ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 2, 2011, pp. 231–248


# ON THE EXISTENCE OF SOLUTIONS TO GENERALIZED QUASI-EQUILIBRIUM PROBLEMS OF TYPE II AND RELATED PROBLEMS

TRUONG THI THUY DUONG AND NGUYEN XUAN TAN

*Dedicated to Tran Duc Van on the occasion of his sixtieth birthday*

ABSTRACT. The generalized quasi-equilibrium problem of type II is formulated and some sufficient conditions on the existence of its solutions are shown. As special cases, we obtain several results on the existence of solutions of ideal quasivariational inclusion problems, quasivariational relation problems of type II, generalized quasi-KKM theorems etc. As corollaries, we show several results on the existence of solutions to other problems in the vector optimization theory concerning multivalued mappings.

## 1. INTRODUCTION

Throughout this paper $X, Y, Z$ and $W$ are supposed to be real Hausdorff locally convex linear topological spaces, $D \subset X, K \subset Z$ and $E \subset W$ are nonempty subsets. Given multivalued mappings $S : D \times K \to 2^D, T : D \times K \to 2^K; P_1 : D \to 2^D, P_2 : D \to 2^E, Q : K \times D \to 2^Z$ and $F : K \times D \times E \to 2^Y$, we are interested in the following problems:

**A**. Find $(\bar{x}, \bar{y}) \in D \times K$ such that
1) $\bar{x} \in S(\bar{x}, \bar{y})$;
2) $\bar{y} \in T(\bar{x}, \bar{y})$;
3) $0 \in F(\bar{y}, \bar{x}, \bar{x}, z)$ for all $z \in S(\bar{x}, \bar{y})$.

This problem is called a generalized quasi-equilibrium problem of type I, denoted by $(GEP)_I$.

**B**. Find $\bar{x} \in D$ such that
$$\bar{x} \in P_1(\bar{x})$$
and
$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$
This problem is called a generalized quasi-equilibrium problem of type II, denoted by $(GEP)_{II}$, in which the multivalued mappings $S, P_1, P_2, T$ and $Q$ are

Received September 7, 2010; in revised form January 27, 2011.
2000 *Mathematics Subject Classification.* Primary.
*Key words and phrases.* Generalized quasi-equilibrium problems of type II, upper and lower quasivariational inclusions, quasi-optimization problems, upper and lower $\mathcal{C}$-quasiconvex multivalued mappings, upper and lower $\mathcal{C}$- continuous multivalued mappings.
This work was supported by the NAFOSTED of Vietnam.

constraints and $F$ is an utility multivalued mapping that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. The generalized quasi-equilibrium problems of type I are studied in [3]. In this paper we consider the existence of solutions of the second problems. The typical instances of generalized quasi-equilibrium problems of type II are as follows:

**i) Quasi-equilibrium problem.** Let $D, K, P_i, i = 1, 2, Q$ be as above and $W = X, E = D$. Let $\mathbb{R}$ be the space of real numbers with the subset of nonnegative numbers $\mathbb{R}_+$ and $\Phi : K \times D \times D \to \mathbb{R}$ be a function with $\Phi(y, x, x) = 0$, for all $y \in K, x \in D$. The generalized quasi-equilibrium problem $(GEP)_{II}$ is defined as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in \Phi(y, \bar{x}, t) - \mathbb{R}_+ \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This problem is known as a quasi-equilibrium problem: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\Phi(y, \bar{x}, t) \geq 0 \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This problem is studied by many authors, for example, in [4], [6], [9], [13] and in the references therein. These problems generalize the well-known equilibrium problem introduced by Blum and Oettli in [2]. There is a difference between $(GEP)_I$ and $(GEP)_{II}$. If we consider a variable $y$ in these problems as parameter, we can see that in $(GEP)_I$ there exists a parameter $\bar{y} \in K, \bar{y} \in T(\bar{x}, \bar{y})$ and $0 \in F(\bar{y}, \bar{x}, \bar{x}, z)$ for all $z \in S(\bar{x}, \bar{y})$. But, in $(GEP)_{II}, 0 \in F(y, \bar{x}, t)$ holds for all $t \in P_2(\bar{x})$ and $y \in Q(\bar{x}, t)$.

**ii) Minty quasivariational problem.** Let $\langle \cdot, \cdot \rangle : X \times Z \to \mathbb{R}$ be a continuous bilinear function. We consider the following Minty quasivariational problem: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\langle y, t - \bar{x} \rangle \geq 0 \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

For $F(y, x, t) = \langle y, t - x \rangle - \mathbb{R}_+, (GEP)_{II}$ reads as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

**iii) Ideal upper quasivariational inclusion problem of type II.** Let $D, K, Y, P_i, i = 1, 2,$ and $Q$ be given as at the beginning of this section. Further, assume that $\mathcal{C} : K \times D \to 2^Y$ is a cone multivalued mapping (for any $(y, x) \in K \times D, \mathcal{C}(y, x)$ is cone in $Y$) and $G$ and $H$ are multivalued mappings on $K \times D \times D$ with values in the space $Y$. We define the multivalued mappings $M : K \times D \to 2^X; F : K \times D \times D \to 2^Y$ by

$$M(y, x) = \{t \in D \mid G(y, x, t) \subseteq H(y, x, x) + \mathcal{C}(y, x)\}, \quad (y, x) \in K \times D$$

and

$$F(y, x, t) = t - M(y, x), \quad (y, x, t) \in K \times D \times D.$$

Problem $(GEP)_{II}$ is formulated as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This shows

$$G(y, \bar{x}, t) \subseteq H(y, \bar{x}, \bar{x}) + \mathcal{C}(y, \bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This is an ideal upper quasivariational inclusion problem studied in [9], [11], [12], [14] and in the references therein.

**iv) Abstract quasivariational relation problem of type II.** Let $D, K, P_1, i = 1, 2, Q$ be as above and $W = X, E = D$. Let $\mathcal{R}(y, x, t)$ be a relation linking $y \in K, x \in D$ and $t \in E$. We define the multivalued mappings $M : K \times D \to 2^X; F : K \times D \times D \to 2^Y$ by

$$M(y, x) = \{t \in D \mid \mathcal{R}(y, x, t) \text{ holds}\};$$

and

$$F(y, x, t) = t - M(y, x), \ (y, x, t) \in K \times D \times D.$$

Problem $(GEP)_{II}$ is formulated as follows: Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \quad \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This becomes that of finding $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$\mathcal{R}(y, \bar{x}, t) \text{ holds for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This is $(VR)$ studied in [10].

**v) Differential inclusion.** Let $D \subset C^1[a, b]$ be a nonempty set, where $C[a, b]$ and $C^1[a, b]$ are the spaces of continuous and continuously diffrentiable functions respectively on the interval $[a, b]$. Let $P_1, P_2$ be given as above. Let $\Omega$ be a nonempty set and $U : D \times D \to 2^\Omega$ a multivalued mapping. Set $K = \Omega \times \mathbb{R}$ and $Q : D \times D \to 2^Y$ by $Q(x, t) = U(x, t) \times [a, b]$. Given a multivalued mapping $G : K \times D \times D \to 2^{C[a,b]}$. Problem of finding $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$x' \in G(y, \xi, \bar{x}, t) \quad \text{for all } t \in P_2(\bar{x}) \text{ and } (y, \xi) \in Q(\bar{x}, t),$$

studied in [5] becomes that of finding $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \xi, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } (y, \xi) \in Q(\bar{x}, t),$$

where $F(y, \xi, x, t) = x' - G(y, \xi, x, t)$ and $x'$ denotes the derivative of $x$.

