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ON THE FIXED-POINT SET AND COMMUTATOR SUBGROUP OF AN AUTOMORPHISM OF A SOLUBLE GROUP

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Abstract. Let ϕ be an automorphism of finite order of a group G. We deduce consequences for the commutator subgroup $[G, \phi]$ of ϕ on G of hypotheses such as finiteness and local finiteness on the fixed-point set $C_G(\phi)$ of ϕ on G. We require various solubility or finiteness conditions on G or at least on $[G, \phi]$.

Throughout this paper ϕ denotes an automorphism of a group G of finite and frequently prime order. We investigate the consequences for

$$[G, \phi] =< g^{-1}.g\phi : g \in G >$$

from hypotheses on $C_G(\phi) = \{g \in G : g\phi = g\}$ such as finite or locally finite. Our starting point is Endimioni and Moravec's paper [2], where they investigate this for G a metabelian group. Specifically they prove, see Theorem 5 of [2] that if G is metabelian, if $C_G(\phi)$ is a π-group for some set π of primes and if $|\phi| = p$ is prime, then $[G, \phi]$ is an extension of a π-group by a nilpotent group of class at most p (even 1 if $p = 2$). Our first two theorems are both generalizations of this. As usual, G' denotes the derived subgroup of G.

Theorem 1. *Let ϕ be an automorphism of the nilpotent-by-abelian group G with $C_{G'}(\phi)$ a periodic π group for some set π of primes and with $\phi^p = 1$ for some prime p. Then $[G, \phi]G'$ is an extension of a π-group by a nilpotent group. Specifically if $P = O_\pi(G')$, then $[G, \phi]G'/P$ is nilpotent of class at most 1 if $p = 2$ and of class at most $((p-1)^d - 1)/(p-2)$ if p is odd, where d denotes the derived length of $[G, \phi]G'/P$.*

Thus the nilpotency class of $[G, \phi]G'/P$ can be bounded in terms of p and the derived length of G only. In particular if G is metabelian then $d \leq 2$ and the nilpotency class of $[G, \phi]G'/P$ is at most p (1 if $p = 2$), which yields a slight generalization of Theorem 5 of [2] (notice that our hypothesis is on $C_{G'}(\phi)$ and not the whole of $C_G(\phi)$). Choosing $\pi = \emptyset$ in Theorem 1 yields the following.

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Corollary 1. *Let ϕ be a fixed-point-free automorphism of prime order of the nilpotent-by-abelian group G. Then $[G,\phi]$ is nilpotent of class bounded in terms of the order of ϕ and the derived length of G.*

For metabelian groups we can deduce stronger conclusions than those of either Theorem 1 or Theorem 5 of [2].

Theorem 2. *Let ϕ be an automorphism of the metabelian group G with $\phi^p = 1$ for some prime p. Set $B = C_{G'}(\phi)^G =< C_{G'}(\phi)^g : g \in G >$. Then $[G,\phi]G'/B$ is nilpotent of class at most p (even 1 if $p = 2$).*

If $C_{G'}(\phi)$ in Theorem 2 is a (periodic) π-group for some set π of primes, then B also is a π-group and hence Theorem 5 of [2] follows from Theorem 2. If $C_{G'}(\phi)$ in Theorem 2 has finite exponent, e say, for example if $C_{G'}(\phi)$ is finite, then B has finite exponent e and $[G,\phi]$ is an extension of a group of finite exponent by a nilpotent group of class at most p (1 if $p = 2$).

Theorem 1 does not extend to soluble groups in general, even to ones of derived length 3. Further, at least for $p = 3$, we cannot replace nilpotent-by-abelian by abelian-by-nilpotent.

Example 1. For each odd prime p there is a polycyclic, abelian-by-finite group G of derived lingth 3 with an automorphism ϕ of order p such that $|C_G(\phi)| = p$ and such that $[G,\phi]$ is not periodic-by-nilpotent.

Example 2. There is a polycyclic, abelian-by-(finite nilpotent) group G with an automorphism ϕ of order 3 such that $|C_G(\phi)| = 2$ and such that $[G,\phi]$ is not periodic-by-nilpotent.

The above suggests that the prime 2 is different, as indeed it is. For $p = 2$ we can prove stronger results, even for soluble groups. In fact we only need some sort of solubility restriction on $[G,\phi]$.

