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WEIGHTED ESTIMATES FOR SOLUTIONS OF $\overline{\partial}$-EQUATIONS ON CLOSED POSITIVE $(1,1)$-CURRENTS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper, we give estimates of Hörmander type and Donnelly-Fefferman type for solutions of the $\overline{\partial}$-equation on a closed positive $(1,1)$-current T in a pseudoconvex domain of $\mathbb{C}^{n+1}$.

1. Introduction

The existence of solutions of $\overline{\partial}$-equations on pseudoconvex domains in $\mathbb{C}^n$ with L^2-estimates plays an important role in complex analysis of several variables. The first basic result in this direction is due to Hörmander (see [11, Lemma 4.4.1]). More precisely, he proved the following remarkable result: *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and φ be a real valued C^2 smooth function on Ω satisfying*

$$c\sum_{j=1}^{n}|\lambda_j|^2 \leqslant \sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}(z)\lambda_j\overline{\lambda}_k, \quad \forall z\in\Omega,\ \forall(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n,$$

where $c>0$ is a continuous function in Ω. Then for any differential form $g\in L^2_{(p,q+1)}(\Omega,\varphi)$ with $\overline{\partial}g=0$ there exists $u\in L^2_{(p,q)}(\Omega,\varphi)$ such that $\overline{\partial}u=g$ and

$$\int|u|^2e^{-\varphi}dV\leqslant 2\int|g|^2e^{-\varphi}/c\,dV,$$

where dV denotes the Lebesgue measure in $\mathbb{C}^n$.

The above theorem allows us to construct holomorphic functions with great flexibility. For instance, we can use this result to give an alternative proof of the fact that holomorphic functions on a complex submanifold of pseudoconvex domains extend to global holomorphic functions.

After that, in [9] H. Donnelly and C. Fefferman solved the $\overline{\partial}$ - equation for $\overline{\partial}$ - closed (p,q)-forms on pseudoconvex domains with estimates through the Kähler

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metric $i\partial\overline{\partial}\varphi$. We recall the following theorem which is one of the key results in [9].

Theorem. *Let φ and ψ be plurisubharmonic functions of class C^2 on a bounded pseudoconvex domain Ω. Suppose that φ satisfies the condition*

$$i\partial\varphi \wedge \overline{\partial}\varphi \leqslant mi\partial\overline{\partial}\varphi,$$

where $m > 0$ is a constant. Then for any $\overline{\partial}$-closed $(0,1)$-form g on Ω, there exists $u \in L^2(\Omega, \psi)$ such that $\overline{\partial}u = g$ and

$$\int |u|^2 e^{-\psi} dV \leqslant Cm \int |g|^2_{i\partial\overline{\partial}\varphi} e^{-\psi} dV,$$

where C is an absolute constant.

Next, by using a Böchner-Kodaira-Nakano's type identity, Berndtsson refined the above result of H. Donnelly and C. Fefferman and showed that we can choose $C = \frac{4}{\delta(1-\delta)^2}$, where $0 < \delta < 1$ is arbitrary (see [3], Theorem 3.1). Later on, by modifying techniques of Berndtsson in [6] Blocki gave better estimates than the mentioned above result of Berndtsson. More recently, in [1] H. Ahn and N. Q. Dieu established Donnelly-Fefferman theorem for $\overline{\partial}$-closed $(0, r)$-forms in q-pseudoconvex domains and in [10] we have given the weighted estimates of Hörmander type and Donnelly-Fefferman type for solutions of $\overline{\partial}$-equations in q-pseudoconvex domains. However, the investigation of solutions of $\overline{\partial}$-equation on currents encounter many difficulties on notions and techniques. Up to now, results on the direction of study are limited. The first article deals with this problem is [5]. In this seminal work, B. Berntsson and N. Sibony have shown the existence of solutions of the $\overline{\partial}$-equation on a closed positive current T of bidegree $(1, 1)$. In particular, their method covers the case where T is the current induced by integration on a pure dimensional complex variety. Following ideas given in [5], in this paper, we give some weighted estimates of Hörmander type and Donnelly-Fefferman type for solutions of the $\overline{\partial}$-equation on a closed positive (1,1)-current T in a pseudoconvex domain of $\mathbb{C}^{n+1}$. Namely, we prove the following two theorems.

Theorem 1.1. *Let D be a pseudoconvex domain in $\mathbb{C}^{n+1}$ and let $T \geqslant 0$ be a $\overline{\partial}$-closed $(1,1)$-current in D. Assume that ω is a Kähler form of a smooth Kähler metric in D and φ is a plurisubharmonic function in D satisfying*

$$h^2\omega \leqslant i\partial\overline{\partial}\varphi,$$

where h is a positive continuous function in D. Then for any (n,q)-form f with $\overline{\partial}f \wedge T = 0$ there is an $(n, q-1)$-form u such that $\overline{\partial}u = f$ on T and

$$\|u\|^2_{\omega,T,\varphi} \leqslant \frac{1}{q} \, \|f/h\|^2_{\omega,T,\varphi}.$$

Theorem 1.2. *Let D be a pseudoconvex domain in $\mathbb{C}^{n+1}$ and let $T \geqslant 0$ be a smooth $\overline{\partial}$-closed $(1,1)$-current. Let φ and ψ be C^2-smooth plurisubharmonic* *functions such that $-e^{-\psi}$ is also a plurisubharmonic function. Assume that $0 < \delta < 1$ and ω is a Kähler form of a smooth complete Kähler metric in D such that*

$$i\partial\overline{\partial}\psi \geqslant \omega.$$

Then for any (n,q)-form f in D with $\overline{\partial} f \wedge T = 0$ there exists an $(n,q-1)$-form u such that $\overline{\partial} u = f$ on T and

$$\begin{aligned} c_{n-q+1} &\int \overline{*u} \wedge *u \wedge \omega_{q-1} \wedge T e^{-\varphi+\delta\psi} \\ &\leqslant \frac{4}{q\delta(1-\delta)^2} c_{n-q} \int \overline{*f} \wedge *f \wedge \omega_q \wedge T e^{-\varphi+\delta\psi}, \end{aligned}$$

where $*$ denotes the Hodge operator on T.

