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ON THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}}$

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Abstract. In this article, we study the global stability and the asymptotic properties of the positive solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

where the parameters A, B, p and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers, while k is a positive integer number. Some numerical examples will be given to illustrate our results.

1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [2, 17, 20, 30]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n), \mu > 0, n \geq 0$. This behavior is ranging according to the value of μ, from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [1, 7-15, 22-49] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, see [3-7, 11, 18, 19] and the references cited therein. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

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The objective of this article is to investigate some qualitative behavior of the positive solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the parameters A, B, p and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers, while k is a positive integer number. Our interest in this article is to study the behavior of solutions of Eq. (1) in the general case where A and B are nonzero positive constants while k is a positive integer number. For the related work see [33-49]. Let us now recall some well know results [16] which will be useful in the sequel.

Definition 1.1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-k}), \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

where F is a continuous function which maps some set J^{k+1} into J where J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 1.2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (2). Then

(i) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called unstable if it is not locally stable.

Definition 1.3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 1.4. A positive semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to $\widetilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } \; l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} < \widetilde{x},$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \widetilde{x}.$$

A negative semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than $\widetilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either} \quad l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} \geq \widetilde{x},$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \widetilde{x}.$$

Definition 1.5. Eq. (2) is said to be permanent if there exist positive real numbers m and M such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (2) there exists a positive integer $N \geq -k$ which depends on the initial conditions, such that

$$m \leq x_n \leq M, \quad \text{for all} \quad n \geq N.$$

The linearized equation of the difference equation (2) about the equilibrium point $\widetilde{x}$ is the linear difference equation

$$z_{n+1} = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_n} z_n + \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_{n-k}} z_{n-k}. \tag{1.3}$$

The characteristic equation associated with Eq. (3) is

$$p(\lambda) = \lambda^{k+1} - p_0 \lambda^k - p_1 = 0, \tag{1.4}$$

where

$$p_0 = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_n}, \qquad p_1 = \frac{\partial F(\widetilde{x}, \widetilde{x})}{\partial x_{n-k}}. \tag{1.5}$$

1.1. **Equilibrium points.** In this section, we examine the positive equilibrium points $\widetilde{x}$ of Eq. (1). The equilibrium points of Eq. (1) are the solutions of the equation

$$\widetilde{x} = (A + B)\widetilde{x} + \frac{1 + \widetilde{x}^2}{(p+1)\widetilde{x}}. \tag{1.6}$$

If $0 < A + B < 1$, and $(p+1)[1 - (A+B)] > 1$, then the equilibrium points of Eq. (1) are

$$\widetilde{x} = \pm \frac{1}{\sqrt{(p+1)[1 - (A+B)] - 1}}. \tag{1.7}$$

Theorem 1.6. ([16] The linearized stability theorem). *Suppose F is a continuously differentiable function defined on an open neighborhood of the equilibrium $\widetilde{x}$. Then the following statements are true.*

(i) *If all the roots of the characteristic equation* (4) *of the linearized equation* (3) *have absolute values less than one, then the equilibrium point $\widetilde{x}$ of Eq.* (2) *is locally asymptotically stable.*

(ii) *If at least one root of Eq.* (4) *has the absolute value greater than one, then the equilibrium point $\widetilde{x}$ of Eq.* (2) *is not locally stable.*

(iii) *If all the roots of Eq.* (4) *have absolute values greater than one, then the equilibrium point $\widetilde{x}$ of Eq.* (2) *is a source.*

1.2. **Linearization.** In this section, we derive the linearized equation of Eq. (1). To this end, we introduce a continuous function $F : (0,\infty)^2 \to (0,\infty)$ which is defined by

$$F(u_0,u_1) = Au_0 + Bu_1 + \frac{1+u_0u_1}{pu_0+u_1}. \tag{1.8}$$

Therefore,

$$\begin{cases} \frac{\partial F(u_0,u_1)}{\partial u_0} = A + \frac{u_1^2-p}{(pu_0+u_1)^2}, \\ \\ \frac{\partial F(u_0,u_1)}{\partial u_1} = B + \frac{pu_0^2-1}{(pu_0+u_1)^2}. \end{cases} \tag{1.9}$$