Theorem 3. *Let ϕ be an automorphism of a group G with $\phi^2 = 1$ and with $C_G(\phi)$ a locally finite π-group for some set π of odd primes.*

a) *Suppose $[G,\phi]$ contains a soluble normal subgroup S of G such that $[G,\phi]/S$ is a locally finite π-group. Then $[G,\phi]' \leq O_\pi(G' \cap [G,\phi])$.*

b) *If $[G,\phi]$ is soluble, then $[G,\phi]'$ is a π-group.*

Theorem 4. *Let ϕ be an automorphism of a group G with $\phi^2 = 1$ and $|C_G(\phi)| = n < \infty$.*

a) *Suppose $[G,\phi]$ contains a soluble normal subgroup S of G such that $[G,\phi]/S$ is a locally finite π-group for some set π of primes. If $\pi \supseteq \{\text{primes}\, q : q \leq n\}$, then $[G,\phi]' \leq O_\pi(G' \cap [G,\phi])$.*

b) *Suppose $[G,\phi]^e$ is soluble for some positive integer e with $[G,\phi]/[G,\phi]^e$ locally finite. If $\pi = \{\text{primes}\, q : q \leq n \text{ or } q|e\}$, then $[G,\phi]'$ is a locally finite π-group.*

c) *Suppose $[G,\phi]$ is soluble and $\pi = \{\text{primes}\, q : q \leq n\}$. Then $[G,\phi]'$ is a (soluble, locally finite) π-group.*

d) *Suppose $[G,\phi]$ is soluble and ϕ acts fixed-point freely on G. Then $[G,\phi]$ is abelian.*

Note that in Parts b) and c) of Theorem 4 the set π is finite. Also if $[G, \phi]$ is soluble-by-finite, then $[G, \phi]$ satisfies the hypothesis of Part b). Part d) very slightly extends Theorem 6 of [2], the latter being the case where G itself is assumed soluble. Note also in Theorem 4 that ϕ-invariant, locally finite subgroups, more generally ϕ-invariant subgroups with finite Hirsch number, of G are always nilpotent-by-finite but need not be abelian-by-finite, see [8].

Theorems 3 and 4 leave a small gap, namely the case where $\phi^2 = 1$ and $C_G(\phi)$ is an infinite 2-group. For locally nilpotent groups the answer to this is both easy and more general. Below $h(p)$ denotes the Higman function, e.g. see Section 5.1 of [3].

Proposition. *Let ϕ be an automorphism of the locally nilpotent group G with $\phi^p = 1$ for some prime p such that $C = C_G(\phi)$ is a π-group for some set π of primes. Set $P = O_\pi(G)$. Then G/P is nilpotent of class at most $h(p)$. If $p = 2$ then $G' \leq P$, ϕ inverts G/P and G is an extension of a π-group by an abelian group.*

The following is a corollary of the Proposition and Theorem 1; it extends the latter.

Corollary 2. *Let ϕ be an automorphism of the (locally nilpotent)-by-abelian group G with $C_{G'}(\phi)$ a periodic π group for some set π of primes and with $\phi^p = 1$ for some prime p. Then $H = [G, \phi]G'/O_\pi(G')$ is nilpotent, of class at most 1 if $p = 2$ and of class at most $((p-1)^d - 1)/(p-2)$ if p is odd, where d denotes the derived length of H.*

A similar type of argument to that proving the Proposition, using Lemma 1 of [9], shows that if ϕ is an automorphism of a group G with $\phi^2 = 1$ and $C_G(G)$ periodic and if G has a local system of (torsion-free)-by-finite subgroups with finite Hirsch numbers, then $[G, \phi]'$ is periodic.

In places in our arguments we can weaken $|\phi|$ prime to $|\phi|$ a power of a prime or sometimes just to $|\phi|$ finite. However it is impossible to do this in general.

Example 3. Let p be any prime. Then there exists a polycyclic, metabelian, abelian-by-finite group G with a fixed-point-free automorphism of order a power of p such that $[G, \phi]$ is not periodic-by-nilpotent.

