ON THE INITIAL VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS WITH THE INITIAL DATUM IN THE SOBOLEV SPACES

D. Q. KHAI AND V. T. T. DUONG

ABSTRACT. In this paper, we study local well-posedness for the Navier-Stokes equations with arbitrary initial data in homogeneous Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ for $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$. The obtained result improves the known ones for $p > d$ and $s = 0$ (see [4, 6]). In the case of critical indexes $s = \frac{d}{p} - 1$, we prove global well-posedness for Navier-Stokes equations when the norm of the initial value is small enough. This result is a generalization of the ones in [5] and [24] in which $(p = d, s = 0)$ and $(p > d, s = \frac{d}{p} - 1)$, respectively.

1. INTRODUCTION

This paper studies the Cauchy problem of the incompressible Navier-Stokes equations (NSE) in the whole space $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \nabla.u = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l(u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ of the fluid element at time t and position x and the pressure $p(t, x)$.
There is an extensive literature on the existence of strong solutions of the Cauchy problem for NSE. The global well-posedness of strong solutions for small initial data in the critical Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is due to Fujita and Kato [20], also in [8], Chemin has proved the case of $H^s(\mathbb{R}^3), (s > 1/2)$. In [21], Kato has proved the case of the Lebesgue space $L^3(\mathbb{R}^3)$. In [23],

2010 *Mathematics Subject Classification.* Primary 35Q30; Secondary 76D05, 76N10.
Key words and phrases. Navier-Stokes equations, existence and uniqueness of local and global mild solutions, critical Sobolev and Triebel spaces.

Koch and Tataru have proved the case of the space BMO^{-1} (see also [7]). In [23], H. Koch has proved the case of the space $\dot{B}^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)_{(p<+\infty)}$, see [23] and the recent ill-posedness result [1] for $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d)$. Results on the existence of mild solutions with value in $L^p(\mathbb{R}^d), (p > d)$ were established in the papers of Fabes, Jones and Rivière [10] and of Giga [12]. Concerning the initial datum in the space L^∞, the existence of a mild solution was obtained by Cannone and Meyer in [4, 6]. Moreover, in [4, 6], they also obtained theorems on the existence of mild solutions with value in the Morrey-Campanato space $M_2^p(\mathbb{R}^d), (p > d)$ and the Sobolev space $H_p^s(\mathbb{R}^d), (p < d, \frac{1}{p} - \frac{s}{d} < \frac{1}{d})$. NSE in the Morrey-Campanato space were also treated by Kato [22] and Taylor [26]. Recently, the authors of this article have considered NSE in Sobolev spaces, Sobolev-Lorentz spaces, mixed-norm Sobolev-Lorentz spaces, and Sobolev-Fourier-Lorentz spaces, see [15, 16, 18], [17], [13], and [14] respectively. In [19], we prove some results on the existence and space-time decay rates of global strong solutions of the Cauchy problem for NSE in weighed $L^\infty(\mathbb{R}^d, |x|^\beta \mathrm{dx})$ spaces. In this paper, we construct mild solutions in the spaces $L^\infty([0,T]; \dot{H}_p^s(\mathbb{R}^d))$ to the Cauchy problem for NSE when the initial datum belongs to the Sobolev spaces $\dot{H}_p^s(\mathbb{R}^d)$, with $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$, we obtain the existence of mild solutions with arbitrary initial value when T is small enough and existence of mild solutions for any $T < +\infty$ when the norm of the initial value in the Triebel-Lizorkin spaces $\dot{F}_{\tilde{q}}^{s-d(\frac{1}{p}-\frac{1}{\tilde{q}}),\infty}$, $(\tilde{q} > \max\{p, q\},$ where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d})$ is small enough. In the case $p > d$ and $s = 0$, this result is stronger than that of Cannone and Meyer [4, 6] under a weaker condition on the initial data. In the case of critical indexes $(p > \frac{d}{2}, s = \frac{d}{p} - 1)$, we obtaine global mild solutions when the norm of the initial value in the Triebel-Lizorkin spaces $\dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d), (\tilde{q} > \max\{d, p\})$ is small enough. This result in one hand if $p = d$ and $s = 0$ is stronger than that of Cannone and Planchon [5] but under a weaker condition on the initial data and in the other hand if $p > d$ and $s = \frac{d}{p} - 1$ is stronger than that of Lemarie-Rieusset but under a weaker condition on the initial data (Proposition 20.2, [24], p. 201). The content of this paper is as follows: in Section 2, we state our main theorem after introducing some notations. In Section 3, we first establish some estimates concerning the heat semigroup with differential. We also recall some auxiliary lemmas and several estimates in the homogeneous Sobolev spaces and Triebel spaces. Finally, in Section 4, we will give the proof of the main theorem.

2. Statement of the results

Now, for $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes u(\tau,.)\big)\mathrm{d}\tau.$$

Above we have used the following notation: for a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1\le k\le d} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

For a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E. We denote by $L^q := L^q(\mathbb{R}^d)$ the usual Lebesgue space for $q \in [1, \infty]$ with the norm $\|.\|_q$, and we do not distinguish between the vector-valued and scalar-valued spaces of functions. We define the Sobolev space by $\dot{H}^s_q := \dot{\Lambda}^{-s}L^q$ equipped with the norm $\big\|f\big\|_{\dot{H}^s_q} := \|\dot{\Lambda}^s f\|_q$. Here $\dot{\Lambda}^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively. $\dot{\Lambda} = \sqrt{-\Delta}$ is the homogeneous Calderon pseudo-differential operator. For vector-valued $f = (f_1, ..., f_M)$, we define $\|f\|_X = \big(\sum_{m=1}^{m=M}\|f_m\|_X^2\big)^{\frac{1}{2}}$. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \le CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. Now we can state our results

Theorem 2.1. *Let s and p be such that*

$$p > \frac{d}{2} \ \textit{and} \ \frac{d}{p} - 1 \le s < \frac{d}{2p}.$$

Set

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d}.$$

(a) *For all* $\tilde{q} > \max\{p, q\}$, *there exists a positive constant* $\delta_{q,\tilde{q},d}$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})} \Big\| \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})} \big|\mathrm{e}^{t\Delta} u_0\big| \Big\|_{L^{\tilde{q}}} \leq \delta_{q,\tilde{q},d}, \tag{2.1}$$

NSE has a unique mild solution $u \in L^\infty([0, T]; \dot{H}^s_p)$ *and the following inequality holds*

$$\Big\| \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{r})} \big|u(t, x)\big| \Big\|_r < +\infty, \ \textit{for all } r > \max\{p, q\}.$$

In particular, the condition (2.1) *holds for arbitrary* $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ *when* $T(u_0)$ *is small enough.*

(b) *If* $s = \frac{d}{p} - 1$ *then for all* $\tilde{q} > \max\{p, d\}$ *there exists a constant* $\sigma_{\tilde{q},d} > 0$ *such that if* $\big\|u_0\big\|_{\dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \leq \sigma_{\tilde{q},d}$ *and* $T = +\infty$ *then the condition* (2.1) *holds.*

In the case of critical indexes ($s = \frac{d}{p} - 1, p > \frac{d}{2}$), we get the following consequence.