## 2. Preliminaries and Definitions

Throughout this paper, as in the introduction, by $X, Z, W$ and $Y$ we denote real Hausdorff locally convex linear topological spaces. Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of $F$ are denoted by

$$\mathrm{dom}F = \{x \in D |\ F(x) \neq \emptyset\},$$

$$\mathrm{Gr}(F) = \{(x, y) \in D \times Y |\ y \in F(x)\},$$

respectively. We recall that $F$ is said to be a closed mapping if the graph $Gr(F)$ of $F$ is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $clF(D)$ of its range $F(D)$ is a compact set in $Y$. A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower semicontinuous) semicontinuous (briefly, u.s.c. (respectively, l.s.c.)) at $\bar{x} \in D$ if for each open set $V$ containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap U \neq \emptyset$), there exists an open neighborhood $U$ of $\bar{x}$ that $F(x) \subseteq V$ (respectively, $F(x) \cap U \neq \emptyset$) for each $x \in U$ and $F$ is said to be u.s.c. (l.s.c.) on $D$ if it is u.s.c. (respectively, l.s.c.) at all $x \in D$. These notions and definitions can be found in [1]. Further, let $Y$ be a topological vector space with a cone $C$. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}$, $C$ is said to be pointed. We recall the following definitions.

Now, let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping (the image of every point of $\mathcal{C}$ is a cone in $Y$). We introduce the following definitions of the $\mathcal{C}$-continuities which are extensions of $C$-continuity notions of multivalued mappings in [8].

**Definition 2.1.** Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping and $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping.

(i) $F$ is said to be upper (lower) $\mathcal{C}$-continuous at $(\bar{y}, \bar{x}, \bar{t}) \in \mathrm{dom}\ F$ if for any neighborhood $V$ of the origin in $Y$ there is a neighborhood $U$ of $(\bar{y}, \bar{x}, \bar{t})$ such that:

$$F(y, x, t) \subseteq F(\bar{y}, \bar{x}, \bar{t})) + V + \mathcal{C}(\bar{y}, \bar{x})$$

$$(\text{respectively}, F(\bar{y}, \bar{x}, \bar{t}) \subseteq F(y, x, t) + V - \mathcal{C}(\bar{y}, \bar{x}))$$

holds for all $(y, x, t) \in U \cap \mathrm{dom}F$.

(ii) If $F$ is upper $\mathcal{C}$-continuous and lower $\mathcal{C}$-continuous at $(\bar{y}, \bar{x}, \bar{t})$ simultaneously, we say that it is $\mathcal{C}$-continuous at $(\bar{y}, \bar{x}, \bar{t})$.

(iii) If $F$ is upper, lower,..., $\mathcal{C}$-continuous at any point of dom$F$, we say that it is upper, lower,..., $\mathcal{C}$-continuous on $D$.

(iv) In the case $\mathcal{C} = \{0\}$, the trivial cone in $Y$, we shall only say $F$ is upper, lower continuous instead of upper, lower 0-continuous. And, $F$ is continuous if it is upper and lower continuous simultaneously.

**Definition 2.2.** Let $G$ be a multivalued mapping from $D$ to $2^Y$ and $C$ a cone in $Y$. We say that:

(i) $G$ is upper $C$-quasiconvex on $D$ if for any $x_1,\ x_2 \in D,\ t \in [0,1]$, either

$$\begin{aligned} &G(x_1) \subseteq G(tx_1 + (1-t)x_2) + C, \\ \text{or}\quad &G(x_2) \subseteq G(tx_1 + (1-t)x_2) + C \end{aligned}$$

holds.

(ii) $G$ is lower $C$-quasiconvex on $D$ if for any $x_1, x_2 \in D, t \in [0,1]$, either

$$\begin{aligned} &G(tx_1 + (1-t)x_2) \subseteq G(x_1) - C, \\ \text{or}\quad &G(tx_1 + (1-t)x_2) \subseteq G(x_2) - C \end{aligned}$$

holds.

**Definition 2.3.** Let $F : K \times D \times D \to 2^Y, Q : D \times D \to 2^K$ be multivalued mappings. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping. We say that

(i) $F$ is diagonally upper $(Q, \mathcal{C})$-quasiconvex at the third variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in \text{co}\{x_1, ..., x_n\}$, there is an index $j \in \{1, ..., n\}$ such that

$$F(y, x, x_j) \subseteq F(y, x, x) + \mathcal{C}(y, x) \text{ for all } y \in Q(x, x_j).$$

(ii) $F$ is diagonally lower $(Q, \mathcal{C})$-quasiconvex at the third variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x \in \text{co}\{x_1, ..., x_n\}$, there is an index $j \in \{1, ..., n\}$ such that

$$F(y, x, x) \subseteq F(y, x, x_j) - \mathcal{C}(y, x) \text{ for all } y \in Q(x, x_j).$$

**Definition 2.4.** Let $F : K \times D \times D \to 2^X, Q : D \times D \to 2^K$ be multivalued mappings. We say that $F$ is $Q$- KKM if for any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \text{co}\{t_1, ..., t_n\}$, there is $t_j \in \{t_1, ..., t_n\}$ such that $0 \in F(y, x, t_j)$, for all $y \in Q(x, t_j)$.

**Definition 2.5.** Let $F : K \times D \times E \to 2^X, Q : D \times E \to 2^K$ be multivalued mappings. We say that $F$ is generalized $Q$- KKM if for any finite set $\{t_1, ..., t_n\} \subset E$ there is a finite set $\{x_1, ..., x_n\} \subseteq D$ such that for any $x \in \text{co}\{x_{i_1}, ..., x_{i_k}\}$, there is $t_{i_j} \in \{t_{i_1}, ..., t_{i_n}\}$ such that $0 \in F(y, x, t_{i_j})$, for all $y \in Q(x, t_{i_j})$.

**Definition 2.6.** Let $\mathcal{R}$ be a binary relation on $K \times D$. We say that $\mathcal{R}$ is closed if for any net $(y_\alpha, x_\alpha)$ converging to $(y, x)$ and $\mathcal{R}(y_\alpha, x_\alpha)$ holds for all $\alpha$, so holds $\mathcal{R}(y, x)$.

**Definition 2.7.** Let $\mathcal{R}$ be a relation on $K \times D \times D$. We say that $\mathcal{R}$ is $Q$- KKM if for any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \text{co}\{t_1, ..., t_n\}$, there is a $t_j \in \{t_1, ..., t_n\}$ such that $\mathcal{R}(y, x, t_j)$ holds, for all $y \in Q(x, t_j)$.

Now, we give some necessary and sufficient conditions on the upper and the lower $\mathcal{C}$- continuities which we shall need in the next section.