Lemma 1. *Let ϕ be an automorphism of a group G and A a ϕ-invariant central subgroup of G. Define subgroups C, $K \leq G$ and a map $\gamma : G \to G$ by $C = C_G(\phi)$, $K/A = C_{G/A}(\phi)$ and $g\gamma = g^{-1}.g\phi = [g, \phi]$. Then the following hold.*

a) $(K : CA) = (K\gamma.A\gamma : A\gamma)$, $(K : CA)(A : K\gamma.A\gamma) = (A : A\gamma)$ *and*

$$(K : A)(A : K\gamma.A\gamma) = (A : A\gamma)(C : C \cap A).$$

b) *Suppose A is periodic and $\phi|_A$ and $C \cap A$ have finite order. Then $(A : A\gamma)$ divides $|C \cap A|$. If also C is finite, then $(K : A)$ divides $|C|$.*

c) *If $\phi|_A$ has finite order m, then*

$$(A\gamma.(C \cap A) : A^m)(A : A\gamma) = (A : A^m)(C \cap A : C \cap A\gamma)$$

and $(K : A)(A : K\gamma.A\gamma)(A\gamma.(C \cap A) : A^m) = (A : A^m)(C : C \cap A\gamma)$.

In particular if $(A : A^m)$ *and* C *are finite, then* $(K : A)$ *is finite and divides* $(A : A^m)|C|$.

The point of Lemma 1 is that if we replace 'A abelian normal' by 'A central' in Lemma 1 of [6] and Lemmas 8 and 9 of [7] then we can replace 'less than or equal to' by 'divides'. This does have some content; for a trivial example consider $G = \mathrm{Sym}(3), A = \mathrm{Alt}(3)$ and ϕ conjugation of G by a 3-cycle. Then $|\phi| = 3 = |C|$, while $(K : A) = (G : A) = 2$, which certainly does not divide $(A : A^3)|C| = 3^2$. (Here C and K are as in Lemma 1).

Proof. a) By Lemma 1 of [6] we have $(K : CA) = (K\gamma.A\gamma : A\gamma)$. Here A is central, so $\gamma|_K$ is a homomorphism of K into A (with kernel C), so $K\gamma$ is a subgroup (and not just a subset) of A. Thus

$$(A : A\gamma) = (A : K\gamma.A\gamma)(K\gamma.A\gamma : A\gamma).$$

The second and third claims of a) follow.

b) Suppose $B \le A$ is finite with $B\phi = B$ and $C \cap A \le B$. Then $\ker(\gamma|_B) = C \cap B = C \cap A$ and so $(B : B\gamma) = |C \cap A|$. Since ϕ has finite order, this yields that $(A : A\gamma) \le |C \cap A|$. Hence we may choose B with $A = A\gamma.B$. Then $(A : A\gamma) = (B : B \cap A\gamma)$, which divides $(B : B\gamma) = |C \cap A|$. If C is finite, then a) yields that $(K : A)$ divides $(A : A\gamma)(C : C \cap A)$, which now divides $|C \cap A|(C : C \cap A) = |C|$.

c) The first claim is immediate from the proof of Lemma 7 of [7]. Then

$$\begin{aligned}
&(K : A)(A : K\gamma.A\gamma)(A\gamma.(C \cap A) : A^m)\\
&\qquad = (A : A\gamma)(C : C \cap A)(A\gamma.(C \cap A) : A^m) \quad \text{by Part a)}\\
&\qquad = (A : A^m)(C \cap A : C \cap A\gamma)(C : C \cap A) \quad \text{by the first claim of Part c)}\\
&\qquad = (A : A^m)(C : C \cap A\gamma),
\end{aligned}$$

which divides $(A : A^m)|C|$ when the latter is finite. The lemma follows. □

Lemma 2. *Let* ϕ *be an automorphism of the group* G *with* $\phi^m = 1$ *for some positive integer* m. *Define maps* γ *and* ψ *of* $G \to G$ *by* $g\gamma = g^{-1}.g\phi$ *and* $g\psi = g.g\phi.g\phi^2.\dots.g\phi^{m-1}$ *for all* $g \in G$. *If* $g, h \in G$ *then* $(g\gamma.h)\psi \in (h^G\psi)^G$.

Proof. Let $g, h \in G$. Then

$$\begin{aligned}
(g\gamma.h)\psi &= \left(\prod_{1\le i\le m} g^{-1}\phi^{i-1}.g\phi^i.h\phi^{i-1}\right)g^{-1}\phi^m.g\\
&= g^{-1}\left(\prod_{1\le i\le m} g\phi^i.h\phi^{i-1}.g^{-1}\phi^i\right)g\\
&= g^{-1}((g\phi.h.g^{-1}\phi)\psi)g \in (h^G\psi)^G.
\end{aligned}$$

□

Lemma 3. *Let* ϕ *be an automorphism of the nilpotent-by-abelian group* G *with* $\phi^m = 1$ *for some power* m *of a prime and with* $C_{G'}(\phi)$ *a* π*-group for some set*

π of primes. Then with $P = O_\pi(G')$ and the maps γ and ψ as in Lemma 2, we have $P \supseteq (G\gamma.G')\psi = ([G,\phi]G')\psi$.