Proposition 2.2. *Let* $p > \frac{d}{2}$. *Then for any* $\tilde{q} > \max\{p, d\}$, *there exists a positive constant* $\delta_{\tilde{q},d}$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$\Big\| \sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})} \big|\mathrm{e}^{t\Delta} u_0\big| \Big\|_{L^{\tilde{q}}} \leq \delta_{\tilde{q},d}, \tag{2.2}$$

NSE has a unique mild solution $u \in L^\infty([0, T]; \dot{H}_p^{\frac{d}{p}-1})$ *and the following inequality holds*

$$\Big\| \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{d}-\frac{1}{r})} \big|u(t, x)\big| \Big\|_r < +\infty, \ \textit{for all } r > \max\{p, d\}.$$

Denoting $w = u - \mathrm{e}^{t\Delta} u_0$ *then* w *satisfies the following inequality*

$$\Big\| \sup_{0<t<T} \big|\dot{\Lambda}^{\frac{d}{\tilde{p}}-1} w(t, x)\big| \Big\|_{L^{\tilde{p}}} < \infty, \ \textit{for all } \tilde{p} > \frac{1}{2}\max\{p, d\}.$$

Moreover, if $p \geq d$ *then*

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})} \big|u(t, .)\big| \in L^p.$$

In particular, the condition (2.2) *holds for arbitrary* $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ *when* $T(u_0)$ *is small enough, and there exists a positive constant* $\sigma_{\tilde{q},d}$ *such that if*

$$\big\|u_0\big\|_{\dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \leq \sigma_{\tilde{q},d} \text{ and } T = +\infty \tag{2.3}$$

then the condition (2.2) *holds.*

Remark 2.3. Proposition 2.2 is the theorem of Canone and Planchon [5] if $p = d$ and the condition (2.3) is replaced by the condition

$$\big\|u_0\big\|_{\dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \leq \sigma_{\tilde{q},d} \text{ and } T = +\infty, \text{ where } d < \tilde{q} < 2d. \tag{2.4}$$

Note that in the case $p = d$, the condition (2.3) is weaker than the condition (2.4) because of the following elementary imbedding maps

$$\dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d)_{(d<\tilde{q}<2d)} \hookrightarrow \dot{F}_{2d}^{-\frac{1}{2},\infty}(\mathbb{R}^d) \hookrightarrow \dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d)_{(\tilde{q}>2d)}.$$

Remark 2.4. The statement about the global existence in Proposition 2.2 is the Lemarie-Rieusset statement (Proposition 20.2, [24], p. 201) if $p > d$ and the condition (2.3) is replaced by the condition

$$\big\|u_0\big\|_{\dot{H}_p^{\frac{d}{p}-1}} < \delta_{d,p}. \tag{2.5}$$

Note that the condition (2.3) is weaker than the condition (2.5) because of the following elementary imbedding maps

$$\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d) \hookrightarrow \dot{F}_p^{\frac{d}{p}-1,\infty}(\mathbb{R}^d) \hookrightarrow \dot{F}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d), (\tilde{q} > p).$$

Lemarie-Rieusset proved the above statement by using Hardy-Littlewood maximal functions theory (as developped for L^d by Canderón [9] and Cannone [4]).

In the case of supercritical indexes $p > \frac{d}{2}$ and $\frac{d}{p} - 1 < s < \frac{d}{2p}$, we get the following consequence.

Proposition 2.5. *Let* $p > \frac{d}{2}$ *and* $\frac{d}{p}-1 < s < \frac{d}{2p}$. *Then for all* $\tilde{q} > \max\{p, q\}$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

there exists a positive constant $\delta_{q,\tilde{q},d}$ *such that for all* $T > 0$ *and for all* $u_0 \in \dot{H}_p^s(\mathbb{R}^d)$ *with* $\mathrm{div}(u_0) = 0$ *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u_0\big\|_{\dot{F}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{q,\tilde{q},d}, \tag{2.6}$$

NSE has a unique mild solution $u \in L^\infty([0,T];\dot{H}^s_p)$ and the following inequality holds

$$\Big\|\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{r})}\big|u(t,x)\big|\Big\|_r < +\infty, \ \textit{for all } r > \max\{p,q\}.$$

Remark 2.6. Proposition 2.5 is the theorem of Canone and Meyer [4, 6] if $s=0$, $p>d$, and the condition (2.6) is replaced by the condition

$$T^{\frac{1}{2}(1-\frac{d}{p})}\big\|u_0\big\|_{L^p} \leq \delta_{p,d}.$$

Note that in the case $s=0$ and $p>d$, the condition (2.6) is weaker than the above condition because of the following elementary imbedding maps

$$L^p(\mathbb{R}^d) \hookrightarrow \dot{F}_{\tilde{q}}^{-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d), (\tilde{q} > p \geq d).$$

3. Tools from harmonic analysis

In this section we prepare some auxiliary lemmas.
The main property we use throughout this paper is that the operator $e^{t\Delta}\mathbb{P}\nabla$ is a matrix of convolution operators with bounded integrable kernels.