The paper is organized as follows. In Section 2, we recall notions needed in the article, mostly from [5]. Section 3 is devoted to the proofs of the main results of the paper. In the course of proving these results, we also need Propositions 3.1 and 3.2 which are slight extensions of Theorem 7.2 and Lemma 8.3 in [5].

2. Preliminaries

First, we recall some notions and results of the theory of currents and the $\overline{\partial}$-equation on currents which will be used throughout the paper. We refer the readers to [5, 8] for more details on these matters.

2.1. Let V be a complex vector space of dimension n. A (q,q)-form u is strongly positive if it belongs to the cone generated by the forms

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_q \wedge \overline{\alpha}_q,$$

where $\alpha_j \in \bigwedge^{1,0}(V^*)$.

A form $u \in \bigwedge^{p,p}(V^*)$ is positive if and only if $u \wedge v$ is positive for every strongly positive form v of bidegree $(n-q, n-q)$. On a domain Ω of $\mathbb{C}^n$ a differential form $u \in C^\infty_{p,p}(\Omega)$ is strongly positive (resp., positive) if for every $z \in \Omega$, $u(z)$ is strongly (resp., positive). The two notions of positivity coincide for $p = 0, 1, n-1, n$.

The space $\mathcal{D}'_{(n-r,n-s)}(\Omega)$ of currents of bidegree (r,s) on Ω is by definition the dual of the space $\mathcal{D}_{(n-r,n-s)}(\Omega)$ of test forms on Ω of bidegree $(n-r, n-s)$, with respect to the usual inductive topology on the space of test forms.

A current T on Ω of bidegree (p,p) can then be represented as a differential form of bidegree (p,p) with distributions as coefficients.

A current T of bidegree (p,p) is positive if $\langle T, u\rangle \geqslant 0$ for all test forms $u \in \mathcal{D}_{(n-p,n-p)}(\Omega)$ that are strongly positive. It is easy to see that the notion is local and the weak limit of positive currents is positive. In this paper, assume that T is a current of bidegree $(1,1)$. Then we say that T is strictly positive and write $T > 0$ if there exists a positive continuous function h such that $T - hi\partial\overline{\partial}|z|^2 \geqslant 0$.

2.2. Assume that T a positive current of bidegree $(1,1)$ on a pseudoconvex domain D in $\mathbb{C}^{n+1}$. Below if we say that a positive $(1,1)$-current T, it means that T is a positive current of bidegree $(1,1)$. Assume that u and f are smooth differential forms on D. We say that $\overline{\partial}u = f$ on T if $\overline{\partial}u \wedge T = f \wedge T$ in the sense of currents.

In the case $\overline{\partial}T = 0$ and u, f are differential forms on D coefficients of which are locally integrable with respect to the trace measure of T (see below) then we also say $\overline{\partial}u = f$ on T if $\overline{\partial}(u \wedge T) = f \wedge T$ in the sense of currents. Moreover, we say that u is $\overline{\partial}$-closed on T if $\overline{\partial}u \wedge T = 0$.

2.3. Let T be a positive (1,1)-current and let $\omega > 0$ be a (1,1)-form. Let $c_q = (-1)^{q(q+1)/2} i^q = (-i)^{q^2}$. These numbers are chosen so that if g is a form of bidegree $(0,q)$ in $\mathbb{C}^n$, then $c_q g \wedge \overline{g}$ is a positive form.

We define the two Hodge operators $* : \bigwedge_T^{n,q} \longrightarrow \bigwedge_T^{n-q,0}$, $f \longmapsto *f$ such that $f = \overline{c_{n-q}}(*f) \wedge \omega_q$ on T, where $\omega_q = \omega^q/q!$ and $* : \bigwedge_T^{n-q,0} \longrightarrow \bigwedge_T^{n,q}$, $g \longmapsto *g$ such that $*g = c_{n-q} g \wedge \omega_q$ on T.

Assume that $(p,q) = (0,q)$ or (n,q) and f is a (p,q)-form such that $*f$ is smooth. Then we put

$$\vartheta f = \varepsilon_{p,q} * \partial * f,$$

where $\varepsilon_{p,q}$ is chosen so that if T is closed, then $(g, \vartheta f)_{\omega,T} = (\overline{\partial} g, f)_{\omega,T}$ for every g which is a $(p,q-1)$-form with support in D. Here, we recall that if f,g are $(0,q)$-forms, then

$$(f,g)_{\omega,T} = c_q \int f \wedge \overline{g} \wedge \omega_{n-q} \wedge T.$$

In the case f,g are (n,q)-forms then

$$(f,g)_{\omega,T} = \overline{c_{n-q}} \int *f \wedge \overline{*g} \wedge \omega_q \wedge T.$$

Assume that φ is a weighted function. Then we put

$$\vartheta_\varphi = e^{\varphi} \vartheta e^{-\varphi}.$$

Now if f,g are $(0,q)$-forms then we define the weighted scalar product

$$(f,g)_{\omega,T,\varphi} = c_q \int f \wedge \overline{g} \wedge \omega_{n-q} \wedge T e^{-\varphi}$$

and in the case f,g are (n,q)-forms then

$$(f,g)_{\omega,T,\varphi} = \overline{c_{n-q}} \int *f \wedge \overline{*g} \wedge \omega_q \wedge T e^{-\varphi}.$$

2.4. Let $T \geqslant 0$ be a positive $(1,1)$-form and let σ_T be the trace of T with respect to ω considered as a form of maximal degree, i.e.,

$$\sigma_T = T \wedge \omega_n.$$

Assume that φ is a weighted function and $f \in \bigwedge_T^{0,q}(\mathbb{C}^n)$. The norm of f on T associated to φ is then defined by

$$|f|^2_{\omega,T,\varphi}\sigma_T = c_q f \wedge \overline{f} \wedge \omega_{n-q} \wedge Te^{-\varphi}.$$

In the case f is a $(q,0)$-form then we put $|f|_{\omega,T,\varphi} = |\overline{f}|_{\omega,T,\varphi}$. When f is a (n,q)-form, we put $|f|^2_{\omega,T,\varphi} := |*f|^2_{\omega,T,\varphi}$.