Lemma 1.7. *The function $F(u_0,u_1)$ is non-decreasing in each of its arguments. That is $F(u_0,u_1)$ non-decreasing in u_0 for a fixed $u_1 > \sqrt{p}$ and non-decreasing in u_1 for a fixed $u_0 > \frac{1}{\sqrt{p}}$.*

From (7) and (9) we have

$$\begin{cases} \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_0} = A + \frac{1}{(p+1)}\left\{1 - p\left[1-(A+B)\right]\right\} = \rho_0, \\ \\ \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_1} = B + \frac{[1-(A+B)]}{(p+1)} = \rho_1. \end{cases} \tag{1.10}$$

The linearized equation of Eq. (1) about the equilibrium points (7) is

$$z_{n+1} - \rho_0\, z_n - \rho_1\, z_{n-k} = 0, \tag{1.11}$$

where ρ_0 and ρ_1 are given by (10).

Theorem 1.8. ([21]) . *Assume that $\rho_0, \rho_1 \in R$ and $k \in \{1,2,\ldots\}$. Then*

$$|\rho_0| + |\rho_1| < 1 \tag{1.12}$$

is a sufficient condition for the asymptotic stability of the difference equation (2). Suppose in addition that one of the following two cases holds:
(i) *k is an odd integer and $\rho_1 > 0$.*
(ii) *k is an even integer and $\rho_0\rho_1 > 0$.*
Then (12) is also a necessary condition for the asymptotic stability of Eq. (2).

Theorem 1.9. ([17]). *Consider the difference equation (2) where the function $F \in C\left(I^{k+1}, R\right)$ and I is an open interval of real numbers. Let $\widetilde{x} \in I$ be an equilibrium point of Eq. (2). Suppose also that*
(i) *F is a nondecreasing function in each of its arguments,*
(ii) *the function F satisfies the negative feedback property*

$$\left[F\left(x,x\right) - x\right]\left(x - \widetilde{x}\right) < 0 \quad for\ all \quad x \in I - \left\{\widetilde{x}\right\}.$$

Then the equilibrium point $\widetilde{x}$ of Eq. (2) is a global attractor for all solutions of Eq. (2).

Theorem 1.10. ([17]). *Let $[a,b]$ be an interval of real numbers and assume that $F : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following two conditions:*

(i) *$F(x,y)$ is non-decreasing in each of its arguments.*

(ii) *If $(m,M) \in [a,b] \times [a,b]$ is a solution of the system $m = F(m,m)$ and $M = F(M,M)$, then $m = M$.*

Then Eq. (2) has a unique equilibrium $\widetilde{x} \in [a,b]$ and every solution of Eq. (2) converges to $\widetilde{x}$.

2. Local stability

In this section, we investigate the local stability of the positive solutions of Eq. (1).

Theorem 2.1. *If $0 < p[1-(A+B)] < \frac{1}{2}$ and $0 < A+B < 1$, then the equilibrium points $\widetilde{x}$ given by (7) are not stable.*

Proof. From (10) and the assumptions of this theorem, we get

$$\begin{aligned} |\rho_0| + |\rho_1| &= \left| A + \frac{1}{(p+1)} \{1 - p[1-(A+B)]\} \right| \\ &\quad + \left| B + \frac{[1-(A+B)]}{(p+1)} \right| \\ &= \frac{2p(A+B) - p + 2}{p+1} > \frac{2\left(p - \frac{1}{2}\right) - p + 2}{p+1} = 1. \end{aligned}$$

This contradicts Theorem 3 and consequently the points $\widetilde{x}$ are not stable. Now, the proof is complete. □

Theorem 2.2. *If $p[1-(A+B)] = 1$ and $0 < A+B < 1$, then the equilibrium points $\widetilde{x}$ are locally asymptotic stable.*

Proof. From (10) and the assumptions of this theorem, we get

$$\begin{aligned} |\rho_0| + |\rho_1| &= \left| A + \frac{1}{(p+1)} \{1 - p[1-(A+B)]\} \right| \\ &\quad + \left| B + \frac{[1-(A+B)]}{(p+1)} \right| \\ &= \frac{p(A+B) + 1}{p+1} < 1, \end{aligned}$$

and by Theorem 3 the proof is complete. □

Theorem 2.3. *If $p[1-(A+B)] > 1$, $0 < A+B < 1$ and*

$$A > \frac{\{p[1-(A+B)] - 1\}}{p+1},$$

then the equilibrium points $\widetilde{x}$ are locally asymptotic stable.