Lemma 3.1. *Let $s > -1$. Then the kernel function of $\dot{\Lambda}^s e^{t\Delta}\mathbb{P}\nabla$ is the function*

$$K_t(x) = \frac{1}{t^{\frac{d+s+1}{2}}}K\big(\frac{x}{\sqrt{t}}\big),$$

where the function K is the kernel function of $\dot{\Lambda}^s e^{\Delta}\mathbb{P}\nabla$ which satisfies the following inequality

$$|K(x)| \lesssim \frac{1}{1+|x|^{d+s+1}}.$$

Proof. See Proposition 11.1 in ([24], p. 107). □

Lemma 3.2. *The kernel function $K_t(x)$ of $\dot{\Lambda}^s e^{t\Delta}\mathbb{P}\nabla$ satisfies the following inequality*

$$|K_t(x)| \lesssim \frac{1}{t^{\gamma_2}|x|^{\gamma_1}}, \ \textit{for } \gamma_1 > 0, \gamma_2 > 0, \ \textit{and } \gamma_1 + 2\gamma_2 = d+s+1.$$

Proof. This is deduced by applying Lemma 3.1 and the Young inequality

$$|K_t(x)| = \Big|\frac{1}{t^{\frac{d+s+1}{2}}}K\Big(\frac{x}{\sqrt{t}}\Big)\Big| \lesssim \frac{1}{t^{\frac{d+s+1}{2}}}\frac{1}{1+(\frac{|x|}{\sqrt{t}})^{d+s+1}}$$

$$= \frac{1}{t^{\frac{d+s+1}{2}} + |x|^{d+s+1}} \lesssim \frac{1}{t^{\gamma_2}|x|^{\gamma_1}}.$$

□

Lemma 3.3. (Sobolev inequalities).
If $s_1 > s_2$, $1 < q_1,\ q_2 < \infty$, *and* $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$, *then we have the following embedding mapping*

$$\dot{H}^{s_1}_{q_1} \hookrightarrow \dot{H}^{s_2}_{q_2}.$$

In this paper we use the definition of the homogeneous Triebel space $\dot{F}^{s,p}_q$ in [2, 3, 11, 25]. The following lemma will provide a different characterization of Triebel spaces $\dot{F}^{s,p}_q$ in terms of the heat semigroup and will be one of the staple ingredients of the proof of Theorem 2.1.

Lemma 3.4.
Let $1 \le p, q \le \infty$ *and* $s < 0$. *Then the two quantities*

$$\Big\|\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big|\mathrm{e}^{t\Delta}f\big|\big)^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}}\Big\|_{L^q} \text{ and } \|f\|_{\dot{F}^{s,p}_q} \text{ are equivalent.}$$

Proof. See [5]. □

Lemma 3.5. (Convolution of the Lorentz spaces).
Let $1 < p < \infty$, $1 \le q \le \infty, 1/p' + 1/p = 1$, *and* $1/q' + 1/q = 1$. *Then convolution is a bounded bilinear operator:*
(a) *from* $L^{p,q} \times L^1$ *to* $L^{p,q}$,
(b) *from* $L^{p,q} \times L^{p',q'}$ *to* L^∞,
(c) *from* $L^{p,q} \times L^{p_1,q_1}$ *to* L^{p_2,q_2}, *for* $1 < p, p_1, p_2 < \infty, 1 \le q, q_1, q_2 \le \infty, 1/p_2 + 1 = 1/p + 1/p_1$, *and* $1/q_2 = 1/q + 1/q_1$.

Proof. See Proposition 2.4 (c) in ([24], p. 20). □

Lemma 3.6. *Let* $\theta < 1$ *and* $\gamma < 1$ *then*

$$\int_0^t (t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^1 (1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau < \infty.$$

The proof of this lemma is elementary and may be omitted. □

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in ([24], p. 227)).

Theorem 3.7. *Let* E *be a Banach space, and* $B : E \times E \to E$ *be a continuous bilinear map such that there exists* $\eta > 0$ *so that*

$$\|B(x, y)\| \le \eta\|x\|\|y\|,$$

for all x *and* y *in* E. *Then for any fixed* $y \in E$ *such that* $\|y\| \le \frac{1}{4\eta}$, *the equation* $x = y - B(x, x)$ *has a unique solution* $\overline{x} \in E$ *satisfying* $\|\overline{x}\| \le \frac{1}{2\eta}$.

4. Proof of Theorem 2.1

In this section we shall give the proof of Theorem 2.1.
We now need three more lemmas. In order to proceed, we define an auxiliary space $\mathcal{G}^{\tilde{q}}_{q,T}$ which is made up of the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}} := \Big\|\sup_{0<t<T} t^{\frac{\alpha}{2}}\big|u(t,x)\big|\Big\|_{L^{\tilde{q}}} < \infty,$$

and

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t} \tau^{\frac{\alpha}{2}}\big|u(\tau,x)\big|\Big\|_{L^{\tilde{q}}} = 0, \tag{4.1}$$

with

$$\tilde{q} \geq q \geq d \text{ and } \alpha = d\Big(\frac{1}{q} - \frac{1}{\tilde{q}}\Big).$$

We recall the definition of the auxiliary space $\mathcal{H}^s_{p,T}$ introduced by Cannone and Planchon [5]. This space is made up of the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{H}^s_{p,T}} := \Big\|\sup_{0<t<T}\big|\dot{\Lambda}^s u(t,x)\big|\Big\|_{L^p} < \infty,$$

and

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\big|\dot{\Lambda}^s u(\tau,x)\big|\Big\|_{L^p} = 0, \tag{4.2}$$

with

$$p > 1 \text{ and } s \geq \frac{d}{p} - 1.$$

The space $\mathcal{H}^s_{p,T}$ is continuously embedded into $L^\infty([0,T];\dot{H}^s_p(\mathbb{R}^d))$ because of the following elementary inequality

$$\sup_{0<t<T}\big\|\dot{\Lambda}^s u(t,x)\big\|_{L^p} \leq \Big\|\sup_{0<t<T}\big|\dot{\Lambda}^s u(t,x)\big|\Big\|_{L^p}.$$

Lemma 4.1. *Suppose that $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $p > 1$ and $\frac{d}{p} - 1 \leq s < \frac{d}{p}$. Then for all $\tilde{q}$ satisfying*

$$\tilde{q} > \max\{p, q\},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

we have

$$\mathrm{e}^{t\Delta}u_0 \in \mathcal{G}^{\tilde{q}}_{q,\infty}.$$

Proof. First, we consider the case $p \le q$. In this case $s \ge 0$, applying Lemma 3.3 to obtain $u_0 \in L^q$. We will prove that

$$\Big\| \sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big| \mathrm{e}^{t\Delta} u_0 \big| \Big\|_{L^{\tilde q}} \lesssim \big\| u_0 \big\|_{L^q}, \text{ for all } \tilde q > q.$$