Proof. Set $N = G'$ and $C = C_N(\phi)$; so N is nilpotent and C is periodic. If X is a finitely generated ϕ-invariant subgroup of N, then $C \cap X$ is a finite π-group. Repeated use of Lemma 1b) yields that ϕ acts fixed-point freely on $X/O_\pi(X)$. But $O_\pi(X) = P \cap X$. Therefore ϕ acts fixed-point freely on N/P. Consequently $P \supseteq N\psi$ by Lemma 14 of [1]. Since G/N is abelian $[G,\phi]N = G\gamma.N$. Also by Lemma 2 we have $(N\psi)^G \supseteq (G\gamma.N)\psi$. Thus $P = P^G = (G\gamma.N)\psi$. The claims follow. □

Lemma 4. *Let ϕ be an automorphism of the metabelian group G with $\phi^m = 1$ for some positive integer m. Set $B = C_{G'}(\phi)^G$. Then with γ and ψ as in Lemma 2 we have $B \supseteq (G\gamma.G')\psi = ([G,\phi]G')\psi$.*

If $C_{G'}(\phi)$ is a π-group, then so is B and we have Lemma 3 for metabelian groups with the restriction on the order of ϕ relaxed. Also the conclusion of Lemma 4 is stronger than that of Lemma 3; for example, if $C_{G'}(\phi)$ is a finite π-group (or even just of finite exponent) then B has finite exponent while $O_\pi(G')$ may not have finite exponent.

Proof. Let $A = G'$. Then A is abelian, $A\psi\gamma =< 1 >$ and hence $A\psi \le C_A(\phi) \le B$. Consequently $B \supseteq (G\gamma.A)\psi$ by Lemma 2 and Lemma 4 follows. □

Proof of Theorem 1. In the notation of Lemma 3 we have $P \supseteq ([G,\phi]G')\psi$. If ϕ acts nontrivially on $H = [G,\phi]G'/P$, then H is nilpotent by [3, 6.4.2] and the proof of the latter theorem yields the bounds as stated. If ϕ acts trivially on H, then H has exponent p (or 1 in the trivial case) again by Lemma 3 and therefore H is nilpotent of class bounded as claimed, see [4, 7.18] for $p > 2$, the case $p = 2$ being well known and very elementary. □

Proof of Theorem 2. Repeat the proof of Theorem 3, but using Lemma 4 in the place of Lemma 3. □

Lemma 5. *Let ϕ be an automorphism of a group G with $\phi^2 = 1$ and with $C_G(\phi)$ a locally finite π-group for some set π of primes. Suppose $[G,\phi]$ contains a soluble normal subgroup S of G such that $[G,\phi]/S$ is a locally finite π-group. Assume either that $2 \notin \pi$ or that $|C_G(\phi)| = n$ is finite with $\pi \supseteq \{primes\, q : q \le n\}$. Then*

$$[G,\phi]' \le O_\pi(G' \cap [G,\phi]).$$

Proof. Of course if $\phi = 1$ the conclusion is vacuous, as indeed it is if $S = \{1\}$, so we assume otherwise. Also $< S, S\phi >$ is soluble, normal and ϕ-invariant, so we may also assume that S is ϕ-invariant. Suppose $C_G(\phi)$ is finite of order n and π contains all primes $q \le n$. Now $O_\pi(S)$ is locally finite and normal in G, so by Proposition 14 of [7] we may pass to $G/O_\pi(S)$ and hence assume that $O_\pi(S) = \{1\}$ in this case. Now assume $2 \notin \pi$. If A is any abelian section of $O_\pi(S)$, then $A = A^2$ and A is 2-torsion-free. Thus by repeated use of Lemma 15 of [7] we deduce that the centralizer of ϕ in $G/O_\pi(S)$ is isomorphic to a section of $C_G(\phi)$. Thus again we may assume that $O_\pi(S) = \{1\}$.