Let now $T \geqslant 0$ be a $(1,1)$-current in $\mathbb{C}^{n+1}$. Such a current can be written as

$$T = \frac{i}{2}\sum_{j,k} T_{j\overline{k}} dz_j \wedge d\overline{z}_k,$$

where the coefficients T_{jk} are measures absolutely continuous with respect to the trace measure σ_T. Let $tr(T)$ be the $(0,0)$-current defined by $tr(T)\omega_{n+1} = \sigma_T$ and written $T = \widetilde{T}tr(T)$ with $\widetilde{T} \geqslant 0$ which is a $(1,1)$-form. If f is a (p,q)-form in $\mathbb{C}^{n+1}$ where $(p,q) = (0,q)$ or (n,q), then we define the L^2-norm of f on T associated to φ by

$$\|f\|^2_{\omega,T,\varphi} = \int |f|^2_{\omega,\widetilde{T},\varphi}\sigma_T.$$

2.5. Let $\varrho \geq 0$ be a radial function with compact support in the unit ball of $\mathbb{C}^n$ such that $\int \varrho(z)dV = 1$. For each $\varepsilon > 0$ put $\varrho_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\varrho(z/\varepsilon)$, $z \in \mathbb{C}^n$. Let $T = \sum_{I,K}{}' T_{I,K} dz_I \wedge d\overline{z}_K$ be a (p,q)-current, where $\sum_{I,K}{}'$ means that the summation is taken over increasing multi-indices only, of lengths p and q, respectively. Then we set

$$T_\varepsilon = T * \varrho_\varepsilon = \sum_{I,K}{}' T_{I,K} * \varrho_\varepsilon dz_I \wedge d\overline{z}_K.$$

Note that T_ε is a smooth differential (p,q)-form and $T_\varepsilon \to T$ in the weak topology of currents as $\varepsilon \to 0$. Now assume that T is a positive (q,q)-current. Then the coefficients T_{IK} of T are measures. For each test $(n-q,n-q)$-form ψ, by Fubini's theorem we have

$$\int T_\varepsilon \wedge \psi = \int T \wedge \psi_\varepsilon.$$

3. Proofs of results

We begin with the following result which is a slight extension of Theorem 7.2 in [5] for the case with weights.

Proposition 3.1. *Let T be a smooth strictly positive $(1,1)$-form in a domain D in $\mathbb{C}^{n+1}$ such that $\overline{\partial}T = 0$ and ω be a complete smooth Kähler metric in D. Then for every C^2 smooth functions k, φ on D, $k > 0$ we have*

$$\int c_{n-q}\overline{*\alpha} \wedge *\alpha \wedge i\partial\overline{\partial}\varphi \wedge \omega_{q-1} \wedge kTe^{-\varphi} - \int c_{n-q}\overline{*\alpha} \wedge *\alpha \wedge i\partial\overline{\partial}k \wedge \omega_{q-1} \wedge Te^{-\varphi}$$

$$\leqslant \|\vartheta_\varphi\alpha\|^2_{\omega,kT,\varphi} + 2\left|\int \overline{\vartheta_\varphi\alpha} \wedge *\alpha \wedge \partial k \wedge Te^{-\varphi}\right| + \|\overline{\partial}\alpha\|^2_{\omega,kT,\varphi}$$

for every (n,q)-form $\alpha \in Dom(\vartheta_\varphi) \cap Dom(\overline{\partial})$ where $* : \bigwedge_T^{n,q} \to \bigwedge_T^{n-q,0}$ denotes the Hodge operator.

Proof. We will use the method of Berndtsson and Sibony in [5]. Note that the two Hodge operators $* : \bigwedge_T^{n,q} \to \bigwedge_T^{n-q,0}$ and $* : \bigwedge_{kT}^{n,q} \to \bigwedge_{kT}^{n-q,0}$ are identical. By Theorem 7.2 in [5], we have the following estimate

$$\int c_{n-q}\overline{*f} \wedge *f \wedge i\partial\overline{\partial}\varphi \wedge \omega_{q-1} \wedge kTe^{-\varphi} - \int c_{n-q}\overline{*f} \wedge *f \wedge \omega_{q-1} \wedge i\partial\overline{\partial}(kT)e^{-\varphi}$$
$$+ \|\vartheta_\varphi f\|^2_{\omega,kT,\varphi} \leqslant 2\mathrm{Re}(\overline{\partial}\vartheta_\varphi f, f)_{\omega,kT,\varphi} + \|\overline{\partial}f\|^2_{\omega,kT,\varphi}$$

which holds for any test form f of bidegree (n,q) with support in D such that $*f$ is smooth. Hence the proposition is proved if we can find a sequence of test forms α_v with support in D such that $*\alpha_v$ is smooth and

$$\alpha_v \to \alpha, \quad \overline{\partial}\alpha_v \to \overline{\partial}\alpha, \quad \vartheta_\varphi\alpha_v \to \vartheta_\varphi\alpha \ \text{ in } L^2(T,\varphi). \tag{3.1}$$

To achieve this, first we approximate α by forms of compact support. Indeed, by the completeness of the smooth Kähler metric ω, Lemma 3.9.2 in [2] implies that there exists an exhausting sequence of cut-off functions χ_v with uniformly bounded gradients. Next, we are going to approximate $\alpha\chi_v$ by smooth forms. By Proposition 5.4 in [5], we can write $\alpha\chi_v = \eta_v \wedge \omega_q$ on T where η_v is a $(n-q,0)$-form. Since T is strictly positive, by Proposition 5.4 in [5], we have

$$\eta_v = \overline{c_{n-q}} * (\alpha\chi_v).$$

Moreover, the following estimate holds

$$\|\eta_v\|_{\omega,T,\varphi} \leqslant \sup_D |\chi_v| \|\alpha\|_{\omega,T,\varphi} < \infty.$$

Thus, η_v corresponds to a unique L^2-form with compact support in $\mathbb{C}^{n+1}$.