Indeed, we have the following estimates

$$t^{\frac{\alpha}{2}} \big| \mathrm{e}^{t\Delta} u_0 \big| = \Big| \frac{t^{\frac{\alpha}{2}}}{(4\pi t)^{d/2}} \mathrm{e}^{\frac{-|.|^2}{4t}} * u_0 \Big| \lesssim \frac{1}{\sqrt{t}^{(d-\alpha)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |u_0|$$

$$= \frac{1}{|.|^{d-\alpha}} \Big(\frac{|.|}{\sqrt{t}} \Big)^{d-\alpha} \mathrm{e}^{\frac{-|.|^2}{4t}} * |u_0| \le \sup_{x \in \mathbb{R}^d} \big(|x|^{d-\alpha} \mathrm{e}^{\frac{-|x|^2}{4}} \big). \frac{1}{|.|^{d-\alpha}} * \big| u_0 \big|$$

$$\lesssim \frac{1}{|.|^{d-\alpha}} * \big| u_0 \big|. \tag{4.3}$$

From the estimate (4.3), applying Lemma 3.5(c) to obtain

$$\Big\| \sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big| \mathrm{e}^{t\Delta} u_0 \big| \Big\|_{L^{\tilde q}} \lesssim \Big\| \frac{1}{|.|^{d-\alpha}} * \big| u_0 \big| \Big\|_{L^{\tilde q}} \lesssim \Big\| \frac{1}{|.|^{d-\alpha}} \Big\|_{L^{\frac{d}{d-\alpha},\infty}} \big\| u_0 \big\|_{L^{q,\tilde q}}$$

$$\lesssim \big\| u_0 \big\|_{L^q}, \text{ (note that } \frac{1}{|.|^s} \in L^{\frac{d}{s},\infty}(\mathbb{R}^d) \text{ with } 0 < s \le d).$$

This proves the result. We now prove that

$$\lim_{t \to 0} \Big\| \sup_{0<\tau<t} \tau^{\frac{\alpha}{2}} \big| \mathrm{e}^{\tau\Delta} u_0 \big| \Big\|_{L^{\tilde q}} = 0.$$

Set $\mathcal{X}_n(x) = 0$ for $x \in \{x : \ |x| < n\} \cap \{x : \ |u_0(x)| < n\}$ and $\mathcal{X}_n(x) = 1$ otherwise. We have

$$t^{\frac{\alpha}{2}} \big| \mathrm{e}^{t\Delta} u_0 \big| \le C \Big(\frac{1}{\sqrt{t}^{(d-\alpha)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |\mathcal{X}_n u_0| + \frac{1}{\sqrt{t}^{(d-\alpha)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |(1-\mathcal{X}_n) u_0| \Big). \tag{4.4}$$

Let $\hat q$ be fixed such that $q < \hat q < \tilde q$ and $\beta = d(\frac{1}{\hat q} - \frac{1}{\tilde q})$. Arguing as in the proof of the estimate (4.3), we derive

$$C \frac{1}{\sqrt{t}^{(d-\alpha)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |\mathcal{X}_n u_0| \le C_1 \frac{1}{|.|^{d-\alpha}} * \big| \mathcal{X}_n u_0 \big|, \tag{4.5}$$

and

$$C \frac{1}{\sqrt{t}^{(d-\alpha)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |(1-\mathcal{X}_n) u_0| = C t^{\frac{\alpha-\beta}{2}} \frac{1}{\sqrt{t}^{(d-\beta)}} \mathrm{e}^{\frac{-|.|^2}{4t}} * |(1-\mathcal{X}_n) u_0|$$

$$\le C \sup_{x \in \mathbb{R}^d} (|x|^{d-\beta} \mathrm{e}^{\frac{-|x|^2}{4}}) t^{\frac{\alpha-\beta}{2}} \frac{1}{|.|^{d-\beta}} * \big| (1-\mathcal{X}_n) u_0 \big|$$

$$\le C_2 n t^{\frac{d}{2}(\frac{1}{q} - \frac{1}{\hat q})} \frac{1}{|.|^{d-\beta}} * \big| 1 - \mathcal{X}_n \big|. \tag{4.6}$$

From the estimates (4.4), (4.5), and (4.6), we have

$$\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|\mathrm{e}^{\tau\Delta}u_0\big|\Big\|_{L^{\tilde{q}}}\leq$$

$$C_1\Big\|\frac{1}{|.|^{d-\alpha}}*\big|\mathcal{X}_nu_0\big|\Big\|_{L^{\tilde{q}}}+C_2nt^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\hat{q}})}\Big\|\frac{1}{|.|^{d-\beta}}*\big|1-\mathcal{X}_n\big|\Big\|_{L^{\tilde{q}}}\leq$$

$$C_3\Big\|\frac{1}{|.|^{d-\alpha}}\Big\|_{L^{\frac{d}{d-\alpha},\infty}}\big\|\mathcal{X}_nu_0\big\|_{L^q}+C_4nt^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\hat{q}})}\Big\|\frac{1}{|.|^{d-\beta}}\Big\|_{L^{\frac{d}{d-\beta},\infty}}\big\|1-\mathcal{X}_n\big\|_{L^{\hat{q}}}\leq$$

$$C_5\big\|\mathcal{X}_nu_0\big\|_{L^q}+C_6nt^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\hat{q}})}\big\|1-\mathcal{X}_n\big\|_{L^{\hat{q}}}. \tag{4.7}$$

For any $\epsilon>0$, we can take n large enough that

$$C_5\big\|\mathcal{X}_nu_0\big\|_{L^q}<\frac{\epsilon}{2}. \tag{4.8}$$

Fixed one of such n, there exists $t_0=t_0(n)>0$ satisfying

$$C_6nt^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\hat{q}})}\big\|1-\mathcal{X}_n\big\|_{L^{\hat{q}}}<\frac{\epsilon}{2},\ \text{for } t<t_0. \tag{4.9}$$

From the estimates (4.7), (4.8), and (4.9), we have

$$\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|\mathrm{e}^{\tau\Delta}u_0\big|\Big\|_{L^{\tilde{q}}}\leq C_5\big\|\mathcal{X}_nu_0\big\|_{L^q}+C_6nt^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\hat{q}})}\big\|1-\mathcal{X}_n\big\|_{L^{\hat{q}}}<\epsilon,\text{for } t<t_0.$$

We now consider the case $p>q$. In this case $s<0$. We prove that

$$\Big\|\sup_{0<t<\infty}t^{\frac{\alpha}{2}}\big|\mathrm{e}^{t\Delta}u_0\big|\Big\|_{L^{\tilde{q}}}\lesssim\big\|u_0\big\|_{\dot{H}^s_p},\ \text{for all } \tilde{q}>p.$$