Let $A \leq S$ be an abelian ϕ-invariant normal subgroup of G that is maximal subject to these constraints. Since $O_\pi(A) = \{1\}$ we have that ϕ acts fixed-point freely on A and hence $[G, \phi, A] = \{1\}$ by Lemma 1 of [2]. If $A < S$ there exists a ϕ-invariant normal subgroup B of G with $B' \leq A < B \leq S$. Then B is nilpotent (of class ≤ 2), $O_\pi(B) \leq O_\pi(S) = \{1\}$ and ϕ acts fixed-point freely on B. Then B is abelian, e.g. by Higman's Theorem ([3, 5.1.1]). The maximal choice of A implies that $A = B$ and consequently $A = S$.

We have now shown that A is central in $[G, \phi]$ and $[G, \phi]/A$ is a locally finite π-group. By a generalization of Schur's Theorem, Corollary 2 of [4, 4.21], we have that $[G, \phi]'$ is a π-group. This implies that $[G, \phi]'$ is a π-group in general, that is even if $O_\pi(S) \neq \{1\}$, and hence $[G, \phi]' \leq G' \cap O_\pi([G, \phi]) = O_\pi(G' \cap [G, \phi])$. □

Remark. The central part of the proof of Lemma 5 yields the following. Let ϕ be an automorphism of a group G with $\phi^2 = 1$ and with $C_G(\phi)$ a locally finite π-group for some set π of primes. Suppose $[G, \phi]$ contains a soluble normal subgroup S of G such that $[G, \phi]/S$ is a locally finite π-group. If $O_\pi(S) = \{1\}$, then $[G, \phi]' \leq O_\pi(G' \cap [G, \phi])$. In particular if $[G, \phi]$ is soluble with $O_\pi([G, \phi]) = \{1\}$, then $[G, \phi]$ is abelian.

Proof of Theorem 3. Part a) follows at once from Lemma 5 and then Part b) follows from setting $S = [G, \phi]$. □

Proof of Theorem 4. Part a) follows at once from Lemma 5. For Part b) note that $[G, \phi]$ and hence $[G, \phi]^e$ are normal in G with $[G, \phi]/[G, \phi]^e$ a locally finite π-group. Thus b) follows from Part a). For Part c) simply choose $e = 1$ in Part b). Finally in Part d) we have $n = 1 = e$ and $\pi = \emptyset$. Thus d) follows from Part c). □

Proof of the Proposition. Let X be a finitely generated, ϕ-invariant subgroup of G. Then X is nilpotent, $C_X(\phi) = C \cap X$ is a finite π-group and $O_\pi(X) = P \cap X$ is the set of all π-elements of X. Repeated use of Lemma 1b) yields that $C_{X/(P \cap X)}(\phi) = \{1\}$. Thus ϕ acts fixed-point freely on $X/(P \cap X)$ and hence by Higman's Theorem (e.g. [3, 5.1.1]) the factor $X/(P \cap X)$ is nilpotent of class at most $h(p)$. Moreover if $p = 2$, we have $h(2) = 1, X' \leq P \cap X$ and ϕ inverts $X/(P \cap X)$. A trivial locallization argument using the finiteness of $|\phi|$ now yields the proposition. □

Proof of Corollary 2. Set $N = G'$ and $P = O_\pi(N)$. Then N/P is nilpotent by the Proposition. Also ϕ acts fixed-point freely on N/P, see the proof of the Proposition. Consequently Theorem 1 applies to G/P and Corollary 2 follows. □

Construction of Example 1. Let ω be a primitive p-th root of unity in the complex numbers $\mathbb{C}$; recall p is an odd prime. Set

$$a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$R = \mathbb{Z}[\omega] \leq \mathbb{C}$ and $X =< a, b >\leq GL(2, R)$; X is dihedral of order $2p$. Let $Y = R^{(2)} \cong \mathbb{Z}^{(2p-2)}$ and let G denote the split extension of Y by X. Clearly G is polycyclic of derived length 3 and Hirsch number $2p-2$ and is abelian-by-finite. Let ϕ denote conjugation by a. Then ϕ is an automorphism of G of order p. Also ϕ acts fixed-point freely on Y, so $C_G(\phi) =< a >$. In particular $|C_G(\phi)| = p < \infty$.

Trivially $[G, \phi]$ contains $[b, a] = a^2$ and $[(1, 0), \phi] = (1, 0)(a - 1) = (\omega - 1, 0)$. Since p is odd, a^2 of order p acts fixed-point freely on the torsion-free group Y. Therefore $< (\omega - 1, 0), a^2 >$ and $[G, \phi]$ are not periodic-by-(locally nilponent).