For $\varepsilon > 0$, we put

$$\alpha_{v,\varepsilon} = (\eta_v) * \varrho_\varepsilon \wedge \omega_q,$$

where $\eta_v * \varrho_\varepsilon$ is the convolution of η_v and ϱ_ε. We claim that the sequence $\{\alpha_{v,\varepsilon}\}$ converges to $\alpha\chi_v$ in $L^2(T,\varphi)$ when $\varepsilon \downarrow 0$. Indeed, we have

$$\|\alpha_{v,\varepsilon} - \alpha\chi_v\|_{\omega,T,\varphi} = \|\eta_v * \varrho_\varepsilon - \eta_v\|_{\omega,T,\varphi}.$$

However, because $\mathrm{supp}\eta_v \Subset \Omega$ we can choose $\widetilde{\Omega} \Subset \Omega$ such that $\mathrm{supp}\eta_v \Subset \widetilde{\Omega} \Subset \Omega$. Since $\varphi \in C^2(D)$, T is smooth, there exists a constant M depending on φ, T, ω such that

$$\|\eta_v * \varrho_\varepsilon - \eta_v\|_{\omega,T,\varphi} \leqslant M\|\eta_v * \varrho_\varepsilon - \eta_v\|_{L^2(\widetilde{\Omega})} \longrightarrow 0,$$

$\varepsilon \to 0$. Hence the desired conclusion follows. On the other hand, because T is smooth, $\overline{\partial}$ and ϑ_φ are differential operators with smooth coefficients depending on η_v. Now it follows from Proposition 1.5.2 in [2] or Lemma 4.1.4 in [11] that

$$\overline{\partial}\alpha_{v,\varepsilon} \to \overline{\partial}(\alpha\chi_v), \quad \vartheta_\varphi\alpha_{v,\varepsilon} \to \vartheta_\varphi(\alpha\chi_v) \ \text{ in } L^2(T,\varphi).$$

Finally, by using a standard diagonal argument, we get the sequence $\{\alpha_v\}$ satisfying (3.1). The proof of the proposition is complete. □

Next, we give an extension of Lemma 8.3 in [5].

Proposition 3.2. *Let D be a bounded domain in $\mathbb{C}^{n+1}$ and let D_1 be an open set in $\mathbb{C}^{n+1}$ with $D \Subset D_1$. Let T be a non-negative $(1,1)$-current in D_1 and let $\omega > 0$ be a smooth $(1,1)$-form in D_1. For $\varepsilon > 0, \delta > 0$ we put*

$$T_{(\varepsilon)} = T_\varepsilon + \varepsilon\omega \quad \textit{and} \quad \omega^\delta = \omega + \delta i\partial\overline{\partial}|z|^2.$$

Assume that φ and h are two continuous functions on D_1. Define

$$\varphi_{(\varepsilon)}(z) = \sup_{|z-\zeta|\leqslant\varepsilon} \varphi(\zeta) \quad \textit{and} \quad h_{(\varepsilon)}(z) = \sup_{|z-\zeta|\leqslant\varepsilon} h(\zeta).$$

Then for every (n,q)-form f on T there is a (n,q)-form $f_{(\varepsilon)}$ on $T_{(\varepsilon)}$ such that

$$(f \wedge T)_{(\varepsilon)} = f_{(\varepsilon)} \wedge T_{(\varepsilon)}.$$

Moreover, for $\delta > 0$ we have

$$\|f_{(\varepsilon)}/h_{(\varepsilon)}\|_{\omega^\delta, T_{(\varepsilon)}, \varphi_{(\varepsilon)}} \leqslant \|f/h\|_{\omega,T,\varphi} \tag{3.2}$$

for $\varepsilon > 0$ sufficiently small.

We need the following lemma.

Lemma 3.3. *With notations and assumptions as in Proposition 3.2, assume that ψ is a $(0,n-q)$-form with compact support in D. Then*

$$c_{n-q}\int \psi \wedge \overline{\psi} \wedge (\omega_q \wedge h^2 T e^{\varphi})_\varepsilon \leqslant c_{n-q}\int h^2_{(\varepsilon)}\psi \wedge \overline{\psi} \wedge \omega^\delta_q \wedge T_{(\varepsilon)} e^{\varphi_{(\varepsilon)}}$$

for every ε sufficiently small.

Proof. Choose the balls $B(z_1,r_1),\ldots,B(z_m,r_m)$ such that $D \Subset \bigcup_{j=1}^m B(z_j,r_j) \Subset D_1$ and

$$\omega \leqslant \omega(z_j) + \frac{\delta}{2} i\partial\overline{\partial}|z|^2 \leqslant \omega + \delta i\partial\overline{\partial}|z|^2 \ \text{ on } \ B(z_j,r_j)$$

for every $j = 1,\ldots,m$. We can choose functions $\chi_j \in C_0^\infty(B(z_j,r_j))$, $j = 1,\ldots,m$ such that $0 \leqslant \chi_j \leqslant 1$ and $\sum_{j=1}^m \chi_j = 1$. Put

$$\varepsilon_0 = \min\{d(\mathrm{supp}\psi_j, \partial B(z_j,r_j)) : \ j = 1,\ldots,m\}.$$

For every $0 < \varepsilon < \varepsilon_0/2$, we have

$$\begin{aligned}
&c_{n-q}\int \psi\wedge\overline{\psi}\wedge(\omega_q\wedge h^2Te^{\varphi})_\varepsilon\\
&= c_{n-q}\sum_{j=1}^m\int \chi_j\psi\wedge\overline{\psi}\wedge(\omega_q\wedge h^2Te^{\varphi})_\varepsilon\\
&= c_{n-q}\sum_{j=1}^m\int (\chi_j\psi\wedge\overline{\psi})_\varepsilon\wedge\omega_q\wedge h^2Te^{\varphi}\\
&\leqslant c_{n-q}\sum_{j=1}^m\int (\chi_j\psi\wedge\overline{\psi})_\varepsilon\wedge\left(\omega(z_j)+\frac{\delta}{2}i\partial\overline{\partial}|z|^2\right)_q\wedge h^2Te^{\varphi}\\
&\leqslant c_{n-q}\sum_{j=1}^m\int \chi_j\psi\wedge\overline{\psi}\wedge\left(\omega(z_j)+\frac{\delta}{2}i\partial\overline{\partial}|z|^2\right)_q\wedge(h^2Te^{\varphi})_\varepsilon\\
&\leqslant c_{n-q}\sum_{j=1}^m\int \chi_j\psi\wedge\overline{\psi}\wedge\left(\omega(z_j)+\frac{\delta}{2}i\partial\overline{\partial}|z|^2\right)_q\wedge h^2_{(\varepsilon)}T_\varepsilon e^{\varphi(\varepsilon)}\\
&\leqslant c_{n-q}\sum_{j=1}^m\int \chi_j\psi\wedge\overline{\psi}\wedge\omega_q^\delta\wedge h^2_{(\varepsilon)}T_\varepsilon e^{\varphi(\varepsilon)}\\
&\leqslant c_{n-q}\int \psi\wedge\overline{\psi}\wedge\omega_q^\delta\wedge h^2_{(\varepsilon)}T_\varepsilon e^{\varphi(\varepsilon)}.
\end{aligned}$$