Proof. From (10) and the assumptions of this theorem, we get

$$\begin{aligned}|\rho_0| + |\rho_1| = &\left|A + \frac{1}{(p+1)}\{1 - p[1-(A+B)]\}\right| \\ &+ \left|B + \frac{[1-(A+B)]}{(p+1)}\right| \\ &< \frac{1}{p+1}\{(A+B)(p+1) + [1-(A+B)]\} \\ &= \frac{p(A+B)+1}{p+1} < 1,\end{aligned}$$

and by Theorem 3 the proof is complete. □

Theorem 2.4. *If* $p[1-(A+B)] > 1$, $0 < A+B < 1$ *and* $\frac{-1}{2(p+1)} < A < \frac{\{p[1-(A+B)]-1\}}{p+1}$, *then the equilibrium points* $\widetilde{x}$ *are locally asymptotic stable.*

Proof. From (10) and the assumptions of this theorem, we get

$$\begin{aligned}|\rho_0| + |\rho_1| = &\left|A + \frac{\{1 - p[1-(A+B)]\}}{(p+1)}\right| + \left|B + \frac{[1-(A+B)]}{(p+1)}\right| \\ = &\frac{\{p[1-(A+B)]-1\}}{p+1} - A + B + \frac{[1-(A+B)]}{p+1} \\ < &1 - (A+B) - \frac{1}{2(p+1)} + B = 1 - A - \frac{1}{2(p+1)} < 1,\end{aligned}$$

and by Theorem 3 the proof is complete. □

3. Periodic solutions

In this section, we investigate the periodic character of the positive solutions of Eq. (1).

Theorem 3.1. (1) *If* k *is an even positive integer, then Eq.* (1) *has no solutions of prime period two for all* $A, B, p \in (0, \infty)$.

(2) *If* k *is an odd positive integer, then Eq.* (1) *has no solutions of prime period two for all* $A, B, p \in (0, \infty)$ *such that* $Ap - B + 1 \neq 0$.

Proof. Assume for the sake of contradiction that there exists distinct positive real numbers Φ and Ψ, such that

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$$

is a prime period two solution of Eq. (1). If k is even, then $x_n = x_{n-k}$. It follows from the difference equation (1) that

$$\Phi = (A+B)\Psi + \frac{1+\Psi^2}{(p+1)\Psi} \quad \text{and} \quad \Psi = (A+B)\Phi + \frac{1+\Phi^2}{(p+1)\Phi}.$$

Consequently, we obtain

$$(p+1)\Phi\Psi = (A+B)(p+1)\Psi^2 + \Psi^2 + 1, \tag{3.1}$$

and

$$(p+1)\,\Phi\Psi = (A+B)\,(p+1)\,\Phi^2 + \Phi^2 + 1. \tag{3.2}$$

By subtracting (13) from (14), we deduce that

$$\left(\Phi^2 - \Psi^2\right)\{1 + (A+B)\,(p+1)\} = 0. \tag{3.3}$$

Since $(A+B)\,(p+1)+1 \neq 0$, then we have $\Phi = \Psi$. This is a contradiction. This proves that Eq. (1) has no solutions of prime period two if k is even. Also, if k is odd, then $x_{n+1} = x_{n-k}$. It follows from the difference equation (1) that

$$\Phi = A\Psi + B\Phi + \frac{1+\Phi\Psi}{p\Psi+\Phi} \quad \text{and} \quad \Psi = A\Phi + B\Psi + \frac{1+\Phi\Psi}{p\Phi+\Psi}.$$

Consequently, we obtain

$$p\Phi\Psi + \Phi^2 = Ap\Psi^2 + A\Phi\Psi + Bp\Phi\Psi + B\Phi^2 + \Phi\Psi + 1, \tag{3.4}$$

and

$$p\Phi\Psi + \Psi^2 = Ap\Phi^2 + A\Phi\Psi + Bp\Phi\Psi + B\Psi^2 + \Phi\Psi + 1. \tag{3.5}$$

By subtracting (16) from (17), we deduce that

$$\left(\Phi^2 - \Psi^2\right)\{1 + Ap - B\} = 0. \tag{3.6}$$

Since $Ap - B + 1 \neq 0$, then we have $\Phi = \Psi$. This is a contradiction. This proves that Eq. (1) has no solutions of prime period two if k is odd. The proof of Theorem 10 is now complete. □

4. Boundedness character

In this section, we investigate the boundedness character of the solutions of Eq. (1).