**Proposition 2.8.** *Let $F : K \times D \times D \to 2^Y$ be a multivalued mapping and $\mathcal{C} : K \times D \to 2^Y$ be a cone upper continuous multivalued mapping with nonempty convex closed values.*

1) *If $F$ is upper $\mathcal{C}$-continuous at $(y_o, x_o, t_o) \in domF$ with $F(y_o, x_o, t_o)) + \mathcal{C}(y_o, x_o)$ being closed, then for any net $(y_\beta, x_\beta, t_\beta) \to (y_o, x_o, t_o), v_\beta \in F(y_\beta, x_\beta, t_\beta) + \mathcal{C}(y_\beta, x_\beta), v_\beta \to v_o$ imply $v_o \in F(y_o, x_o, t_o) + \mathcal{C}(y_o, x_o)$.*

*Conversely, if $F$ is compact and for any net $(y_\beta, x_\beta, t_\beta) \to (y_o, x_o, t_o)$, $v_\beta \in F(y_\beta, x_\beta, t_\beta) + \mathcal{C}(y_\beta, x_\beta), v_\beta \to v_o$ imply $v_o \in F(y_o, x_o, t_o) + \mathcal{C}(y_o, x_o)$, then $F$ is upper $\mathcal{C}$-continuous at $(y_o, x_o, t_o)$.*

*2) If $F$ is compact and lower $\mathcal{C}$-continuous at $(y_o, x_o, t_o) \in domF$, then for any net $(y_\beta, x_\beta, t_\beta) \to (y_o, x_o, t_o), v_o \in F(y_o, x_o, t_o) + \mathcal{C}(y_o, x_o)$, there is a net $\{v_\beta\}, v_\beta \in F(y_\beta, x_\beta, t_\beta)$, which has a convergent subnet $\{v_{\beta_\gamma}\}, v_{\beta_\gamma} - v_o \to c \in \mathcal{C}(y_o, x_o)(i.e \quad v_{\beta_\gamma} \to v_o + c \in v_o + \mathcal{C}(y_o, x_o)$.*

*Conversely, if $F(y_o, x_o, t_o)$ is compact and for any net $(y_\beta, x_\beta, t_\beta) \to (y_o, x_o, t_o)$, $v_o \in F(y_o, x_o, t_o) + \mathcal{C}(y_o, x_o)$, there is a net $\{v_\beta\}, v_\beta \in F(y_\beta, x_\beta, t_\beta)$, which has a convergent subnet $\{v_{\beta_\gamma}\}, v_{\beta_\gamma} - v_o \to c \in \mathcal{C}(y_o, x_o)$, then $F$ is lower $\mathcal{C}$–continuous at $(y_o, x_o, t_o)$.*

*Proof.* We proceed the proof of this proposition exactly as the one of Proposition 2.3 in [7]. □

In the sequel, if we say that the set $A$ is open in $D$, this means that this set is open in the relative topology of the topology on $X$ restricted to $D$. The proofs of the main results in our paper are based on the following theorems (in [15]).

**Theorem 2.9.** *Let $D$ be a nonempty convex compact subset of $X$ and $F : D \to 2^D$ be a multivalued mapping satisfying the following conditions:*

*1. For all $x \in D, x \notin F(x)$ and $F(x)$ is convex;*

*2. For all $y \in D, F^{-1}(y)$ is open in $D$.*

*Then there exists $\bar{x} \in D$ such that $F(\bar{x}) = \emptyset$.*

**Theorem 2.10.** *Let $D$ be a nonempty convex compact subset of $X$ and $F : D \to 2^D$ be a multivalued mapping with F(x) being nonempty for any $x \in D$. Assume that $F^{-1}(y)$ is open in $D$ for any $y \in D$. Then there exists $\bar{x} \in D$ such that $\bar{x} \in coF(\bar{x})$, where $co(A)$ denotes the convex hull of $A$.*

One can easily see that the conclusion of Theorem 2.10 follows immediately from Theorem 2.9. Indeed, we assume that for any $x \in D, x \notin coF(x)$. Since for all $y \in D, F^{-1}(y)$ is open in $D$, so is $(coF)^{-1}(y)$ (see the proof of Theorem 3.1 below). Applying Theorem 2.9, we conclude that there exists $\bar{x} \in D$ such that $coF(\bar{x}) = \emptyset$ and we have a contradiction.

## 3. Main Results

Throughout this section, unless otherwise specified, by $X$, $Z$ and $Y$ we denote real Hausdorff locally convex linear topological spaces. Let $D \subseteq X, K \subseteq Z$ be nonempty subsets and $C \subseteq Y$ be a convex closed cone, $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping. Given multivalued mappings $P_i : D \to 2^D, i = 1, 2$, $Q : D \times D \to 2^K$ and $F : K \times D \times D \to 2^Y$, we first prove the following theorem

**Theorem 3.1.** *The following conditions are sufficient for $(GEP)_{II}$ to have a solution:*

*(i) $D$ is a nonempty convex compact subset;*

*(ii) $P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in D|\ x \in P_1(x)\}$;*

(iii) $P_2 : D \to 2^D$ *is a multivalued mapping with nonempty* $P_2(x)$ *and open* $P_2^{-1}(x)$ *and the convex hull* $coP_2(x)$ *of* $P_2(x)$ *is contained in* $P_1(x)$ *for each* $x \in D$;

(iv) $Q : D \times D \to 2^K$ *is a multivalued mapping such that for any fixed* $t \in D$ *the multivalued mapping* $Q(.,t) : D \to 2^K$ *is l.s.c.;*

(v) *For any fixed* $t \in D$, *the set*

$$B = \{x \in D|\ 0 \notin F(y,x,t) \textit{ for some } \ y \in Q(x,t)\}$$

*is open in* $D$*;*

(vi) $F : K \times D \times D \to 2^Y$ *is a* $Q - KKM$ *multivalued mapping.*

*Proof.* We define the multivalued mapping $M : D \to 2^D$ by

$$M(x) = \{t \in D| \quad 0 \notin F(y,x,t) \text{ for some } y \in Q(x,t)\}.$$

Observe that if for some $\bar{x} \in D, \bar{x} \in P_1(\bar{x})$, it gives $M(\bar{x}) \cap P_2(\bar{x}) = \emptyset$, then

$$0 \in F(y,\bar{x},t) \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x},t)$$

and hence the proof of the theorem is completed. Thus, our aim is to show the existence of such a point $\bar{x}$. Indeed, we assume on the contrary that for any $x \in P_1(x)$, $M(x) \cap P_2(x) \neq \emptyset$. We consider the multivalued mapping $H : D \to 2^D$ defined by

$$H(x) = \begin{cases} (coM)(x) \cap (coP_2)(x), & \text{if } x \in P_1(x) \\ P_2(x), & \text{otherwise }, \end{cases}$$

where $(coN)(x) = coN(x)$. Next, we claim that if for any $y \in D, N^{-1}(y)$ is open, then so is $(coN)^{-1}(y)$. Indeed, assume that $y \in D$ and $x \in (coN)^{-1}(y)$, then $y \in co(N(x)), y = \sum_{i=1}^n \alpha_i y_i$ with $0 \le \alpha_i \le 1, \sum_{i=1}^n \alpha_i = 1, y_i \in N(x)$. This gives $x \in N^{-1}(y_i)$, for all $i = 1,..,n$. Since $N^{-1}(y_i), i = 1,...,n$ are open, there is a neighborhood $U(x)$ of $x$ such that $U(x) \subseteq N^{-1}(y_i)$ for all $i = 1,...,n$. This implies $y_i \in N(z)$ for all $z \in U(x)$ and $i = 1,...,n$. Therefore, $y = \sum_{i=1}^n \alpha_i y_i \in (coN)(z)$ for $z \in U(x)$ and then $U(x) \subseteq (coN)^{-1}(y)$. So $(coN)^{-1}(y)$ is open.