Therefore, the proof is complete. □

Proof of Proposition 3.2. We use the arguments as in the proof of Lemma 8.3 in [5]. First, the existence of $f_{(\varepsilon)}$ follows from Proposition 5.4 in [5]. Next, we prove (3.2). Assume

$$\|f/h\|_{\omega,T,\varphi} = 1.$$

By the definition after Proposition 5.2 in [5], we have that if g is a (n,q)-form, then

$$\|g\|_{\omega,T,\varphi} = \sup\{|\int g\wedge\psi\wedge Te^{-\varphi}| : \quad \|\psi\|_{\omega,T,\varphi}\leqslant 1\} = \|*g\|_{\omega,T,\varphi}, \tag{3.3}$$

where ψ is a test $(0,n-q)$-form. Hence it is enough to prove that for $\varepsilon > 0$ sufficiently small and for any test form ψ of bidegree $(0,n-q)$ satisfying

$$c_{n-q}\int h^2_{(\varepsilon)}\psi\wedge\overline{\psi}\wedge\omega_q^\delta\wedge T_{(\varepsilon)}e^{\varphi(\varepsilon)}\leqslant 1$$

we have

$$\left|\int f_{(\varepsilon)}\wedge\psi\wedge T_{(\varepsilon)}\right|\leqslant 1.$$

For this, we use the identity $(f \wedge T)_\varepsilon = f_{(\varepsilon)} \wedge T_{(\varepsilon)}$ and Schwarz's inequality to obtain

$$\begin{aligned} &\left|\int f_{(\varepsilon)} \wedge \psi \wedge T_{(\varepsilon)}\right|^2 = \left|\int (f\wedge T)_\varepsilon \wedge \psi\right|^2 \\ &= \left|\int f \wedge T \wedge \psi_\varepsilon\right|^2 = \left|\int \frac{f}{h} \wedge T \wedge he^{\varphi}\psi_\varepsilon e^{-\varphi}\right|^2 \\ &= \left|\int * \left(\frac{f}{h}\right) \wedge he^{\varphi}\psi_\varepsilon \wedge \omega_q \wedge T \wedge e^{-\varphi}\right|^2 \\ &\leqslant \|f/h\|^2_{\omega,T,\varphi} \, \|h\psi_\varepsilon e^{\varphi}\|^2_{\omega,T,\varphi} = c_{n-q}\int h^2\psi_\varepsilon \wedge \overline{\psi}_\varepsilon \wedge \omega_q \wedge Te^{\varphi} \end{aligned}$$

because $\|f/h\|_{\omega,T,\varphi} = 1$.
Hence by Lemma 8.2 in [5], we get

$$\begin{aligned} c_{n-q}\int h^2\psi_\varepsilon \wedge \overline{\psi}_\varepsilon \wedge \omega_q \wedge Te^{\varphi} &\leqslant c_{n-q}\int h^2(\psi\wedge\overline{\psi})_\varepsilon \wedge \omega_q \wedge Te^{\varphi} \\ &= c_{n-q}\int \psi \wedge \overline{\psi} \wedge (\omega_q \wedge h^2 Te^{\varphi})_\varepsilon. \end{aligned}$$

Applying Lemma 3.3 we get

$$c_{n-q}\int h^2\psi_\varepsilon \wedge \overline{\psi}_\varepsilon \wedge \omega_q \wedge Te^{\varphi} \leqslant c_{n-q}\int h^2_{(\varepsilon)}\psi \wedge \overline{\psi} \wedge \omega_q^{\delta} \wedge T_{(\varepsilon)}e^{\varphi(\varepsilon)} \leqslant 1$$

for every ε sufficiently small. This finishes the proof of the proposition. □

Now we prove the weighted estimates of Hörmander type for solutions of the $\overline{\partial}$-equation on a positive $(1,1)$-current (compared with Lemma 4.4.1 in [11]).

Proof of Theorem 1.1. We use similar reasonings as in the proof of Theorem 8.1 and Theorem 8.5 in [5]. Assume first that D is bounded, T, h, φ, f extend to a neighborhood of $\overline{D}$, T is strictly positive, T and φ are smooth in $\overline{D}$. Moreover, assume that ω is a complete smooth Kähler metric in D. Note that if α is a test form of bidegree (n,q), then $\vartheta_\varphi\alpha$ is a $(n,q-1)$-form on T. By applying Hahn-Banach's theorem to the antilinear form $\vartheta_\varphi\alpha \longmapsto (f,\alpha)_{\omega,T,\varphi}$, it suffices to show that

$$|(f,\alpha)_{\omega,T,\varphi}|^2 \leqslant \frac{1}{q}\ \|f/h\|^2_{\omega,T,\varphi}.\|\vartheta_\varphi\alpha\|^2_{\omega,T,\varphi} \tag{3.4}$$

for every test form α of bidegree (n,q).

The proof of (3.4) follows by standard arguments that we shall present briefly. We write

$$\alpha = \alpha^1 + \alpha^2,$$

where $\overline{\partial}\alpha^1 \wedge T = 0$ in the sense of currents and α^2 is orthogonal to the kernel of $\overline{\partial}$ in $L^2(T, e^{-\varphi})$. In particular, α^2 is orthogonal to any form of type $\overline{\partial}v$, where v

is a test $(n, q-1)$-form. Thus, $\vartheta_\varphi \alpha^2 = 0$ in the sense of currents and we have $\vartheta_\varphi \alpha = \vartheta_\varphi \alpha^1$. Since $(f,\alpha)_{\omega,T,\varphi} = (f,\alpha^1)_{\omega,T,\varphi}$, it is enough to check

$$|(f,\alpha^1)_{\omega,T,\varphi}|^2 \leqslant \frac{1}{q}\, \|f/h\|^2_{\omega,T,\varphi}.\|\vartheta_\varphi \alpha^1\|^2_{\omega,T,\varphi}. \tag{3.5}$$