We have

$$\mathrm{e}^{t\Delta}u_0=\mathrm{e}^{t\Delta}\dot{\Lambda}^{-s}\dot{\Lambda}^su_0=\frac{1}{t^{\frac{d-s}{2}}}K\Big(\frac{\cdot}{\sqrt{t}}\Big)*(\dot{\Lambda}^su_0),$$

where

$$\hat{K}(\xi)=\frac{1}{(2\pi)^{\frac{d}{2}}}\mathrm{e}^{-|\xi|^2}|\xi|^{-s}\ \text{and}\ |K(x)|\lesssim\frac{1}{(1+|x|)^{d-s}}.$$

From the above inequality, we have

$$\begin{aligned}t^{\frac{\alpha}{2}}\big|\mathrm{e}^{t\Delta}u_0\big|&\leq\Big|\frac{1}{\sqrt{t}^{(d-s-\alpha)}}K\Big(\frac{\cdot}{\sqrt{t}}\Big)\Big|*|\dot{\Lambda}^su_0|\\&=\Big|\frac{1}{|.|^{d-s-\alpha}}\Big(\frac{|.|}{\sqrt{t}}\Big)^{d-s-\alpha}K\Big(\frac{\cdot}{\sqrt{t}}\Big)\Big|*|\dot{\Lambda}^su_0|\\&\leq\sup_{x\in\mathbb{R}^d}\big(\big||x|^{d-s-\alpha}K(x)\big|\big)\frac{1}{|.|^{d-s-\alpha}}*\big|\dot{\Lambda}^su_0\big|\lesssim\frac{1}{|.|^{d-s-\alpha}}*\big|\dot{\Lambda}^su_0\big|.\end{aligned} \tag{4.10}$$

From the estimate (4.10), applying Lemma 3.5(c), we have

$$\Big\|\sup_{0<t<\infty}t^{\frac{\alpha}{2}}\big|\mathrm{e}^{t\Delta}u_0\big|\Big\|_{L^{\tilde{q}}}\lesssim\Big\|\frac{1}{|.|^{d-s-\alpha}}*\big|\dot{\Lambda}^su_0\big|\Big\|_{L^{\tilde{q}}}\leq$$

$$\Big\|\frac{1}{|.|^{d-s-\alpha}}*\big|\dot{\Lambda}^su_0\big|\Big\|_{L^{\tilde{q},p}}\lesssim\Big\|\frac{1}{|.|^{d-s-\alpha}}\Big\|_{L^{\frac{d}{d-s-\alpha},\infty}}\big\|\dot{\Lambda}^su_0\big\|_{L^p}\simeq\big\|u_0\big\|_{\dot{H}^s_p}.$$

This proves the result. We now claim that

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|\mathrm{e}^{\tau\Delta}u_0\big|\Big\|_{L^{\tilde q}}=0,\quad \text{for all } \tilde q>p.$$

Set $\mathcal{X}_{n,s}(x)=0$ for $x\in\{x:\ |x|<n\}\cap\{x:\ |\dot\Lambda^s u_0(x)|<n\}$ and $\mathcal{X}_{n,s}(x)=1$ otherwise. Let $\hat q$ be fixed such that $p<\hat q<\tilde q$ and $\beta=d(\frac{1}{p}-\frac{1}{\hat q})$. For any $\epsilon>0$, by an arguing similar to the case $q>p$, there exist a sufficiently large n and a sufficiently small $t_0=t_0(n)$ such that

$$\begin{aligned}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|\mathrm{e}^{\tau\Delta}u_0\big|\Big\|_{L^{\tilde q}}&\le C_1\Big\|\frac{1}{|.|^{d-s-\alpha}}\Big\|_{L^{\frac{d}{d-s-\alpha},\infty}}\big\|\mathcal{X}_{n,s}\dot\Lambda^s u_0\big\|_{L^p}\\&+C_2nt^{\frac{\beta}{2}}\Big\|\frac{1}{|.|^{d-s-\alpha+\beta}}\Big\|_{L^{\frac{d}{d-s-\alpha+\beta},\infty}}\big\|1-\mathcal{X}_{n,s}\big\|_{L^{\hat q}}<\epsilon,\ \text{for } t<t_0.\end{aligned}$$

□

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by

$$B(u,v)(t)=\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\mathrm{d}\tau. \tag{4.11}$$

Lemma 4.2. *Let p and s be such that*

$$p>\frac{d}{2}\ \textit{and}\ \frac{d}{p}-1\le s<\frac{d}{2p}.$$

Then the bilinear operator B is continuous from $\mathcal{G}^{\tilde q}_{q,T}\times\mathcal{G}^{\tilde q}_{q,T}$ into $\mathcal{H}^s_{p,T}$, where

$$\frac{1}{q}=\frac{1}{p}-\frac{s}{d},\ q<\tilde q<2p,$$

and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{H}^s_{p,T}}\le CT^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{G}^{\tilde q}_{q,T}}\big\|v\big\|_{\mathcal{G}^{\tilde q}_{q,T}}, \tag{4.12}$$

where C is a positive constant and independent of T.

Proof. From the equality (4.11), applying Lemma 3.2 to obtain

$$\begin{aligned}\big|\dot\Lambda^s B(u,v)(t)(x)\big|&\le\int_0^t\big|\dot\Lambda^s\mathrm{e}^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,x)\otimes v(\tau,x)\big)\big|\mathrm{d}\tau\\&=\int_0^t\Big|K_{t-\tau}(x)*\big(u(\tau,x)\otimes v(\tau,x)\big)\Big|\mathrm{d}\tau\\&\lesssim\int_0^t\Big|\frac{1}{(t-\tau)^{\gamma_2}.|x|^{\gamma_1}}*\big(u(\tau,x)\otimes v(\tau,x)\big)\Big|\mathrm{d}\tau \qquad (4.13)\end{aligned}$$

where

$$\gamma_1>0,\gamma_2>0,\gamma_1+2\gamma_2=d+1+s.$$

Using the estimate (4.13) for

$$\gamma_1 = d\Big(1+\frac{1}{p}-\frac{2}{\tilde{q}}\Big), \gamma_2 = \frac{1}{2}-\frac{d}{2p}+\frac{s}{2}+\frac{d}{\tilde{q}},$$

and applying Lemma 3.6 to obtain

$$\begin{aligned}
&\big|\dot{\Lambda}^s B(u,v)(t)(x)\big| \\
&\lesssim \frac{1}{|x|^{d(1+\frac{1}{p}-\frac{2}{\tilde{q}})}} * \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}-\frac{d}{2p}+\frac{s}{2}+\frac{d}{\tilde{q}}}}\big|u(\tau,x)\otimes v(\tau,x)\big|\mathrm{d}\tau \\
&\lesssim \frac{1}{|x|^{d(1+\frac{1}{p}-\frac{2}{\tilde{q}})}} * \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}-\frac{d}{2p}+\frac{s}{2}+\frac{d}{\tilde{q}}}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|v(\eta,x)\big|\mathrm{d}\tau \\
&= \frac{1}{|x|^{d(1+\frac{1}{p}-\frac{2}{\tilde{q}})}} * \Big(\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|v(\eta,x)\big|\Big)\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}-\frac{d}{2p}+\frac{s}{2}+\frac{d}{\tilde{q}}}}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq t^{\frac{1}{2}(1+s-\frac{d}{p})}\frac{1}{|x|^{d(1+\frac{1}{p}-\frac{2}{\tilde{q}})}} * \Big(\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|v(\eta,x)\big|\Big). \qquad (4.14)
\end{aligned}$$