Further, we show that $H$ verifies the hypotheses of Theorem 2.9 in Section 2. Indeed, since for any $x \in D$ with $x \in P_1(x)$, $M(x) \cap P_2(x) \neq \emptyset$, we conclude that $H(x) \neq \emptyset$ and then $D = \bigcup_{x \in D} H^{-1}(x)$. From the assumption (v) for any $x \in D, M^{-1}(x)$ is open, it follows that

$$H^{-1}(x) = (coM)^{-1}(x) \cap (coP_2)^{-1}(x) \cup (P_2^{-1}(x) \setminus D_0),$$

where $D_0 = \{x \in D : x \in P_1(x)\}$ is a closed subset in $D$. Hence $H^{-1}(x)$ is an open set in $D$, for every $x \in D$. Further, if there is a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x}) = coM(\bar{x}) \cap coP_2(\bar{x})$, then one can find $t_1,...,t_n \in M(\bar{x})$ such that $\bar{x} = \sum\limits_1^n \alpha_i t_i, \alpha_i \ge 0, \sum\limits_1^n \alpha_i = 1$. By the definition of $M$, we have

$$0 \notin F(y,x,t_i) \text{ for some } y \in Q(x,t_i)\}, \text{ for all } i = 1,...,n.$$

Together with the fact that the multivalued mapping $F$ is $Q-$KKM, one can find an index $j = 1,...,n$ such that

$$0 \in F(y,x,t_j) \text{ for all } y \in Q(x,t_j)$$

and we get a contradiction. Theorefore, we conclude that for any $x \in D, x \notin H(x)$. An application of Theorem 2.9 in Section 2 implies that there exists a point $\bar{x} \in D$ with $H(\bar{x}) = \emptyset$. If $\bar{x} \notin P_1(\bar{x})$, then $H(\bar{x}) = P_2(\bar{x}) = \emptyset$, which is impossible. Therefore, we conclude that $\bar{x} \in P_1(\bar{x})$ and $H(\bar{x}) = coM(\bar{x}) \cap coP_2(\bar{x}) = \emptyset$. Thus, we have a contradition and the proof of the theorem is complete. □

Several applications of the above theorem to the existence of solutions of quasi-equilibrium, variational inclusion problems,..., can be shown in the following corollaries.

**Corollary 3.2.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $\Phi : K \times D \times D \to R$ be a real diagonally $(Q, R_+)$- quasiconvex function with $\Phi(y, x, x) = 0$ for all $y \in K, x \in D$. In addition, assume that for any fixed $t \in D$ the function $\Phi(.,.,t) : K \times D \to R$ is upper semicontinuous. Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$\Phi(y, \bar{x}, t) \geq 0 \textit{ for all } t \in P_(\bar{x}) \textit{ and } y \in Q(\bar{x}, t).$$

*Proof.* Setting $F(y, x, t) = \Phi(y, x, t) - R_+$, for any $(y, x, t) \in K \times D \times D$, we can see that for any fixed $t \in D$ the set

$$\begin{aligned} B =& \{x \in D |\ 0 \notin F(y, x, t) \text{ for some } y \in Q(x, t)\} \\ =& \{x \in D |\ \Phi(y, x, t) < 0\} \end{aligned}$$

is open in $D$. Since $\Phi$ is diagonally upper $(Q, R_+)$-quasiconvex in the third variable, for any finite set $\{t_1, ..., t_n\} \subseteq D, x \in co\{t_1, ..., t_n\}$, there is an index $j \in \{1, ..., n\}$ such that

$$\Phi(y, x, t_j) \in \Phi(y, x, x) + R_+ \text{ for all } y \in Q(x, t_j).$$

This implies that $\Phi(y, x, t_j) \geq 0$ and so $0 \in F(y, x, t_j)$ for all $y \in Q(x, t_j)$. This shows that $F$ is a $Q-$ KKM multivalued mapping from $K \times D \times D$ to $2^R$. Therefore, $P_1, P_2, Q$ and $F$ satisfy all conditions in Theorem 3.1. This implies that there is a point $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This is equivalent to

$$\Phi(y, \bar{x}, t) \geq 0 \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t),$$

and the proof is complete. □

In the following corollary we assume that $\mathcal{C} : K \times D \to 2^Y$ is a given cone upper continuous multivalued mapping with convex closed values.

**Corollary 3.3.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem* 3.1. *Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values and $G(y, x, x) \subseteq H(y, x, x) + \mathcal{C}(y, x)$ for any $(y, x) \in K \times D$. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with nonempty convex closed values. In addition, assume:*

(i) *For any fixed* $t \in D$, *the multivalued mapping* $G(\cdot,\cdot,t) : K \times D \to 2^Y$ *is lower* $(-\mathcal{C})-$*continuous and the multivalued mapping* $N : K \times D \to 2^Y$, *defined by* $N(y,x) = H(y,x,x)$, *is upper* $\mathcal{C}-$ *continuous;*

(ii) $G$ *is diagonally upper* $(Q,\mathcal{C})$*-quasiconvex in the third variable.*

*Then, there exists* $\bar{x} \in D$ *such that* $\bar{x} \in P_1(\bar{x})$ *and*

$$G(y,\bar{x},t) \subseteq H(y,\bar{x},\bar{x}) + \mathcal{C}(y,\bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x},t).$$

*Proof.* We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y,x) = \{t \in D|\ G(y,x,t) \subseteq H(y,x,x) + \mathcal{C}(y,x)\},\ (y,x) \in K \times D$$

and

$$F(y,x,t) = t - M(y,x),\ (y,x,t) \in K \times D \times D.$$

For any fixed $t \in D$, we set

$$\begin{aligned} A &= \{x \in D|\ 0 \in F(y,x,t) \text{ for all } y \in Q(x,t)\} \\ &= \{x \in D|\ t \in M(y,x) \text{ for all } y \in Q(x,t)\} \\ &= \{x \in D|\ G(y,x,t) \subseteq H(y,x,x) + \mathcal{C}(y,x) \text{ for all } y \in Q(x,t)\}. \end{aligned}$$

We claim that this subset is closed in $D$. Indeed, assume that a net $\{x_\alpha\} \subset A$ and $x_\alpha \to x$. Take arbitrary $y \in Q(x,t)$. Since $Q(\cdot,t)$ is a lower semicontinuous mapping and $x_\alpha \to x$, there exists a net $\{y_\alpha\}, y_\alpha \in Q(x_\alpha,t)$ such that $y_\alpha \to y$. For any neighborhood $V$ of the origin in $Y$ there is an index $\alpha_0$ such that for all $\alpha \leq \alpha_0$ the following inclusions hold:

$$\begin{aligned} &G(y,x,t) \subseteq G(y_\alpha,x_\alpha,t) + V + \mathcal{C}(y_\alpha,x_\alpha) \\ &\subseteq H(y_\alpha,x_\alpha,x_\alpha) + V + \mathcal{C}(y_\alpha,x_\alpha) \subseteq H(y,x,x) + 2V + \mathcal{C}(y,x). \end{aligned}$$

This and the compact values of $H$ imply that

$$G(y,x,t) \subseteq H(y,x,x) + \mathcal{C}(y,x),$$

and therefore, $x \in A$. This follows that $A$ is closed in $D$ and the set

$$B = D \setminus A = \{x \in D|\ 0 \notin F(y,x,t) \text{ for some } y \in Q(x,t)\}$$

is open in $D$.

Further, since $G(y,x,x) \subseteq H(y,x,x) + \mathcal{C}(y,x)$ for any $(y,x) \in K \times D$ and $G$ is diagonally upper $(Q,\mathcal{C})$-quasiconvex in the third variable, we conclude that for any finite set $\{t_1,...,t_n\} \subseteq D, x \in co\{t_1,...,t_n\}$, there is an index $j \in \{1,...,n\}$ such that

$$G(y,x,t_j) \subseteq G(y,x,x) + \mathcal{C}(y,x) \subseteq H(y,x,x) + \mathcal{C}(y,x) \text{ for all } y \in Q(x,t_j).$$

This follows that $0 \in F(y,x,t_j)$ and then $F$ is a $Q-$KKM multivalued mapping. Thus, to complete the proof of the corollary, it remains to apply Theorem 3.1 to deduce that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y,\bar{x},t) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x},t).$$

This is equivalent to

$$G(y, \bar{x}, t) \subseteq H(y, \bar{x}, \bar{x}) + \mathcal{C}(y, \bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

□

Analogically, we obtain the following corollary.