Theorem 4.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.* (1) *with $0 < A+B < 1$. Then the following statements are true.*

(i) *Suppose $p < 1$ and for some $N \geq 0$, the intial conditions*

$$x_{N-k+1}, \ldots, x_{N-1}, x_N \in [p, 1],$$

then

$$x_n \in \left[(A+B)\,p + \frac{1}{2}\left(1+p^2\right), \frac{3}{p}\right], \quad \textit{for all} \quad n \geq N. \tag{4.1}$$

(ii) *Suppose $p > 1$ and for some $N \geq 0$, the intial conditions*

$$x_{N-k+1}, \ldots, x_{N-1}, x_N \in [1, p],$$

then

$$x_n \in \left[\frac{1}{p}\,(A+B+1), (A+B)\,p + \frac{1}{2}\left(1+p^2\right)\right], \quad \textit{for all} \quad n \geq N. \tag{4.2}$$

Proof. First of all, if for some $N \geq 0$, the initial conditions $x_{N-k+1}, \ldots, x_{N-1}, x_N \in [p, 1]$ and $p < 1$, then

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}} \geq (A+B)\,p + \frac{1+p^2}{p^2+p} \\ &\geq (A+B)\,p + \frac{1}{2}\left(1+p^2\right), \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}} \leq (A+B) + \frac{2}{1+p} \leq 1 + \frac{2}{1+p} \\ &\leq \frac{1}{p} + \frac{2}{p} = \frac{3}{p}, \end{aligned}$$

and hence the proof of part (i) is complete. Secondly, if for some $N \geq 0$, the initial conditions $x_{N-k+1}, \ldots, x_{N-1}, x_N \in [1, p]$ and $p > 1$, then

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}} \geq (A+B) + \frac{2}{p+1} \\ &\geq (A+B) + \frac{1}{p} \geq \frac{1}{p}\,(A+B+1)\,, \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + \frac{1 + x_n x_{n-k}}{px_n + x_{n-k}} \leq (A+B)\,p + \frac{1+p^2}{p^2+p} \\ &\leq (A+B)\,p + \frac{1+p^2}{2p} \leq (A+B)\,p + \frac{1}{2}\left(1+p^2\right), \end{aligned}$$

and hence the proof of part (ii) is complete. Therefore, the proof of Theorem 11 is now complete. □

4.1. **Semi-cycle analysis.**

Theorem 4.2. *Assume that* $F \in C\left[(0,\infty)^2; (0,\infty)\right]$ *is a continuous function such that* $F(x, y)$ *is non-decreasing in each of its arguments. Let* $\widetilde{x}$ *be an equilibrium point of Eq.* (1). *Then except possibly for the first semi-cycle, every oscillatory solution of Eq.* (1) *has semi-cycle of length at least* k.

Proof. The proof is obvious when $k = 1$. We just give the proof for $k = 2$. The proof is similar for $k \geq 3$ which is omitted here. Let $\{x_n\}$ be a solution of Eq. (1) with at least three semi-cycles. Then there exists $N \geq 0$ such that either

$$x_{N-1} < \widetilde{x} \leq x_{N+1},$$

or

$$x_{N-1} \geq \widetilde{x} > x_{N+1}.$$

We first assume that

$$x_{N-1} < \widetilde{x} \leq x_{N+1}.$$

Since the function $F(x,y)$ is non-decreasing in each of its arguments, then we get

$$\begin{aligned} x_{N+2} = F(x_{N+1}, x_{N-1}) &= Ax_{N+1} + Bx_{N-1} + \frac{1 + x_{N+1}x_{N-1}}{px_{N+1} + x_{N-1}} \\ &= A\widetilde{x} + Bx_{N-1} + \frac{1 + \widetilde{x}x_{N-1}}{p\widetilde{x} + x_{N-1}} = F(\widetilde{x}, x_{N-1}) \leq F(\widetilde{x}, \widetilde{x}) = \widetilde{x}, \end{aligned}$$