For this, we apply Proposition 3.1 with $k \equiv 1$ to get

$$\int c_{n-q}\overline{*\alpha}^1 \wedge *\alpha^1 \wedge i\partial\overline{\partial}\varphi \wedge \omega_{q-1} \wedge Te^{-\varphi} \leqslant \|\vartheta_\varphi \alpha^1\|^2_{\omega,T,\varphi}.$$

Moreover, since $i\partial\overline{\partial}\varphi \geqslant h^2\omega$ on T we obtain

$$\begin{aligned} q\|h\alpha^1\|^2_{\omega,T,\varphi} &= \int c_{n-q}\overline{*\alpha}^1 \wedge *\alpha^1 \wedge h^2\omega \wedge \omega_{q-1} \wedge Te^{-\varphi} \\ &\leqslant \int c_{n-q}\overline{*\alpha}^1 \wedge *\alpha^1 \wedge i\partial\overline{\partial}\varphi \wedge \omega_{q-1} \wedge Te^{-\varphi} \\ &\leqslant \|\vartheta_\varphi \alpha^1\|^2_{\omega,T,\varphi}. \end{aligned}$$

Hence by using Schwarz's inequality, the inequality (3.5) follows. This is the proof of the theorem for the case of smooth weight functions φ. In the general case, it is enough to consider the smoothing sequence $\varphi * \varrho_\varepsilon, \varepsilon > 0$ and applying the first case proved above with h, φ and ω replaced by $\inf_{|z-\xi|\leqslant\varepsilon} h(z)$, ω_ε and φ_ε with the note that

$$\inf_{|z-\xi|\leqslant\varepsilon} h^2(z)\omega_\varepsilon(z) \leqslant (h^2\omega)_\varepsilon(z) \leqslant i\partial\overline{\partial}\varphi_\varepsilon(z).$$

After that, by using the Lebesgue dominated convergence theorem, we get the desired statement for h, φ and ω by passing to the limit $\varepsilon \to 0$.

Now we consider the case of general T (not necessarily smooth). We can assume that φ is smooth. Consider an exhaustion of D by relatively compact strictly pseudoconvex subdomains, and it is enough to prove the theorem in each such subdomain. In other words, we may assume D is strictly pseudoconvex and that all data T, h, φ, f extend to a neighborhood of $\overline{D}$. The proof of Lemma 3.9.1 in [2] implies that there exists a positive plurisubharmonic function ψ in D such that

$$\eta = i\partial\overline{\partial}\psi$$

defines a complete Kähler metric in D. Put

$$\omega^\lambda = \omega + \lambda\eta \ \text{ and } \ \varphi^\lambda = \varphi + (2\lambda \sup_D h)\psi,$$

where $\lambda > 0$. It suffices to prove the theorem with φ and ω replaced by φ^λ and ω^λ. After that, by using the Lebesgue dominated convergence theorem, we get the desired statement for φ and ω by letting $\lambda \to 0$. To this end, we observe that ω^λ is a complete Kähler metric in D. Fix $\delta > 0$ and assume that

$$\|f/h\|_{\omega,T,\varphi} = 1.$$

Put

$$T_{(\varepsilon)} = T_\varepsilon + \varepsilon\omega^\lambda,\ \omega^{\lambda,\delta} = \omega^\lambda + \delta i\partial\overline{\partial}|z|^2,\ \varphi^{\delta,\lambda} = \varphi^\lambda + (2\delta \sup_D h)(|z|^2 + 1)$$

and

$$\varphi_{(\varepsilon)}^{\delta,\lambda} = \varphi_\varepsilon + (2\lambda \sup_D h)\psi + (2\delta \sup_D h)(|z|^2 + 1).$$

We write

$$(f \wedge T)_\varepsilon = f_{(\varepsilon)} \wedge T_{(\varepsilon)}.$$

Moreover, if $\omega^\delta = \omega + \delta i\partial\overline{\partial}|z|^2$, then by Proposition 3.2, we have

$$\|f_{(\varepsilon)}/h_{(\varepsilon)}\|_{\omega^\delta, T_{(\varepsilon)}, \varphi_{(\varepsilon)}} \leqslant 1$$

for every ε sufficiently small. Hence we have

$$\lim_{\lambda\to 0} \|f_{(\varepsilon)}/h_{(\varepsilon)}\|_{\omega^{\lambda,\delta}, T_{(\varepsilon)}, \varphi^\lambda_{(\varepsilon)}} = \|f_{(\varepsilon)}/h_{(\varepsilon)}\|_{\omega^\delta, T_{(\varepsilon)}, \varphi_{(\varepsilon)}} \leqslant 1.$$

On the other hand, we have $\varphi^\lambda_{(\varepsilon)} < \varphi^{\lambda,\delta}$ and $h_{(\varepsilon)}\omega^{\lambda,\delta} \leqslant i\partial\overline{\partial}\varphi^{\lambda,\delta}$ when ε sufficiently small, so by the previous result there is a $(n, q-1)$-form $u_{(\lambda,\delta,\varepsilon)}$ on $T_{(\varepsilon)}$ such that

$$\overline{\partial}u_{(\lambda,\delta,\varepsilon)} \wedge T_{(\varepsilon)} = f_{(\varepsilon)} \wedge T_{(\varepsilon)} = (f \wedge T)_{(\varepsilon)}$$

and

$$\begin{aligned}\|u_{(\lambda,\delta,\varepsilon)}\|^2_{\omega^{\lambda,\delta}, T_{(\varepsilon)}, \varphi^{\lambda,\delta}} &\leqslant \frac{1}{q}\|f_{(\varepsilon)}/h_{(\varepsilon)}\|^2_{\omega^{\lambda,\delta}, T_{(\varepsilon)}, \varphi^{\lambda,\delta}} \\ &\leqslant \frac{1}{q}\|f_{(\varepsilon)}/h_{(\varepsilon)}\|^2_{\omega^{\lambda,\delta}, T_{(\varepsilon)}, \varphi^\lambda_{(\varepsilon)}}.\end{aligned}$$

Hence by repeating the arguments in the end of the proof of Theorem 8.1 in [5] and by letting $\lambda \to 0,\ \varepsilon \to 0,\ \delta \to 0$ respectively, we finish the proof of the theorem. □

Next, we will give a result which extends Theorem 1.4 in [10] to all closed positive $(1,1)$-currents T.