**Corollary 3.4.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values and $H(y, x, x) \subseteq G(y, x, x) - \mathcal{C}(y, x)$ for any $(y, x) \in K \times D$. Let $\mathcal{C} : K \times D \to 2^Y$ be a cone multivalued mapping with nonempty convex closed values. In addition, assume:*

(i) *For any fixed $t \in D$ the multivalued mapping $G(.,.,t) : K \times D \to 2^Y$ is upper $(-\mathcal{C})$- continuous and the multivalued mapping $N : K \times D \to 2^Y$ defined by $N(y, x) = H(y, x, x)$ is lower $\mathcal{C}$- continuous.*

(ii) *$G$ is diagonally lower $(Q, \mathcal{C})$-quasiconvex in the third variable.*

*Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$H(y, \bar{x}, \bar{x}) \subseteq G(y, \bar{x}, t) - \mathcal{C}(y, \bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Proof.* The proof is similar to the previous one of Corollary 3.3. □

**Corollary 3.5.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings with compact values. Let $\mathcal{C} : K \times D \to 2^Y$ be an upper continuous cone multivalued mapping with nonempty convex closed values. In addition, assume:*

(i) *For any fixed $t \in D$, the multivalued mapping $G(.,.,t) : K \times D \to 2^Y$ is upper $(-\mathcal{C})$- continuous. The multivalued mapping $N : K \times D \to 2^Y$ defined by $N(y, x) = H(y, x, x), (y, x) \in K \times D$, is upper $\mathcal{C}$- continuous.*

(ii) *For any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \mathrm{co}\{t_1, ..., t_n\}$, there is an index $j \in \{1, ..., n\}$ such that*

$$G(y, x, t_j) \not\subseteq H(y, x, x) + \mathrm{int}\mathcal{C}(y, x) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$G(y, \bar{x}, t) \not\subseteq H(y, \bar{x}, \bar{x}) + \mathrm{int}\mathcal{C}(y, \bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Proof.* We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y, x) = \{t \in D |\ G(y, x, t) \not\subseteq H(y, x, x) + \mathrm{int}\mathcal{C}(y, x)\},\ (y, x) \in K \times D$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D$, we set

$$\begin{aligned} A &= \{x \in D |\ 0 \in F(y, x, t) \text{ for all } y \in Q(x, t)\} \\ &= \{x \in D |\ t \in M(y, x) \text{ for all } y \in Q(x, t)\} \end{aligned}$$

$$\{x \in D |\ G(y, x, t) \not\subseteq H(y, x, x) + \mathrm{int}\mathcal{C}(y, x) \text{ for all } y \in Q(x, t)\}.$$

We claim that this subset is closed in $D$. Indeed, assume that a net $\{x_\alpha\} \subset D$ and $x_\alpha \to x$. Take arbitrary $y \in Q(x, t)$. Since $Q(\cdot, t)$ is a lower semicontinuous mapping and $x_\alpha \to x$, there exists a net $\{y_\alpha\}, y_\alpha \in Q(x_\alpha, t)$ such that $y_\alpha \to y$. For any neighborhood $V$ of the origin in $Y$ there is an index $\alpha_0$ such that for all $\alpha \leq \alpha_0$ the following inclusions hold:

$$G(y_\alpha, x_\alpha, t) \subseteq G(y, x, t) + V - \mathcal{C}(y, x),$$
$$H(y, x, x) \subseteq H(y_\alpha, x_\alpha, x_\alpha) + V + \mathcal{C}(y_\alpha, x_\alpha).$$

For $x_\alpha \in A$, we have

$$G(y_\alpha, x_\alpha, t) \not\subseteq H(y_\alpha, x_\alpha, x_\alpha) + \text{int}\mathcal{C}(y_\alpha, x_\alpha)$$

and then

$$G(y, x, t) + V - \mathcal{C}(y, x) \not\subseteq H(y_\alpha, x_\alpha, x_\alpha) + \mathcal{C}(y_\alpha, x_\alpha) + \text{int}\mathcal{C}(y_\alpha, x_\alpha).$$

Therefore, we conclude

$$G(y, x, t) + V - \mathcal{C}(y, x) \not\subseteq H(y, x, x) + V + \text{int}\mathcal{C}(y, x)$$

and so

$$G(y, x, t) + V \not\subseteq H(y, x, x) + \text{int}\mathcal{C}(y, x)$$

for arbitrary neighborhood $V$ of the origin in $Y$.

Now, suppose that

$$G(y, x, t) \subseteq H(y, x, x) + \text{int}\mathcal{C}(y, x).$$

For arbitrary neighborhood $V_\alpha$ of the origin in $Y$ there exist $a_\alpha \in G(y, x, t), v_\alpha \in V_\alpha$ and $a_\alpha + v_\alpha \notin H(y, x, t) + \text{int}\mathcal{C}(y, x)$. Without loss of generality, we may assume that $a_\alpha \to a$ and $v_\alpha \to 0$ and then $a_\alpha + v_\alpha \to a \in G(y, x, t) \subseteq H(x, x, x) + \text{int}\mathcal{C}(y, x)$. But, $H(x, x, x) + \text{int}\mathcal{C}y, x)$ is an open subset, there exists $\alpha_0$ such that for all $\alpha \leq \alpha_0$. Hence $a_\alpha + v_\alpha \in H(y, x, t) + \text{int}\mathcal{C}(y, x)$ and we have a contradiction. Thus, we conclude

$$G(y, x, t) \not\subseteq H(y, x, x) + \text{int}\mathcal{C}(y, x).$$

This shows that $x \in A$ and $A$ is a closed subset in $D$. Hence, for any fixed $t \in D$, the set

$$B = D \setminus A = \{x \in D |\ 0 \notin F(y, x, t) \text{ for some } y \in Q(x, t)\}$$

is open.

Further, Condition (ii) implies that the multivalued mapping $F$ is $Q-$KKM. Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.1 to deduce that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ for all } \ t \in P_2(\bar{x}) \ \text{ and } \ y \in Q(\bar{x}, t).$$

This is equivalent to

$$G(y, \bar{x}, t) \not\subseteq H(y, \bar{x}, \bar{x}) + \text{int}\mathcal{C}(y, \bar{x}) \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

□

Analogically, we obtain the following corollary.