From the estimate (4.14), applying Lemma 3.5(c) and Hölder's inequality in order to obtain

$$\begin{aligned}
&\Big\|\sup_{0<\tau<t}\big|\dot{\Lambda}^s B(u,v)(\tau)\big|\Big\|_{L^p} \leq \Big\|\sup_{0<\tau<t}\big|\dot{\Lambda}^s B(u,v)(\tau)\big|\Big\|_{L^{p,\frac{\tilde{q}}{2}}} \\
&\lesssim t^{\frac{1}{2}(1+s-\frac{d}{p})}\Big\|\frac{1}{|x|^{d(1+\frac{1}{p}-\frac{2}{\tilde{q}})}}\Big\|_{L^{\frac{1}{1+\frac{1}{p}-\frac{2}{\tilde{q}}},\infty}}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|v(\eta,x)\big|\Big\|_{L^{\frac{\tilde{q}}{2}}} \\
&\lesssim t^{\frac{1}{2}(1+s-\frac{d}{p})}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|u(\eta,x)\big|\Big\|_{L^{\tilde{q}}}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big|v(\eta,x)\big|\Big\|_{L^{\tilde{q}}}. \qquad (4.15)
\end{aligned}$$

Let us now check the validity of the condition (4.2) for the bilinear term $B(u,v)(t)$. In fact, from the estimate (4.15) it follows that

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\big|\dot{\Lambda}^s B(u,v)(\tau)\big|\Big\|_{L^p} = 0,$$

whenever

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|u(\tau,x)\big|\Big\|_{L^{\tilde{q}}} = \lim_{t\to 0}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha}{2}}\big|v(\tau,x)\big|\Big\|_{L^{\tilde{q}}} = 0.$$

The estimate (4.12) is deduced from the inequality (4.15). □

Lemma 4.3. *Let q and q_1 be such that $d \leq q < q_1 < +\infty$. Then the bilinear operator B is continuous from $\mathcal{G}^{q_1}_{q,T} \times \mathcal{G}^{q_1}_{q,T}$ into $\mathcal{G}^{q_2}_{q,T}$ for all q_2 satisfying $\frac{1}{q_2} \in \big(0, \frac{1}{q}\big] \cap \big(\frac{2}{q_1}-\frac{1}{d}, \frac{2}{q_1}\big)$, and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{G}^{q_2}_{q,T}} \leq CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{G}^{q_1}_{q,T}}\big\|v\big\|_{\mathcal{G}^{q_1}_{q,T}}, \qquad (4.16)$$

where C is a positive constant and independent of T.

Proof. From the equality (4.11), applying Lemma 3.2 to obtain

$$\begin{aligned}\big|B(u,v)(t)(x)\big| &\le \int_0^t \big|\mathrm{e}^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,x)\otimes v(\tau,x)\big)\big|\mathrm{d}\tau\\ &= \int_0^t \Big|K_{t-\tau}(x) * \big(u(\tau,x)\otimes v(\tau,x)\big)\Big|\mathrm{d}\tau\\ &\lesssim \int_0^t \Big|\frac{1}{(t-\tau)^{\gamma_2}.|x|^{\gamma_1}} * \big(u(\tau,x)\otimes v(\tau,x)\big)\Big|\mathrm{d}\tau \qquad (4.17)\end{aligned}$$

where

$$\gamma_1>0, \gamma_2>0, \gamma_1+2\gamma_2=d+1.$$

Set

$$\alpha_1 = d(\frac{1}{q}-\frac{1}{q_1}), \alpha_2 = d(\frac{1}{q}-\frac{1}{q_2}).$$

Using the estimate (4.17) for

$$\gamma_1 = d\Big(1+\frac{1}{q_2}-\frac{2}{q_1}\Big), \gamma_2 = \frac{1}{2}-\frac{d}{2q_2}+\frac{d}{q_1},$$

and applying Lemma 3.6 to obtain

$$\big|B(u,v)(t)(x)\big| \lesssim \frac{1}{|x|^{d(1+\frac{1}{q_2}-\frac{2}{q_1})}} * \int_0^t \frac{1}{(t-\tau)^{\frac12-\frac{d}{2q_2}+\frac{d}{q_1}}}\big|u(\tau,x)\otimes v(\tau,x)\big|\mathrm{d}\tau$$

$$\lesssim \frac{1}{|x|^{d(1+\frac{1}{q_2}-\frac{2}{q_1})}} * \int_0^t \frac{1}{(t-\tau)^{\frac12-\frac{d}{2q_2}+\frac{d}{q_1}}}\tau^{-\alpha_1}\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|v(\eta,x)\big|\mathrm{d}\tau$$

$$= \frac{1}{|x|^{d(1+\frac{1}{q_2}-\frac{2}{q_1})}} * \Big(\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|v(\eta,x)\big|\Big)\int_0^t \frac{1}{(t-\tau)^{\frac12-\frac{d}{2q_2}+\frac{d}{q_1}}\tau^{\alpha_1}}\mathrm{d}\tau$$

$$\simeq t^{\frac12(1-\frac{d}{q})-\frac{\alpha_2}{2}}\frac{1}{|x|^{d(1+\frac{1}{q_2}-\frac{2}{q_1})}} * \Big(\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|v(\eta,x)\big|\Big). \qquad (4.18)$$