and hence

$$x_{N+2} \leq \widetilde{x}. \tag{4.3}$$

Also for $\widetilde{x} < x_N$, we have

$$\begin{aligned} x_{N+3} = F(x_{N+2}, x_N) &= Ax_{N+2} + Bx_N + \frac{1 + x_{N+2}x_N}{px_{N+2} + x_N} \\ &= A\widetilde{x} + Bx_N + \frac{1 + \widetilde{x}x_N}{p\widetilde{x} + x_N} = F(\widetilde{x}, x_N) \geq F(\widetilde{x}, \widetilde{x}) = \widetilde{x}, \end{aligned}$$

and hence

$$x_{N+3} \geq \widetilde{x}. \tag{4.4}$$

From (21) and (22) we have

$$x_{N+2} \leq \widetilde{x} \leq x_{N+3}. \tag{4.5}$$

Similarly, we can prove this theorem if $x_{N-1} \geq \widetilde{x} > x_{N+1}$ which is omitted here. The proof of Theorem 12 is now complete. □

5. Global stability

In this section, we investigate the global stability of the positive solutions of Eq. (1).

Theorem 5.1. *Consider the difference Eq.* (1). *If* $p\,[1 - (A + B)] > 1$ *and* $0 < A + B < 1$, *then the equilibrium points*

$$\widetilde{x} = \frac{\pm 1}{\sqrt{(p+1)\,[1 - (A + B)] - 1}}$$

of Eq. (1) *are global attractors.*

Proof. We shall prove this theorem using two different ways because they are both interesting to the readers. First of all, we consider the function

$$F(x, y) = Ax + By + \frac{1 + xy}{px + y}. \tag{5.1}$$

If the function (24) satisfies the two conditions (i), (ii) of Theorem 4, then the equilibrium points $\widetilde{x}$ of Eq. (1) are global attractors. With reference to Lemma

2, the condition (i) is obvious. It remains to prove the condition (ii) as follows:

$$\begin{aligned}
&[F(x,x)-x]\,(x-\widetilde{x}) \\
&\quad=\left[(A+B)\,x+\frac{1+x^2}{x\,(p+1)}-x\right]\left[x-\frac{\pm 1}{\sqrt{(p+1)\,[1-(A+B)]-1}}\right] \\
&\quad=\left\{\frac{1+x^2\,(A+B)-x^2p\,[1-(A+B)]}{p+1}\right\} \\
&\qquad\pm\left\{\frac{x^2\,\{(p+1)\,[1-(A+B)]-1\}-1}{x\,(p+1)\,\sqrt{(p+1)\,[1-(A+B)]-1}}\right\}.
\end{aligned}$$

Since $p\,[1-(A+B)]>1$ and $0<A+B<1$, then we have

$$\begin{aligned}
[F(x,x)-x]\,(x-\widetilde{x}) &<\frac{1+x^2\,[(A+B)-1]}{p+1}\pm\frac{x\sqrt{(p+1)\,[1-(A+B)]-1}}{p+1} \\
&\quad\mp\frac{1}{x\,(p+1)\,\sqrt{(p+1)\,[1-(A+B)]-1}} \\
&<\frac{\pm x^2\,\{(p+1)\,[1-(A+B)]-1\}}{(p+1)\,x\sqrt{(p+1)\,[1-(A+B)]-1}}+ \\
&\quad+\frac{x\sqrt{(p+1)\,[1-(A+B)]-1}\mp 1}{(p+1)\,x\sqrt{(p+1)\,[1-(A+B)]-1}} \\
&=\frac{\pm 1}{(p+1)}\left\{\frac{\left(x\sqrt{(p+1)\,[1-(A+B)]-1}\pm\frac{1}{2}\right)^2-\frac{5}{4}}{x\sqrt{(p+1)\,[1-(A+B)]-1}}\right\}.
\end{aligned}\tag{5.2}$$

From (25) we discuss the following two cases:

Case 1. If $0<x\le\frac{\sqrt{5}-1}{2\sqrt{(p+1)[1-(A+B)]-1}}$ and $\widetilde{x}=\frac{1}{\sqrt{(p+1)[1-(A+B)]-1}}$, then the inequality (25) reduces to

$$\begin{aligned}
[F(x,x)-x]\,(x-\widetilde{x}) &<\frac{1}{(p+1)}\left\{\frac{\left(x\sqrt{(p+1)\,[1-(A+B)]-1}+\frac{1}{2}\right)^2-\frac{5}{4}}{x\sqrt{(p+1)\,[1-(A+B)]-1}}\right\} \\
&<0.
\end{aligned}\tag{5.3}$$

This proves that the positive equilibrium point

$$\widetilde{x}=\frac{+1}{\sqrt{(p+1)\,[1-(A+B)]-1}}$$

of Eq. (1) is a global attractor.