**Corollary 3.6.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $G, H : K \times D \times D \to 2^Y$ be multivalued mappings. Let $\mathcal{C} : K \times D \to 2^Y$ be an upper continuous cone multivalued mapping with nonempty convex closed values. In addition, assume:*

(i) *For any fixed $t \in D$ the multivalued mapping $G(.,.,t) : K \times D \to 2^Y$ is lower $\mathcal{C}$- continuous and the multivalued mapping $N : K \times D \to 2^Y$ defined by $N(y, x) = H(y, x, x)$ is upper $(-\mathcal{C})$- continuous and has compact values.*

(ii) *For any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \mathrm{co}\{t_1, ..., t_n\}$ there is an index $j \in \{1, ..., n\}$ such that*

$$H(y, x, x) \not\subseteq G(y, x, t_j) - \mathrm{int}\mathcal{C}(y, x) \text{ holds for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$, and*

$$H(y, \bar{x}, \bar{x}) \not\subseteq G(y, \bar{x}, t) - \mathrm{int}\mathcal{C}(y, \bar{x}) \text{ for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Proof.* The proof is similar to the previous one of Corollary 3.6. □

**Corollary 3.7.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $\mathcal{R}$ be a relation linking $y \in K, x \in D, t \in D$. In addition, assume:*

(i) *For any fixed $t \in D$ the relation $\mathcal{R}(.,.,t)$ linking elements $y \in K, x \in D$is closed;*

(ii) *$\mathcal{R}$ is Q-KKM.*

*Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$\mathcal{R}(y, \bar{x}, t) \text{ holds for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*Proof.* We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y, x) = \{t \in D |\ \mathcal{R}(y, x, t) \text{ holds}\}$$

and

$$F(y, x, t) = t - M(y, x),\ (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D$, we set

$$\begin{aligned} A &= \{x \in D |\ \mathcal{R}(y, x, t) \text{ holds for all } y \in Q(x, t)\} \\ &= \{x \in D |\ 0 \in F(y, x, t) \text{ for all } y \in Q(x, t)\}. \end{aligned}$$

By arguments as in the proof of Corollary 3.3, we conclude that this subset is closed in $D$, therefore, the set

$$B = D \setminus A = \{x \in D |\ 0 \notin F(y, x, t) \text{ for some } y \in Q(x, t)\}$$

is open in $D$.

Further, it is easy to check that $\mathcal{R}$ is $Q$-KKM, so is the multivalued mapping $F$. Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.1 to deduce that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\mathcal{R}(y, \bar{x}, t) \text{ holds for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

□

**Corollary 3.8.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $G : K \times D \to 2^Y$ be a multivalued mapping. In addition, assume:*

(i) *For any fixed $t \in D$ the multivalued mapping $G(., t) : K \to 2^Y$ is closed;*

(ii) *For any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \mathrm{co}\{t_1, ..., t_n\}$ there is an index $j \in \{1, ..., n\}$ such that*

$$x \in G(y, t_j) \textit{ holds for all } \; t \in P_2(\bar{x}) \; \textit{ and } \; y \in Q(\bar{x}, t).$$

*Then, there exists $\bar{x} \in D$ such that*

$$\bar{x} \in P_1(\bar{x}) \cap \Big\{ \bigcap_{t \in P_2(\bar{x})} \bigcap_{y \in Q(\bar{x}, t)} G(y, t) \Big\}.$$

*Proof.* We define the multivalued mapping $F : K \times D \times D \to 2^X$ by

$$F(y, x, t) = x - G(y, t), \; (y, x, t) \in K \times D \times D.$$

For any fixed $t \in D$, we claim that the set

$$\begin{aligned} A &= \{x \in D | \; 0 \in F(y, x, t) \; \text{ for all } y \in Q(x, t)\} \\ &= \{x \in D | \; x \in G(y, t) \text{ for all } y \in Q(x, t)\} \end{aligned}$$

is closed in $D$. Indeed, assume that a net $\{x_\alpha\} \subset A$ and $x_\alpha \to x$. Take arbitrary $y \in Q(x, t)$. Since $Q(., t)$ is a lower semicontinuous mapping and $x_\alpha \to x$ there exists a net $\{y_\alpha\}, y_\alpha \in Q(x_\alpha, t)$ such that $y_\alpha \to y$. Therefore, $x_\alpha \in G(y_\alpha, t), x_\alpha \to x; y_\alpha \to y$, the closedness of $G(., t)$ implies that $x \in G(y, t)$. This means that $x \in A$ and $A$ is closed. Hence, the set

$$B = D \setminus A = \{x \in D | \; 0 \notin x - G(y, t) = F(y, x, t) \text{ for some } \; y \in Q(x, t)\}$$

is open in $D$.

Further, Condition (ii) implies that for any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \mathrm{co}\{t_1, ..., t_n\}$ there is an index $j \in \{1, ..., n\}$ such that

$$0 \in F(y, x, t_j) \text{ holds for all } \; t \in P_2(\bar{x}) \; \text{ and } \; y \in Q(\bar{x}, t).$$

This implies that $F$ is $Q-$ KKM.

Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.1 to conclude that there is $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$0 \in F(y, \bar{x}, t) \text{ holds for all } \; t \in P_2(\bar{x}) \; \text{ and } \; y \in Q(\bar{x}, t).$$

This yields

$$\bar{x} \in P_1(\bar{x}) \cap \Big\{ \bigcap_{t \in P_2(\bar{x})} \bigcap_{y \in Q(\bar{x}, t)} G(y, t) \Big\}.$$

□

As a special case of the above corollary, we obtain the following corollary which is the KKM Theorem.

**Corollary 3.9.** *Let $D$ be a compact convex subset of $X$. Then, for any KKM mapping $G : D \to 2^D$ with nonempty closed values, one has $\bigcap_{t \in D} G(t) \neq \emptyset$.*

*Proof.* It is obvious. □

**Corollary 3.10.** *Let $D, K, P_1, P_2$ and $Q$ be as in Theorem 3.1. Let $\mathcal{F} \subseteq K \times D \times D$ be a subset satisfying the following conditions:*
(i) *For any fixed $t \in D$ the set*

$$B = \{x \in D|\ (y, x, t) \notin \mathcal{F} \ \text{ for some } \ y \in Q(x, t)\,\}$$

*is open in $D$;*
(ii) *For any finite set $\{t_1, ..., t_n\} \subset D$ and $x \in \mathrm{co}\{t_1, ..., t_n\}$ there is an index $j \in \{1, ..., n\}$ such that*

$$(y, x, t_j) \in \mathcal{F} \ \text{ for all } \ t \in P_2(x) \ \text{ and } \ y \in Q(x, t).$$

*Then, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and*

$$(y, \bar{x}, t) \in \mathcal{F} \text{ for all } \ t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*(This implies that $\bigcup_{t \in P_2(x)} Q(\bar{x}, t) \times \{\bar{x}\} \times \{t\} \subseteq \mathcal{F}$).*

*Proof.* We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y, x) = \{t \in D|\ (y, x, t) \in \mathcal{F}\},\ (y, x) \in K \times D,$$

and

$$F(y, x, t) = t - M(y, x),\ \ (y, x, t) \in K \times D \times D.$$

By Condition (i), for any fixed $t \in D$, the set

$$\begin{aligned} B &= \{x \in D|\ 0 \notin F(y, x, t) \ \text{ for some } \ y \in Q(x, t)\,\} \\ &= \{x \in D|\ (y, x, t) \notin \mathcal{F} \ \text{ for some } \ y \in Q(x, t)\,\} \end{aligned}$$

is open in $D$. Further, Condition (ii) implies that the multivalued mapping $F$ is $Q-$ KKM. Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.1. □

Further, let $D, E, K$ be subsets as in the introduction and $P : D \to 2^E, Q : D \times E \to 2^K$ and $F : K \times D \times E \to 2^Y$ be multivalued mappings. We are interested in the following generalized quasi-equilibrium problem: Find $\bar{x} \in D$ such that

$$0 \in F(y, \bar{x}, t) \text{ for all } t \in P(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

We define the multivalued mapping $T : E \to 2^D$ by

$$T(t) = (D \setminus P^{-1}(t)) \cup \{x \in D|\ 0 \in F(y, x, t) \ \text{ for all } \ y \in Q(x, t)\}.$$