Construction of Example 2. The binary tetrahedral group T is the split extension of a quaternion group Q of order 8 by a cyclic group $< c >$ of order 3. Also T can be regarded as a subgroup of the multiplicative group of the real quaternion division algebra D, see [5, page 63]. Let $R = \mathbb{Z}[T] \leq D$, let G denote the subgroup $< Q, R >$ of the split extension of R by T and let ϕ denote the automorphism of G induced by conjugation by c.

Clearly ϕ has order 3 and acts fixed-point freely on R, so $C_G(\phi) = C_Q(c) = < -1 >$. Thus $|C_G(\phi)| = 2 < \infty$. Also $[Q, c] = Q$ and $[R, \phi] = R(c - 1) \neq \{0\}$. But Q acts fixed-point freely on R and trivially $R\cap < R(c - 1), Q >\geq R(c - 1)$ is torsion-free. Therefore $< R(c - 1), Q >$ is not periodic-by-(locally nilpotent) and consequently neither is $[G, \phi]$. Trivially G is polycyclic and abelian-by-(finite nilpotent).

Construction of Example 3. If $p = 2$, let $X =< a, c >$, where a has odd order, c has order 4 and $a^c = a^{-1}$. If p is odd, choose a prime q with p dividing $q-1$; such q always exist, indeed infinitely many do for each p by Dirichlet's Theorem. Let p^r be the largest power of p to divide $q - 1$. Set $X =< a, c >$, where a has order q, c has order p^{r+1} and c normalizes $< a >$ and acts on it as an automorphism of order p. Again this is always possible.

In both cases X embeds into the multiplicative group of a division ring D of characteristic zero by Amitsur's Theorem, see [5, 2.1.5]. Set $R = \mathbb{Z}[X] \leq D$ and let G denote the subgroup $< a, R >$ of the split extension of R by X. If ϕ denotes the automorphism of G induced by conjugation by c, then ϕ has order p^{r+1} (4 if $p = 2$) and acts fixed-point freely on R, on $< a >$ and hence on G. Finally $[a, \phi] \in < a > \setminus\{1\}$, $[R, \phi] = R(c-1) \neq \{0\}$ and $[G, \phi]$ is not periodic-by-(locally nilpotent) as in the previous two constructions. Trivially G is polycyclic, metabelian and abelian-by-finite.

In the above construction, if q is such that p^2 does not divide $q - 1$ (e.g. if $p \leq 36$ with $q = 3, 7, 11, 29, 23, 53, 103, 191, 47, 59, 311$ in ascending order of p), then we can choose ϕ as in Example 3 but now with order p^2. Such a q would exist for the prime p if $\zeta(p)-\zeta(p^2) < p/(p^2-1)$, but I have been unable to confirm this inequality (ζ denotes the Riemann zeta function; always $\zeta(p)-\zeta(p^2) \leq p/(p^2-1)$).

Concluding Remarks

Finally we consider what little we can say about $G/[G, \phi]$. Suppose A is an abelian normal subgroup of the group G. Let ϕ be an automorphism of G with $A\phi = A$ and $\phi^m = 1$ (actually $(\phi|_A)^m = 1$ would suffice) for some positive integer m. As above we define maps $\gamma, \psi : G \to G$ by $g\gamma = g^{-1}.g\phi$ and $g\psi = g.g\phi.g\phi^2.\dots.g\phi^{m-1}$ for all $g \in G$.

If $a \in A$, then $a^m \equiv a\psi$ modulo $A\gamma$ and $a\psi\gamma = 1$. Hence $A^m \leq A\gamma.A\psi \leq [A,\phi].C_A(\phi)$. Thus we have the series $\{1\} \leq [G,\phi] \leq [G,\phi]A^m \leq [G,\phi]A \leq G$, where $[G,\phi]A^m/[G,\phi]$ is isomorphic to a section of $C_A(\phi), [G,\phi]A/[G,\phi]A^m$ is abelian with exponent dividing m and $G/[G,\phi]A$ is a section of G/A. In fact if η denotes the natural projection of G onto $G/[G,\phi]$, then

$$[G,\phi]A^m/[G,\phi] \cong A^m\eta \leq C_A(\phi)\eta.$$

In particular if G is metabelian $A = G'$ and $C_A(\phi)$ is a π-group for some set π of primes, then $[G,\phi]A^m/[G,\phi]$ is an abelian π-group and $G/[G,\phi]A$ is abelian. Thus in this case $G/[G,\phi]$ is (an abelian π-group)-by-(abelian of exponent dividing m)-by-abelian. This is a very slight generalization of Theorem 4 of [2].

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