Case 2. If $x \geq \frac{1+\sqrt{5}}{2\sqrt{(p+1)[1-(A+B)]-1}}$ and $\widetilde{x} = \frac{-1}{\sqrt{(p+1)[1-(A+B)]-1}}$, then the inequality (25) reduces to

$$[F(x,x)-x]\,(x-\widetilde{x}) < \frac{-1}{(p+1)}\left\{\frac{\left(x\sqrt{(p+1)\,[1-(A+B)]-1}\right)^2-\frac{5}{4}}{x\sqrt{(p+1)\,[1-(A+B)]-1}}\right\}$$

$$< 0. \tag{5.4}$$

This proves that the negative equilibrium point

$$\widetilde{x} = \frac{-1}{\sqrt{(p+1)\,[1-(A+B)]-1}}$$

of Eq. (1) is a global attractor. The proof of Theorem 13 is now complete.

Secondly, since the function $F(x,y)$ given by (24) is nondecreasing in each of its arguments, then if (m,M) is a solution of the system

$$m = F(m,m) \qquad \text{and} \qquad M = F(M,M),$$

then we get

$$m = (A+B)\,m + \frac{1+m^2}{(p+1)\,m},$$

and

$$M = (A+B)\,M + \frac{1+M^2}{(p+1)\,M}.$$

Consequently, we have

$$(p+1)\,m^2 = (A+B)\,(p+1)\,m^2 + m^2 + 1, \tag{5.5}$$

$$(p+1)\,M^2 = (A+B)\,(p+1)\,M^2 + M^2 + 1. \tag{5.6}$$

By subtracting (28) from (29) we get

$$(m-M)\,(m+M)\,\{(p+1)\,[1-(A+B)]-1\} = 0. \tag{5.7}$$

Since $(p+1)\,[1-(A+B)] > 1$, then we deduce from (30) that $m = M$. According to Theorem 5, the equilibrium points $\widetilde{x}$ are global attractors. Therefore, the proof of Theorem 13 is now complete. □

On combining Theorem 8 or 9 together with Theorem 13, we have the following result:

Theorem 5.2. *If* $p\,[1-(A+B)] > 1$, $0 < A+B < 1$ *and either* $A > \frac{\{p[1-(A+B)]-1\}}{p+1}$ *or* $\frac{-1}{2(p+1)} < A < \frac{\{p[1-(A+B)]-1\}}{p+1}$, *then the equilibrium points*

$$\widetilde{x} = \frac{\pm 1}{\sqrt{(p+1)\,[1-(A+B)]-1}}$$

of Eq. (1) *are globally asymptotically stable.*

6. Numerical examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear difference equation (1).

Example 1. Figure 1 shows that the solution of Eq. (1) has no positive solutions of prime period two if $k = 4,\ x_{-4} = 1, x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ A = 300,\ B = 100,\ p = 50.$

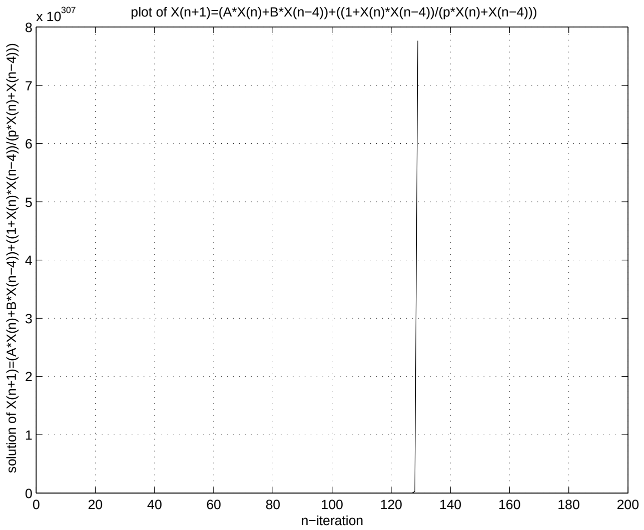

Figure 1. $(x_{n+1} = 300x_n + 100x_{n-4} + \frac{1+x_n x_{n-4}}{50x_n+x_{n-4}})$

