium measure of $J = J(P)$ for any fixed $a \in \mathbb{C}$. Moreover, the capacity of $J(P)$ is 1 and $J(P)$ is completely invariant under $T : z \to P(z)$, i.e., $T(J) = T^{-1}(J) = J$. On the other hand, as a transformation on $J = J(P)$ the transformation T is strongly mixing and measure preserving, $\mu(T^{-1}(E)) = \mu(E)$ for $E \subset J$ in this case. Recently, we prove that [7]

$$\mathrm{cap}(J) = \sqrt[d-1]{\frac{1}{|a|}}$$

for all $P = az^d + \cdots$ and the equilibrium measure of $J = J(P)$ is invariant under T. Recall that the logarithmic energy of a probability measure μ is

$$I(\mu) = \iint \ln \frac{1}{|z-w|} d\mu(z)\, d\mu(w).$$

The equilibrium measure is the only probability measure with minimal energy [11]. Moreover, for the equilibrium measure μ of $J(P)$ we have $I(\mu) = -\ln \mathrm{cap}(J)$ and

$$\int \ln \frac{1}{|z-w|} d\mu(z) = I(\mu),$$

for q.e. $w \in J$. In [7] we proved the following theorems.

Theorem 1. Let $P = az^d + \cdots$ denote a polynomial of degree $d > 1$. Then the Julia set $J = J(P)$ of P is compact, the equilibrium measure of J is invariant under P and

$$\mathrm{cap}(J) = \sqrt[d-1]{\frac{1}{|a|}}. \tag{1}$$

Theorem 2. Let $\{\varphi_n = \kappa_n z^n + \cdots : \kappa_n > 0\}_{n=0}^{\infty}$ denote the orthonormal polynomial basis of $L^2(J, \mu)$. Here J is the Julia set of a polynomial $P(z) =$

$az^d + bz^{d-1} + \cdots$ of degree $d > 1$ and μ is the equilibrium measure of J. Then

$$\varphi_n \circ P = \frac{a^n}{|a|^n}\varphi_{nd}, \qquad \kappa_{nd} = \kappa_n |a|^n, \qquad \kappa_{d^n} = \kappa_1 |a|^{d^{n-1}+\cdots+d+1}. \tag{2}$$

Moreover,

$$\varphi_1(z) = \kappa_1\big(z + \frac{b}{ad}\big), \quad \varphi_{d^n} = \kappa_1\big(\underbrace{P \circ P \circ \cdots \circ P}_{n} + \frac{b}{ad}\big)\big(\frac{|a|}{a}\big)^{d^{n-1}+\cdots+d+1}.$$

For $P(z) = az^d + bz^{d-1} + \cdots$ of degree $d > 1$, every root of $\underbrace{P \circ P \circ \cdots \circ P}_{n} + \frac{b}{ad}$ is lying in the convex hull of $J(P)$. Consequently, the Julia set of P is real if and only if the roots of $\underbrace{P \circ P \circ \cdots \circ P}_{n} + \frac{b}{ad}$ are real and distinct.

Theorem 3. If two polynomials $P_1 = a_1 z^d + \cdots$ and $P_2 = a_2 z^d + \cdots$ have the same degree and Julia set then there is a point ξ in the unit circle such that $a_1 = \xi a_2$ and $P_1 = \xi P_2 + P_1(0) - \xi P_2(0)$. In this case the Julia set is invariant under the transformation $z \to \xi z + P_1(0) - \xi P_2(0)$. If $\xi = 1$ then $P_1 = P_2$. If J is real then $\xi = \pm 1$.

Theorem 4. If two polynomials $P_1 = a^{d-1} z^d + \cdots$ and $P_2 = a^{d^n - 1} z^{d^n} + \cdots$ have the same Julia set then $P_2 = \underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{n}$.

Remark. Similar results are proved in [1, 10] for polynomials having the same Julia sets. However, our results make more insight about these polynomials than earlier results. Theorem 2 really recovers the polynomial of a given degree from its Julia set.

2. Main Results

Let μ denote the equilibrium measure of the Julia set of $P = az^k + \cdots$. Define the Cauchy transform $S\mu$ of μ by letting

$$S\mu(z) = \int \frac{d\mu(t)}{z - t}.$$

Then $S\mu$ is holomorphic on the Fatou set of P and tending to 0 as $z \to \infty$. For $P(z) = z^k$, the Julia set is the unit circle and $S\mu = 0$ on the whole complex plane. For $P = T_k$, the Chebisev polynomial of the first kind, $J(P) = [-1, 1]$ and $S\mu(z) = \frac{1}{\sqrt{z^2-1}}$. Now we prove

Theorem A. For the Cauchy transform of μ the equilibrium measure of the Julia set of $P = az^2 + bz + c$ we have

$$S\mu\left(z\right) - S\mu\left(-z - \frac{b}{a}\right) = P'\left(z\right) S\mu\left(P\left(z\right)\right). \tag{3}$$

For polynomial $P_a = 1 - az^2$, we have

$$S\mu_a\left(z\right) + azS\mu_a\left(1 - az^2\right) = 0, \tag{4}$$

where μ_a denotes the equilibrium measure of $J(P_a)$. Therefore, $S\mu_a = 0$ on any attractive cycle of P_a.