Corollary 3.4. *Let D, T, ω be as in Theorem 1.1. Assume that μ is a positive C^2-function, h is a positive continuous function and φ is a plurisubharmonic function in D satisfying*

$$i\partial\overline{\partial}\mu \leqslant \mu(i\partial\overline{\partial}\varphi - h\omega)$$

in the sense of currents. Then for any (n,q)-form f with $\overline{\partial}f \wedge T = 0$ there is a $(n, q-1)$-form u such that $\overline{\partial}u = f$ on T and

$$\begin{aligned}&c_{n-q+1}\int \overline{*u} \wedge *u \wedge \omega_{q-1} \wedge Te^{-\varphi}\mu \\ &\qquad\leqslant \frac{1}{q}c_{n-q}\int \overline{*f} \wedge *f \wedge \omega_q \wedge Te^{-\varphi}\frac{\mu}{h}.\end{aligned}$$

Proof. Put $\psi = -\ln\mu$. Then by the proof of Theorem 1.3 in [10], it follows that $\varphi + \psi$ is a strictly plurisubharmonic function in D and

$$h\omega \leqslant i\partial\overline{\partial}(\varphi+\psi)$$

in the sense of currents. So by Theorem 1.1, there is a $(n, q-1)$-form u on T such that

$$\overline{\partial}u \wedge T = f \wedge T$$

and

$$\begin{aligned} c_{n-q+1} \int \overline{*u} \wedge *u \wedge \omega_{q-1} \wedge Te^{-\varphi-\psi} \\ \leqslant \frac{1}{q} c_{n-q} \int \overline{*f} \wedge *f \wedge \omega_q \wedge Te^{-\varphi-\psi} \frac{1}{h}. \end{aligned}$$

The proof of the corollary is complete. □

We now prove weighted estimates of Donnelly-Fefferman type for solutions of the $\overline{\partial}$-equation on smooth positive $(1,1)$-currents.

Proof of Theorem 1.2. We will consider two cases. First, we assume that T is strictly positive. By Proposition 3.1 with k being replaced by $e^{-\delta\psi}$, we get

$$\begin{aligned} & -c_{n-q} \int \overline{*\alpha^1} \wedge *\alpha^1 \wedge i\partial\overline{\partial}e^{-\delta\psi} \wedge \omega_{q-1} \wedge Te^{-\varphi} \\ & \leqslant \|\vartheta_\varphi \alpha^1\|^2_{\omega, e^{-\delta\psi}T, \varphi} + 2\left|\int \overline{\vartheta_\varphi \alpha^1} \wedge *\alpha^1 \wedge \partial e^{-\delta\psi} \wedge Te^{-\varphi}\right| \\ & = \|\vartheta_\varphi \alpha^1\|^2_{\omega, T, \varphi+\delta\psi} + 2\left|\int \overline{\vartheta_\varphi \alpha^1} \wedge *\alpha^1 \wedge (\delta\partial\psi) \wedge Te^{-\varphi-\delta\psi}\right| \\ & \leqslant \left(1 + \frac{2\delta}{1-\delta}\right) \|\vartheta_\varphi \alpha^1\|^2_{\omega, T, \varphi+\delta\psi} + \frac{1-\delta}{2\delta} \|\delta\partial\psi \wedge *\alpha^1\|^2_{\omega, T, \varphi+\delta\psi} \end{aligned} \tag{3.6}$$

for every (n, q)-form α^1 in D such that $\overline{\partial}\alpha^1 \wedge T = 0$ and $\alpha^1 \in Dom(\vartheta_\varphi)$.

On the other hand, because $-e^{-\psi}$ is plurisubharmonic, we have

$$i\partial\overline{\partial}(-e^{-\psi}) = e^{-\psi}(i\partial\overline{\partial}\psi - i\partial\psi \wedge \overline{\partial}\psi) \geqslant 0. \tag{3.7}$$

Hence we obtain

$$\begin{aligned} & \|\delta\partial\psi \wedge *\alpha^1\|^2_{\omega, T, \varphi+\delta\psi} \\ & = c_{n-q} \int \overline{*\alpha^1} \wedge *\alpha^1 \wedge \delta^2 i\partial\psi \wedge \overline{\partial}\psi \wedge \omega_{q-1} \wedge Te^{-\varphi-\delta\psi} \\ & \leqslant c_{n-q}\delta^2 \int \overline{*\alpha^1} \wedge *\alpha^1 \wedge i\partial\overline{\partial}\psi \wedge \omega_{q-1} \wedge Te^{-\varphi-\delta\psi}. \end{aligned}$$

Combining this with (3.6) and (3.7), we obtain

$$\begin{aligned}\delta(1-\delta)c_{n-q}&\int \overline{*\alpha^1}\wedge *\alpha^1\wedge i\partial\overline{\partial}\psi\wedge\omega_{q-1}\wedge Te^{-\varphi-\delta\psi}\\ &\leqslant\left(1+\frac{2\delta}{1-\delta}\right)\|\vartheta_\varphi\alpha^1\|^2_{\omega,T,\varphi+\delta\psi}\\ &\quad+\frac{\delta(1-\delta)}{2}c_{n-q}\int\overline{*\alpha^1}\wedge *\alpha^1\wedge i\partial\overline{\partial}\psi\wedge\omega_{q-1}\wedge Te^{-\varphi-\delta\psi}.\end{aligned}$$

Thus, it follows that

$$\|\alpha^1\|^2_{\omega,T,\varphi+\delta\psi}\leqslant\frac{4}{q\delta(1-\delta)^2}\|\vartheta_\varphi\alpha^1\|^2_{\omega,T,\varphi+\delta\psi}.$$