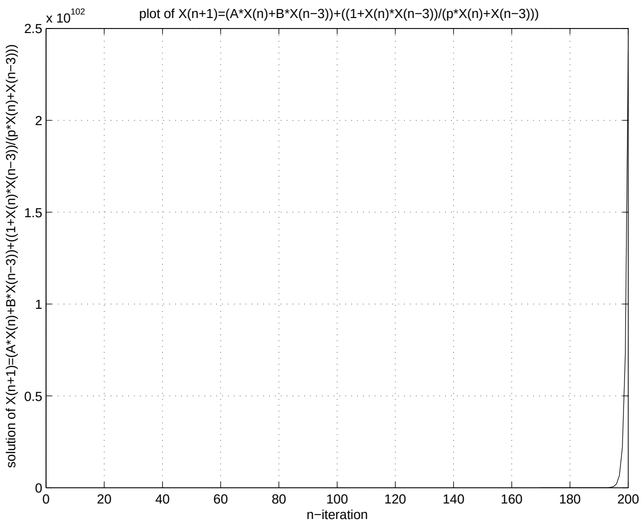

Figure 2. $(x_{n+1} = 3x_n + 10x_{n-4} + \frac{1+x_n x_{n-4}}{5x_n+x_{n-4}})$

Example 2. Figure 2 shows that the solution of Eq. (1) has no positive solutions of prime period two if $k = 3,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ A = 3,\ B = 10,\ p = 5$.

Example 3. Figure 3 shows that the solution of Eq. (1) is global stability if $k = 3,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ A = 0.5,\ B = 0.25,\ p = 5$.

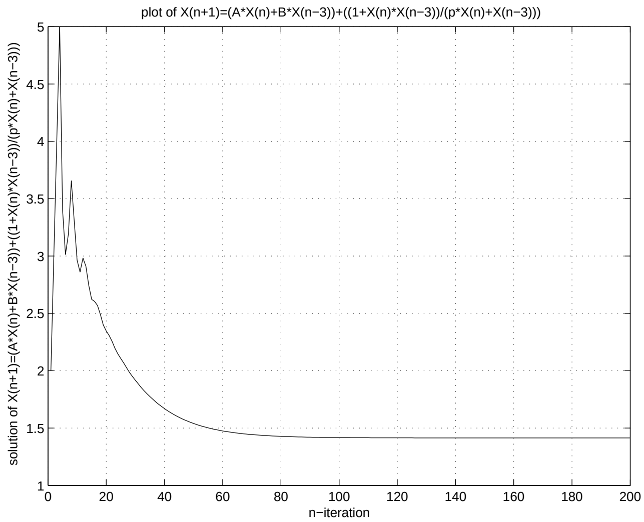

FIGURE 3. $(x_{n+1} = 0.5x_n + 0.25x_{n-4} + \frac{1+x_n x_{n-4}}{5x_n + x_{n-4}})$

Example 4. Figure 4 shows that the solution of Eq. (1) is not stable if $k = 3, x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ A = 0.5,\ B = 0.25,\ p = 1$.

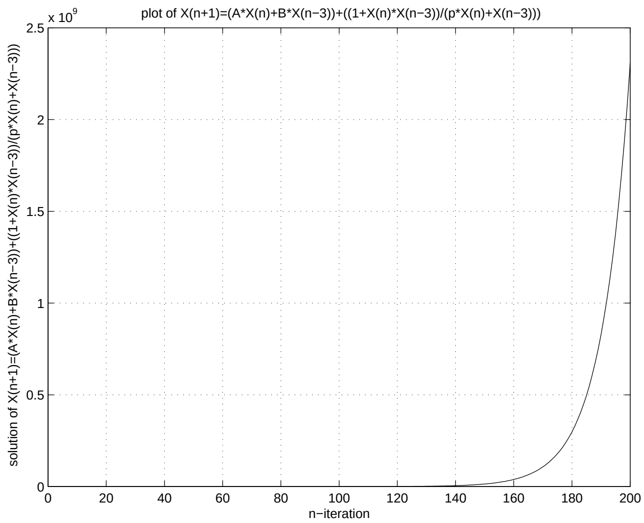

FIGURE 4. $(x_{n+1} = 0.5x_n + 0.25x_{n-4} + \frac{1+x_n x_{n-4}}{x_n + x_{n-4}})$

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