Proof. For $z \in \mathbb{C}$ satisfying $P'\left(z\right) \neq 0$

$$\frac{P'\left(z\right)}{P\left(z\right) - P(t)} = \frac{1}{z-t} + \frac{1}{z + \frac{b}{a} + t}$$

so

$$\begin{aligned} P'\left(z\right) S\mu\left(P\left(z\right)\right) &= \int \frac{d\mu\left(t\right)}{z-t} + \int \frac{d\mu\left(t\right)}{z+t+\frac{b}{a}} \\ &= S\mu\left(z\right) - S\mu\left(-z - \frac{b}{a}\right) \end{aligned}$$

For $P_a = 1 - az^2$, we note that $S\mu_a(z) = -S\mu_a(-z)$ so

$$2S\mu_a\left(z\right) = P_a'\left(z\right) S\mu_a\left(P_a\left(z\right)\right).$$

If $S\mu_a(z) \neq 0$ for some attractive fixed point z of P_a^n then it follows from the above formula that

$$\frac{d}{dz} P_a^n(z) = 2,$$

which contradicts the attractivity of z. The proof is now complete. ■

Remark. Levin [9] had some results related the Cauchy transform of a limiting measure, but our Theorem is not following directly from his results.

3. Iterations of $1 - ax^2$ on $(-1, 1)$

Our methods on orthonormal polynomials with respect to the invariant measures can be applied for iterations of $P_a : x \to 1 - ax^2$ on $(-1, 1)$. Carleson and co-author proved in [5] that for $a \in (0, 2)$ in a set of positive Lebesgue measure, the transformation P_a has no attractive cycle but it has an absolutely continuous invariant measure μ_a with density $g_a/\pi \in L^p(-1, 1)$ for all $p \in (1, 2)$.

Theorem B. If the transformation P_a of $[-1, 1]$ has an absolutely continuous invariant measure $\mu_a = g_a dx/\pi$ then roots of the iterations P_a^n are distinct and lying in $[-1, 1]$. Therefore, the Julia set of P_a is real (a subset of $[-1, 1]$). Moreover, the Cauchy transform of μ_a will satisfy the identity

$$S\mu_a(z) + azS\mu_a\left(1 - az^2\right) = 0. \tag{5}$$

Therefore, $S\mu_a = 0$ on any attractive cycle of P_a if $S\mu$ is well defined on that cycle. Moreover,

$$\tilde{g}_a(x) + ax\tilde{g}_a(1 - ax^2) = 0 \qquad \text{for a.e. } x \in [-1, 1]. \tag{6}$$

Here, $\tilde{g}$ denote the Hilbert transform of g. If $g_a \in L^p(-1, 1)$ for some $p > 1$ then

$$g_a(x) = \frac{1}{\sqrt{1 - x^2}}\left(1 + \frac{1}{\pi}\int_{-1}^{1}\frac{\tilde{g}_a(s)\sqrt{1 - s^2}}{s - x}ds\right) \tag{7}$$

and

$$g_2(x) = \frac{1}{\sqrt{1 - x^2}}. \tag{8}$$

Proof. Apply the orthonormal polynomials techniques [2] we get that the iterations P_a^n are orthogonal with respect to the measure μ_a so any root of them is lying in $[-1, 1]$. We omit the details here. We remind the readers that $P_a : z \to 1 - az^2$ is a transformation of the compact interval $[-1.1]$. Theorem 2 implies that the Julia set of P_a is (real) subset of $[-1, 1]$. But not every real quadratic polynomial having an absolutely continuous invariant measure should have real Julia set. The Cauchy transform of μ_a will satisfy the identity (5). The proof is completely similar as the proof of (3). On the other hand, the Cauchy transform is exactly the Hilbert transform of the density function. Hence, the formula (6) is directly following from (5). The formula (7) is proved in [8]. If $a = 2$, it follows from (6) that $\tilde{g}_2 = 0$. Hence, (8) follows directly from (7). ■

Remark. If $a = 1$ then $P_1 = 1 - x^2$ and $P_1 \circ P_1 \circ P_1 = 1 - (x^4 - 2x^2)^2$ has two complex roots $(\pm\sqrt{1-\sqrt{2}})$ so P_1 has no absolutely continuous invariant measure. However,

$$\frac{\delta_1 + \delta_0}{2}$$

is a discrete invariant measure of P_1.

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