Moreover, for any test form α of bidegree (n,q) and by using arguments as in the proof of Theorem 1.1, we can write

$$\alpha=\alpha^1+\alpha^2,$$

where $\overline{\partial}\alpha^1\wedge T=0$ and α^2 is orthogonal to the kernel of $\overline{\partial}$ in $L^2_{(n,q)}(\omega,T,\varphi)$. Then, in particular, α^2 is orthogonal to any form of type $\overline{\partial}\beta$ so $\vartheta_\varphi\alpha^2=0$ in the sense of currents. Hence $\vartheta_\varphi\alpha=\vartheta_\varphi\alpha^1$. On the other hand, because $\overline{\partial}f\wedge T=0$ then

$$(f,\alpha)_{\omega,T,\varphi}=(f,\alpha^1)_{\omega,T,\varphi}.$$

We consider the linear form L defined on the range of ϑ_φ by

$$L(\vartheta_\varphi\alpha)=(f,\alpha)_{\omega,T,\varphi}=(f,\alpha^1)_{\omega,T,\varphi}.$$

Since $\|\vartheta_\varphi\alpha^1\|_{\omega,T,\varphi+\delta\psi}=\|\vartheta_\varphi\alpha\|_{\omega,T,\varphi+\delta\psi}$, we have

$$\begin{aligned}|L(\vartheta_\varphi\alpha)|&\leqslant\|f\|^2_{\omega,T,\varphi-\delta\psi}\|\alpha^1\|^2_{\omega,T,\varphi+\delta\psi}\\ &\leqslant\frac{4}{q\delta(1-\delta)^2}\|f\|^2_{\omega,T,\varphi-\delta\psi}\|\vartheta_\varphi\alpha\|^2_{\omega,T,\varphi+\delta\psi}.\end{aligned}$$

An application of Hahn-Banach's theorem implies that there exists a $(n,q-1)$-form v such that

$$(v,\vartheta_\varphi\alpha)_{\omega,T,\varphi+\delta\psi}=L(\vartheta_\varphi\alpha)=(f,\alpha)_{\omega,T,\varphi}$$

and

$$\|v\|^2_{\omega,T,\varphi+\delta\psi}\leqslant\frac{4}{q\delta(1-\delta)^2}\|f\|^2_{\omega,T,\varphi-\delta\psi}.$$

By choosing $u=e^{-\delta\psi}v$, we complete the proof for the first case.

Next, the general case where T is an arbitrary smooth positive $(1,1)$ - current. We may normalize so that

$$\|f\|_{\omega,T,\varphi-\delta\psi}=1.$$

For $\eta>0$, we define

$$T^{(\eta)}=T+\eta i\partial\overline{\partial}|z|^2.$$

Then $T^{(\eta)}$ is a strictly smooth positive $(1,1)$-current. In view of Proposition 5.4 in [5], we can write

$$f \wedge T = f^{(\eta)} \wedge T^{(\eta)},$$

where $f^{(\eta)}$ is a (n,q)-form which is $\overline{\partial}$ - closed on $T^{(\eta)}$. Moreover, since $\|\psi\|_{\omega,T^{(\eta)},\varphi-\delta\psi} \geqslant \|\psi\|_{\omega,T,\varphi-\delta\psi}$ for every $(0,n-q)$-form, by the definition (3.3) of (n,q)-forms we have

$$\begin{aligned}\|f^{(\eta)}\|_{\omega,T^{(\eta)},\varphi-\delta\psi} &= \sup\{|\int f^{(\eta)} \wedge \psi \wedge T^{(\eta)} e^{-\varphi+\delta\psi}| : \|\psi\|_{\omega,T^{(\eta)},\varphi-\delta\psi} \leqslant 1\} \\ &= \sup\{|\int f \wedge \psi \wedge T e^{-\varphi+\delta\psi}| : \|\psi\|_{\omega,T^{(\eta)},\varphi-\delta\psi} \leqslant 1\} \\ &\leqslant \sup\{|\int f \wedge \psi \wedge T e^{-\varphi+\delta\psi}| : \|\psi\|_{\omega,T,\varphi-\delta\psi} \leqslant 1\} \\ &= \|f\|_{\omega,T,\varphi-\delta\psi} = 1.\end{aligned}$$

So from the first case proved above, it follows that there is an $(n,q-1)$-form $u^{(\eta)}$ on $T^{(\eta)}$ such that

$$\overline{\partial} u^{(\eta)} \wedge T^{(\eta)} = f^{(\eta)} \wedge T^{(\eta)} = f \wedge T$$

and

$$\|u^{(\eta)}\|^2_{\omega,T^{(\eta)},\varphi-\delta\psi} \leqslant \frac{4}{q\delta(1-\delta)^2} \|f^{(\eta)}\|^2_{\omega,T^{(\eta)},\varphi-\delta\psi} \leqslant \frac{4}{q\delta(1-\delta)^2}.$$

By letting $\eta \to 0$ and repeating the arguments in the end of the proof of Theorem 8.1 in [5], we finish the proof of the theorem. □

Remark 3.5. For a pseudoconvex domain $D \subset \mathbb{C}^{n+1}$ with $D \neq \mathbb{C}^{n+1}$, the assumption on the smoothness of T in Theorem 1.2 is necessary in view of our smoothing technique. More precisely, if we replace T by the smooth $(1,1)$ forms $T_\varepsilon = T * \varrho_\varepsilon$ on domains D_ε then the condition about the completeness of ω on D_ε are not satisfied. Indeed, suppose otherwise then there exists a domain $D_1 \Subset D$ and a complete Kähler form ω_1 on D_1 such that

$$i\partial\overline{\partial}\psi \geqslant \omega_1. \tag{3.8}$$

Since ψ is smooth on $\overline{D}_1 \Subset D$ then there exists $M > 0$ such that

$$Mi\partial\overline{\partial}|z|^2 \geqslant i\partial\overline{\partial}\psi. \tag{3.9}$$

On the other hand, because ω_1 is complete then there exists a sequence of cut-off functions $\{\chi_j\}$ such that $0 \leqslant \chi_j \leqslant 1$, $\mathrm{supp}\chi_j \Subset D_1$, $\{\chi_j\} \uparrow 1$ on D_1 and $|d\chi_j|_{\omega_1} \leqslant C$ where C is a constant. Hence from (3.8) and (3.9), we get

$$C \geqslant |d\chi_j|_{\omega_1} \geqslant |d\chi_j|_{i\partial\overline{\partial}\psi} \geqslant |d\chi_j|_{Mi\partial\overline{\partial}|z|^2} = \frac{1}{M}|d\chi_j|_{i\partial\overline{\partial}|z|^2}.$$

By using the Taylor expansion of χ_j for sufficiently larger j at a point of the boundary of D_1, we get a contradiction.

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