**Theorem 3.11.** *The following conditions are sufficient for the above problem to have a solution:*
(i) *$D$ is a nonempty compact subset;*
(ii) *$\bigcap_{t \in E} clT(t) \neq \emptyset$ implies $\bigcap_{t \in E} T(t) \neq \emptyset$;*
(iii) *$F : K \times D \times E \to 2^Y$ is a generalized $Q - KKM$ multivalued mapping.*

*Proof.* Since $F$ is a generalized $Q - KKM$ multivalued mapping, it implies that for any $t \in E$ there is $x \in D$ such that $0 \in F(y, x, t)$ for all $y \in Q(x, t)$. This shows that $T(t)$ is nonempty for any $t \in E$. Now, we claim that for any finite set

$\{t_1, ..., t_n\} \subset E$, one has $\bigcap_{i=1}^{n} clT(t_i) \neq \emptyset$. Indeed, assume on the contrary that there exists a finite set $A = \{t_1, ..., t_n\} \subset E$ such that

$$\bigcap_{i=1}^{n} clT(t_i) = \emptyset. \tag{1}$$

Since the multivalued mapping $F$ is generalized $Q-$KKM, there exists a finite subset $\{x_1, ..., x_n\} \subset D$ such that for any subset $\{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_n\}$ and for any $x \in \text{co}\{x_{i_1}, ..., x_{i_k}\}$ there exists $t_{i_j} \in \{t_1, ..., t_n\}$ such that $0 \in F(y, x, t_{i_j})$ for all $y \in Q(x, t_{i_j})$. We take $B = \text{co}\{x_1, ..., x_n\} \subset D$. It is clear that $B$ is a nonempty convex compact subset in $D$. We define the multivalued mapping $S : B \to 2^B$ by

$$S(x) = \text{co}\{x_i |\ x \notin clT(t_i)\}. \tag{2}$$

It implies from (1) that for any $x \in B, S(x)$ is a nonempty convex compact subset in $D$. Further, we prove that for each $z \in B$ there is an open subset $O_z$ in $B$ such that $O_z \subseteq S^{-1}(z)$. Indeed, if $S^{-1}(z) = \emptyset$, we take $O_z = \emptyset$. Otherwise, let $x \in S^{-1}(z)$. We define

$$\Delta_x = \bigcup_{i \in I_x} clT(t_i), \text{ where } I_x = \{i \in \{1, ..., n\} |\ x \notin clT(t_i)\}.$$

It is easy to see that $\Delta_x$ is closed in $E$ and then $O_x = E \setminus \Delta_x$ is open in $E$.

Now, we affirm that $O_x \subseteq S^{-1}(z)$. In fact, let $v \in O_x = E \setminus \Delta_x$. It implies that $v \notin \Delta_x$ and hence $v \notin clT(t_i)$ for every $i \in I_x$. Then, by the definition of $S$, we have $S(x) \subseteq S(v)$. Since $z \in S(x)$, it follows that $z \in S(v)$ and therefore $v \in S^{-1}(z)$ for all $v \in O_x$. Lastly, we take $O_z = \bigcup_{x \in S^{-1}(z)} O_x$ which is also open in $B$ and $O_z \subseteq S^{-1}(z)$ for every $z \in B$. Further, we show that $B = \bigcup_{z \in B} O_z$. Indeed, for $z \in B$, there is $v \in B$ such that $v \in S(z)$ and then $z \in S^{-1}(v)$. It has been shown $z \in O_z \subseteq S^{-1}(v)$ and $O_v = \bigcup_{z \in S^{-1}(v)} O_z$. Therefore, we conclude that $B \subseteq \bigcup_{z \in B} O_z$. Since the other inclusion is obviously valid, the result holds.

Thus, we apply Theorem 2.10 in Section 2 to conclude that there exists $\bar{x} \in B$ such that $\bar{x} \in S(\bar{x})$. We have

$$\bar{x} \in S(\bar{x}) = co\{x_i |\ i \in I_{\bar{x}}\} \subseteq \bigcup_{i \in I_{\bar{x}}} clT(t_i).$$

At time, by the definition of $I_{\bar{x}}$ it results that $x \notin \bigcup_{i \in I_{\bar{x}}} clT(t_i)$. So, we obtain a contradiction. Thus, we deduce that the family $\{clT(t), t \in E\}$ has the finite intersection property. Applying Theorem 6 in [1], we conclude that $\bigcap_{t \in E} clT(t) \neq \emptyset$ and using Condition (ii), we then have $\bigcap_{t \in E} T(t) \neq \emptyset$.

Consequently, take $\bar{x} \in \bigcap_{t \in E} T(t)$, it gives $\bar{x} \in T(t)$ for all $t \in E$. For arbitrary $t \in P(\bar{x})$, it yields $\bar{x} \notin D \setminus P^{-1}(t)$. By the definition of $T$, we affirm that $0 \in F(y, \bar{x}, t)$ for all $y \in Q(x, t)$ and the proof of the theorem is complete. □

**Remark 3.12.** (1) We recall that a multivalued mapping $T : E \to 2^D$ is said to be intersectionally closed on $E$ if $\bigcap_{t \in E} clT(t) = cl(\bigcap_{t \in E} T(t))$. Therefore, every intersectionally closed multivalued mapping satisfies Condition (ii).

(2) Let $Q$ and $F$ be as above. If for any fixed $t \in E$ the multivalued mapping $F(\cdot,\cdot,t) : K \times D \to 2^Y$ is closed and the multivalued mapping $Q(\cdot,t) : D \to 2^K$ is lower semicontinuous, then the set

$$\{x \in D|\ 0 \in F(y,x,t) \text{ for all } \ y \in Q(x,t)\}$$

is closed. Therefore, in this case, the multivalued mapping $T$ defined as above satisfies Condition (ii) if $P^{-1}(t)$ is open for each $t \in E$.

(3) Let both $\Lambda$ and $X$ be topological spaces. We recall that a multivalued mapping $G : \Lambda \to 2^X$ is said to be outer-continuous at $\lambda_0 \in \Lambda$ if

$$\limsup_{\lambda \to \lambda_0} G(\lambda) \subseteq G(\lambda_0),$$

where $\limsup_{\lambda \to \lambda_0} G(\lambda)$ denotes the Kuratowski-Painlevé outer limit of $G$ at $\lambda_0$, that is $x \in \limsup_{\lambda \to \lambda_0} G(\lambda)$ if and only if there is a net $\lambda_\alpha$ converging to $\lambda_0, \lambda_\alpha \neq \lambda_0$ and $x_\alpha \in G(\lambda_\alpha)$ such that $x_\alpha$ converges to $x$.. Further, we assume that $Q$ and $F$ are given as above. Moreover, for any fixed $t \in E$ the multivalued mapping $F(\cdot,\cdot,t) : K \times D \to 2^Y$ is outer-continuous and the multivalued mapping $Q(\cdot,t) : D \to 2^K$ is lower semicontinuous, then the set

$$A = \{x \in D|\ 0 \in F(y,x,t) \text{ for all } \ y \in Q(x,t)\}$$

is closed.

Indeed, assume that a net $\{x_\alpha\} \subset A$ converges to $x$. We have

$$0 \in F(y_\alpha, x_\alpha, t) \text{ for all } \ y \in Q(x_\alpha, t).$$

For each $y \in Q(x,t)$ there exists $y_\alpha \in Q(x_\alpha,t)$ and $y_\alpha$ converges to $y$. Therefore, we have $0 \in F(y_\alpha, x_\alpha, t)$ and $(y_\alpha, x_\alpha)$ converges to $(y,x)$. This gives $0 \in \limsup_{(y_\alpha,x_\alpha)\to(y,x)} F(y_\alpha,x_\alpha,t) \subseteq F(y,x,t)$ for all $y \in Q(x,t)$ and so $x \in A$. This shows that $A$ is closed.