From the estimate (4.18), applying Lemma 3.5(c) and Hölder's inequality in order to obtain

$$\Big\|\sup_{0<\tau<t}t^{\frac{\alpha_2}{2}}\big|B(u,v)(\tau)\big|\Big\|_{L^{q_2}} \le \Big\|\sup_{0<\tau<t}t^{\frac{\alpha_2}{2}}\big|B(u,v)(\tau)\big|\Big\|_{L^{q_2,\frac{q_2}{2}}}$$

$$\lesssim t^{\frac12(1-\frac{d}{q})}\Big\|\frac{1}{|x|^{d(1+\frac{1}{q_2}-\frac{2}{q_1})}}\Big\|_{L^{\frac{1}{1+\frac{1}{q_2}-\frac{2}{q_1}},\infty}}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|u(\eta,x)\big|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|v(\eta,x)\big|\Big\|_{L^{\frac{q_1}{2}}}$$

$$\lesssim t^{\frac12(1-\frac{d}{q})}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|u(\eta,x)\big|\Big\|_{L^{\tilde q}}\Big\|\sup_{0<\eta<t}\eta^{\frac{\alpha_1}{2}}\big|v(\eta,x)\big|\Big\|_{L^{q_1}}. \qquad (4.19)$$

Let us now check the validity of the condition (4.1) for the bilinear term $B(u,v)(t)$. In fact, from the estimate (4.19) it follows that

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}t^{\frac{\alpha_2}{2}}\big|B(u,v)(\tau)\big|\Big\|_{L^{q_2}} = 0,$$

whenever

$$\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha_1}{2}}\big|u(\tau,x)\big|\Big\|_{L^{q_1}}=\lim_{t\to 0}\Big\|\sup_{0<\tau<t}\tau^{\frac{\alpha_1}{2}}\big|v(\tau,x)\big|\Big\|_{L^{q_1}}=0$$

The estimate (4.16) is deduced from the inequality (4.19). □

Proof of Theorem 2.1

(a) Applying Lemma 4.3 for $q_1=q_2=\tilde{q}$, we deduce that B is continuous from $\mathcal{G}^{\tilde{q}}_{q,T}\times\mathcal{G}^{\tilde{q}}_{q,T}$ to $\mathcal{G}^{\tilde{q}}_{q,T}$ and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}}\le C_{q,\tilde{q},d}T^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}}=C_{q,\tilde{q},d}T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}},$$

where $C_{q,\tilde{q},d}$ is a positive and independent of T. From Theorem 3.7 and the above inequality, we deduce that for any $u_0\in\dot{H}^s_p$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{\mathcal{G}^{\tilde{q}}_{q,T}}=T^{\frac{1}{2}(1+s-\frac{d}{p})}\sup_{0<t<T}t^{\frac{\alpha}{2}}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}}\le\frac{1}{4C_{q,\tilde{q},d}},$$

where

$$\alpha=d\Big(\frac{1}{q}-\frac{1}{\tilde{q}}\Big)=d\Big(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}}\Big),$$

NSE has a solution u on the interval $(0,T)$ so that $u\in\mathcal{G}^{\tilde{q}}_{q,T}$. We prove that $u\in\bigcap\limits_{r>\max\{p,q\}}\mathcal{G}^r_{q,T}$. Indeed, applying Lemma 4.3, we have $B(u,u)\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{q}\big]\cap\big(\frac{2}{\tilde{q}}-\frac{1}{d},\frac{2}{\tilde{q}}\big)$. Applying Lemma 4.1, we have $e^{t\Delta}u_0\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{\max\{p,q\}}\big)$. Since $u=e^{t\Delta}u_0-B(u,u)$, it follows that $u\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{\max\{p,q\}}\big)\cap\big(\frac{2}{\tilde{q}}-\frac{1}{d},\frac{2}{\tilde{q}}\big)$. Applying again Lemmas 4.3 and 4.1, in exactly the same way, since $u\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{\max\{p,q\}}\big)\cap\big(\frac{2}{\tilde{q}}-\frac{1}{d},\frac{2}{\tilde{q}}\big)$, it follows that $u\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{\max\{p,q\}}\big)\cap\big(\frac{1}{d}-2^2(\frac{1}{d}-\frac{1}{\tilde{q}}),\frac{2^2}{\tilde{q}}\big)$. By induction, we get $u\in\mathcal{G}^r_{q,T}$ for all r satisfying $\frac{1}{r}\in\big(0,\frac{1}{\max\{p,q\}}\big)\cap\big(\frac{1}{d}-2^n(\frac{1}{d}-\frac{1}{\tilde{q}}),\frac{2^n}{\tilde{q}}\big)$ with $n\ge 1$. Since $\frac{1}{d}-\frac{1}{\tilde{q}}>0$, it follows that there exists sufficiently large n satisfying $\big(0,\frac{1}{\max\{p,q\}}\big)\cap\big(\frac{1}{d}-2^n(\frac{1}{d}-\frac{1}{\tilde{q}}),\frac{2^n}{\tilde{q}}\big)=\big(0,\frac{1}{\max\{p,q\}}\big)$. Therefore $u\in\mathcal{G}^r_{q,T}$ for all $r>\max\{p,q\}$. This proves the result.

We now prove that $u\in L^\infty([0,T];\dot{H}^s_p)$. Indeed, from $u\in\mathcal{G}^r_{q,T}$ for all $r>\max\{p,q\}$, applying Lemma 4.2 to obtain $B(u,u)\in\mathcal{H}^s_{p,T}\subseteq L^\infty\big([0,T];\dot{H}^s_p\big)$. On the other hand, since $u\in\dot{H}^s_p$, it follows that $\mathrm{e}^{t\Delta}u_0\in L^\infty\big([0,T];\dot{H}^s_p\big)$. Therefore

$$u=\mathrm{e}^{t\Delta}u_0-B(u,u)\in L^\infty\big([0,T];\dot{H}^s_p\big).$$

Finally, we will show that the condition (2.1) is valid when T is small enough. From the definition of $\mathcal{G}^{\tilde{q}}_{q,T}$ and Lemma 4.1, we deduce that the left-hand side of the condition (2.1) converges to 0 when T goes to 0. Therefore the condition (2.1) holds for arbitrary $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
(b) From Lemma 3.4, the two quantities $\big\|u_0\big\|_{\dot{F}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}}$ and $\sup\limits_{0<t<\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}}$ are equivalent. Thus, there exists a positive constant $\sigma_{\tilde{q},d}$ such that the condition (2.1) holds for $T=\infty$ whenever $\big\|u_0\big\|_{\dot{F}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \le \sigma_{\tilde{q},d}$. $\square$

Proof of Proposition 2.2

By Theorem 2.1, we only need to prove that $w \in \mathcal{H}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$ for all $\tilde{p} > \frac{1}{2}\max\{p,d\}$ and if $p \ge d$ then $\sup\limits_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|u(t,.)\big| \in L^p$. Indeed, applying Lemma 4.2, we deduce that the bilinear operator B is continuous from $\mathcal{G}^r_{d,T}\times\mathcal{G}^r_{d,T}$ into $\mathcal{H}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$ for all $\tilde{p} > \frac{d}{2}$ and r satisfying $d<r<2\tilde{p}$; hence from $u \in \bigcap\limits_{r>\max\{p,d\}}\mathcal{G}^r_{d,T}$ and $2\tilde{p} > \max\{p,d\}$, we have $w = -B(u,u) \in \mathcal{H}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$.
We now prove that if $p \ge d$ then $\sup\limits_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|u(t,.)\big| \in L^p$. Indeed, we notice that, if $u_0 \in \dot{H}^{\frac{d}{p}-1}_p$, then $u_0 = \dot{\Lambda}^{1-\frac{d}{p}}v_0$ with $v_0 \in L^p$; hence $t^{\frac{1}{2}(1-\frac{d}{p})}\big|\mathrm{e}^{t\Delta}u_0\big| \lesssim M_{v_0}$, where M_{v_0} is the Hardy-Littlewood maximal function of v_0 (hence $M_{v_0} \in L^p$). On the other hand, from $u \in \bigcap\limits_{r>p}\mathcal{G}^r_{d,T}$, we apply Lemma 4.3 to obtain $B(u,u) \in \mathcal{G}^p_{d,T}$, hence $\sup\limits_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|B(u,u)(t,.)\big| \in L^p$. Thus

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|u(t,.)\big| \le \sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|\mathrm{e}^{t\Delta}u_0\big| + \sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{p})}\big|B(u,u)(t,.)\big| \in L^p.$$

$\square$

Proof of Proposition 2.5

By Lemma 3.4, we deduce that two quantities

$$\big\|u_0\big\|_{\dot{F}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}} \text{ and } \Big\|\sup_{0<t<\infty} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}|\mathrm{e}^{t\Delta}u_0|\Big\|_{L^{\tilde{q}}}$$

are equivalent. Thus

$$\Big\|\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\big|\mathrm{e}^{t\Delta}u_0\big|\Big\|_{L^{\tilde{q}}} \lesssim \big\|u_0\big\|_{\dot{F}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}_{\tilde{q}}}.$$

The Proposition 2.5 is proved by applying Theorem 2.1 and the above inequality. □

Acknowledgements. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.50.

References

[1] J. Bourgain and N. Pavloviéc, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal., **255** (9) (2008) 2233-2247.

[2] G. Bourdaud, *Ce qu'il faut savoir sur les espaces de Besov*, Prépublication de l'Universitéde Paris 7 (janvier 1993).

[3] G. Bourdaud, *Réalisation des espaces de Besov homogènes*, Ark. Mat. **26** (1) (1988) 41-54.

[4] M. Cannone, Ondelettes, Paraproduits et Navier-Stokes, Diderot Editeur, Paris, 1995, 191 p.

[5] M. Cannone and F. Planchon, *On the non stationary Navier-Stokes equations with an external force*, Adv. in Diff. Eq., **4** (5) (1999), 697-730.

[6] M. Cannone and Y. Meyer, *Littlewood-Paley decomposition and the Navier-Stokes equations*, Methods Appl. Anal., **2** (1995), 307-319.

[7] J. Chemin, I. Gallagher, *Wellposedness and stability results for the NavierStokes in* $\mathbb{R}^3$, Ann. Inst. H. Poincare Anal. Non Lineaire **26** (2) (2009) 599-624.

[8] J. M. Chemin, *Remarques sur l'existence globale pour le système de Navier-Stokes incompressible*, SIAM J. Math. Anal., **23** (1992), 20-28.

[9] C. Calderón, *Initial values of Navier-Stokes equations*, Proc. A.M.S., **117** (1993), 761-766.

[10] E. Fabes, B. Jones and N. Riviere, *The initial value problem for the Navier-Stokes equations with data in* L^p, Arch. Rat. Mech. Anal., **45** (1972), 222-240.

[11] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, 79, AMS, Providence (1991).

[12] Y. Giga, *Solutions of semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differ. Eq., **62** (1986), 186-212.

[13] D. Q. Khai and N. M. Tri, *Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations*, J. Math. Anal. Appl., **417** (2014) 819-833.

[14] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces*, J. Math. Anal. Appl., **437** (2016) 754-781.

[15] D. Q. Khai and N. M. Tri, *On the Hausdorff dimension of the singular set in time for weak solutions to the nonstationary Navier-Stokes equation on torus*, Vietnam Journal of Mathematics, **43** (2015) 283-295.

[16] D. Q. Khai and N. M. Tri, *On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces*, J. Math. Sci. Univ. Tokyo, **23** (2016) 499-528.

[17] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with data in homogeneous Sobolev-Lorentz spaces*, preprint, arXiv:1601.01742.

[18] D. Q. Khai and N. M. Tri, *The existence and decay rates of strong solutions for Navier-Stokes Equations in Bessel-potential spaces*, preprint, arXiv:1603.01896.

[19] D. Q. Khai and N. M. Tri *The existence and space-time decay rates of strong solutions to Navier-Stokes Equations in weighed $L^\infty(|x|^\gamma \mathrm{d}x) \cap L^\infty(|x|^\beta \mathrm{d}x)$ spaces*, preprint, arXiv:1601.01441

[20] T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova, **32** (1962), 243-260.

[21] T. Kato, *Strong L^p solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Zeit., **187** (1984), 471-480.

[22] T. Kato, *Strong solutions of the Navier-Stokes equations in Morrey spaces*, Bol. Soc. Brasil. Math., **22** (1992), 127-155.

[23] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157** (1) (2001) 22-35.

[24] P. G. Lemarie-Rieusset, Recent Developments in the Navier-Stokes Problem, Chapman and Hall/CRC Research Notes in Mathematics, vol. 431, Chapman and Hall/CRC, Boca Raton, FL, 2002, 395 p.

[25] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, 1976.

[26] M. E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes equations and other evolution equations*, Comm. P. D. E., **17** (1992), 1407-1456.

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Cau Giay, Hanoi, Vietnam.
E-mail address: khaitoantin@gmail.com

Faculty of Basic Sciences, Quang Ninh University of Industry, Yen Tho, Dong Trieu, Quang Ninh, Vietnam.
E-mail address: vuthuyduong309@gmail.com