Further, let $G : K \times D \to 2^Y, P : D \to 2^E$ and $Q : D \times E \to 2^E$ be multivalued mappings. We define the multivalued mapping $S : E \to 2^D$ by

$$S(t) = (D \setminus P^{-1}(t)) \cup \{x \in D|\ x \in G(y,t) \ \text{ for all } \ y \in Q(x,t)\}.$$

We have

**Corollary 3.13.** *Let $D, K, E, P$ and $Q$ be as above. Let $G : K \times D \to 2^Y$ be a multivalued mapping. In addition, assume:*

(i) *$D$ is compact;*

(ii) *$\bigcap_{t\in E} clS(t) \neq \emptyset$ implies $\bigcap_{t\in E} S(t) \neq \emptyset$;*

(iii) *For any finite set $\{t_1,...,t_n\} \subset E$ there is a finite set $\{x_1,...,x_n\} \subseteq D$ such that for any $x \in \mathrm{co}\{x_{i_1},...,x_{i_k}\}$ there is a $t_{i_j} \in \{t_{i_1},...,t_{i_n}\}$ such that $x \in G(y,t_{i_j})$ for all $y \in Q(x,t_{i_j})$.*

*Then, there exists $\bar{x} \in D$ such that*

$$\bar{x} \in \bigcap_{t\in P(\bar{x})} \bigcap_{y\in Q(\bar{x},t)} G(y,t).$$

*Proof.* We define the multivalued mapping $F : K \times D \times E \to 2^X$ by

$$F(y,x,t) = x - G(y,t),\ (y,x,t) \in K \times D \times E.$$

Condition (iii) implies that for any finite set $\{t_1, ..., t_n\} \subset E$ there is a finite set $\{x_1, ..., x_n\} \subseteq D$ such that for each $x \in co\{x_{i_1}, ..., x_{i_k}\}$ there is a $t_{i_j} \in \{t_{i_1}, ..., t_{i_n}\}$ such that $x \in G(y, t_{i_j})$ for all $y \in Q(x, t_{i_j})$. This implies that $F$ is $Q-$ KKM.

Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.11 to conclude that there is $\bar{x} \in D$ such that

$$0 \in F(y, \bar{x}, t) \text{ holds for all } t \in P(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This yields

$$\bar{x} \in \bigcap_{t \in P_2(\bar{x})} \bigcap_{y \in Q(\bar{x},t)} G(y,t).$$

This completes the proof of the corollary. □

**Corollary 3.14.** *Let $D, K, E, P$ and $Q$ be as above with $P^{-1}(t)$ being open for any $t \in E$. Let $\mathcal{F} \subseteq K \times D \times E$ be a subset satisfying the following conditions:*

(i) *$D$ is nonempty compact;*

(ii) *For any fixed $t \in E$ the set*

$$B = \{x \in D |\ (y,x,t) \notin \mathcal{F} \text{ for some } y \in Q(x,t)\}$$

*is open in $D$;*

(iii) *For any finite set $\{t_1, ..., t_n\} \subset E$ there is a finite set $\{x_1, ..., x_n\} \subseteq D$ such that for any $x \in co\{x_{i_1}, ..., x_{i_k}\}$ there is a $t_{i_j} \in \{t_{i_1}, ..., t_{i_n}\}$ such that $(y, x, t_{i_j}) \in \mathcal{F}$ for all $y \in Q(x, t_{i_j})$.*

*Then, there exists $\bar{x} \in D$ such that*

$$(y, \bar{x}, t) \in \mathcal{F} \text{ for all } t \in P(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

*(This implies that $\bigcup_{t \in P(\bar{x})} Q(\bar{x}, t) \times \{\bar{x}\} \times \{t\} \subseteq \mathcal{F}$).*

*Proof.* We define the multivalued mappings $M : K \times D \to 2^X, F : K \times D \times D \to 2^D$ by

$$M(y,x) = \{t \in D |\ (y,x,t) \in \mathcal{F},\ \},\ (y,x) \in K \times D,$$

and

$$F(y,x,t) = t - M(y,x),\ (y,x,t) \in K \times D \times D.$$

By Condition (i), for any fixed $t \in D$, the set

$$\begin{aligned} B &= \{x \in D |\ 0 \notin F(y,x,t) \text{ for some } y \in Q(x,t)\} \\ &= \{x \in D |\ (y,x,t) \notin \mathcal{F} \text{ for some } y \in Q(x,t)\} \end{aligned}$$

is open in $D$. Therefore, for any $t \in E$, the set

$$T(t) = (D \setminus P^{-1}(t)) \cup \{x \in D |\ 0 \in F(y,x,t) \text{ for all } y \in Q(x,t)\}$$

is closed in $D$. Thus, Condition (ii) of Theorem 3.11 is satisfied. Further, Condition (ii) implies that the multivalued mapping $F$ is $Q-$ KKM. Therefore, to complete the proof of the corollary, it remains to apply Theorem 3.11. □

## Acknowledgements

The authors would like to thank the anonymous referee for some useful suggestions and corrections.

## References

[1] C. Berge, *Topological Spaces*, Dover Publications, NY, 1997.
[2] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1-4) (1994), 123 145.
[3] T. T. T. Duong and N. X. Tan, On the generalized quasi-equilibrium problem of type I and related problems, *Adv. Nonlinear Var. Inequal* **13** (1) (2010), 29-47.
[4] K. Fan, *A minimax inequality and application,* Inequalities III, O. Shisha (Ed), Acad. Press, New-York, 1972.
[5] N. X. Hai and P. Q. Khanh, The solution existence of general variational inclusion problems, *J. Math. Anal. Appl.* **328** (2007), 1268-1277.
[6] L. J. Lin, Z. T. Yu, and G. Kassay, Existence of equilibria for monotone multivalued mappings and its applications to vectorial equilibria, *J. Optim. Theory Appl.* **114** (2002), 189-208.
[7] L. J. Lin and N. X. Tan, Quasi-equilibrium problems of Blum-Oettli type and related problems, *Acta Math. Vietnam.* **34** (2009), 111-123.
[8] D. T. Luc, *Theory of Vector Optimization*, Lect. Notes in Eco. and Math. Systems, Springer Verlag, Berlin, Germany, Vol. 319, 1989.
[9] D. T. Luc and N. X. Tan, Existence conditions in variational inclusions with constraints, *Optimization* **53** (2004), 505- 515.
[10] D. T. Luc, An abstract problem in variational analysis, *J. Optim. Theory Appl.* **138** (2008), 65-76.
[11] N. B. Minh and N. X. Tan, Some sufficient conditions for the existence of equilibrium points concerning multivalued mappings, *Vietnam J. Math.* **28** (2000), 295-310.
[12] N. B. Minh and N. X. Tan, On the existence of solutions of quasivariational inclusion problems of Stampacchia type, *Adv. Nonlinear Var. Inequal.* **8** (2005), 1–16.
[13] S. Park, Fixed points and quasi-equilibrium problems, *Nonlinear Oper. Theory. Math. and Com. Model.* **32** (2000), 1297-1304.
[14] N. X. Tan, On the existence of solutions of quasi-variational inclusion problems, *J. Optim. Theory Appl.* **123** (2004), 619-638.
[15] N. C. Yannelis and N. D. Prabhaker, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Eco.* **12** (1983), 233-245.

Vinh Technical Teachers Training University
*E-mail address*: thuyduongktv@yahoo.com.vn

Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam
*E-mail address*: nxtan@math.ac.vn