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ON STABILITY OF PROPERTIES OF GENERAL RELATIONAL SYSTEMS UNDER POWERS

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ABSTRACT. In the paper, we study powers of general relational systems. We solve the problem of finding sufficient conditions under which some properties of relational systems are preserved by powers of these systems. The properties considered are defined with respect to a decomposition of the corresponding index set and they are a common generalization of certain properties of binary and ternary relational systems. We also associate new relational systems with given ones and then study their powers.

1. PRELIMINARIES

Let n be a positive integer. Let A, H be nonempty sets and let A^H be the set of all mappings from H into A. A *sequence* in A is any mapping from H into A, denoted by $(a_h \mid h \in H)$. If $H = \{1, \dots, n\}$, we write $(a_1, \dots, a_n)$ instead of $(a_h \mid h \in H)$. We usually identify the set of all mappings from $\{1, \dots, n\}$ into A with the cartesian power $A^n = \underbrace{A \times \ldots \times A}_{n-terms}$. By a *relation* R in the general sense we mean a set of mappings $R \subseteq A^H$. The pair (A, R) is called a *relational system* of type H. The sets A and H are called the *carrier set* and the *index set* of R, respectively. In case of $H = \{1, \dots, n\}, R \subseteq A^H$ is called an *n-ary relation* on A and (A, R) is called an *n-ary relational system.*

Let $\mathbf{A} = (A, R),\ \mathbf{B} = (B, S)$ be n-ary relational systems. A mapping $f : B \to A$ is called a *homomorphism* of B into A provided that if $(x_1, \dots, x_n) \in S$ then $(f(x_1), \dots, f(x_n)) \in R$. We denote by $Hom(\mathbf{B}, \mathbf{A})$ the set of all homomorphisms of B into A. If $f : B \to A$ is a bijection and both $f : B \to A$ and $f^{-1} : A \to B$ are homomorphisms, then f is called an *isomorphism* of B into A. We will write $B \cong A$ if there exists an isomorphism of B into A.

An n-ary relational system $\mathbf{A} = (A, R)$ is said to be

(i) *reflexive* provided that $(x_1, \dots, x_n) \in R$ whenever $x_1 = \cdots = x_n \in A$.

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(ii) *diagonal* provided that, whenever (x_i^j) is an $n \times n$-matrix over A, from $(x_i^1, \ldots, x_i^n) \in R$ for each $i = 1, \ldots, n$ and $(x_1^j, \ldots, x_n^j) \in R$ for each $j = 1, \ldots, n$ it follows that $(x_1^1, \ldots, x_n^n) \in R$.

Let $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, S)$ be n-ary relational systems. The *power* of $\mathbf{A}$ and $\mathbf{B}$ is the n-ary relational system $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$ where for any $f_1, \ldots, f_n \in Hom(\mathbf{B}, \mathbf{A}), (f_1, \ldots, f_n) \in r$ if and only if $(x_1, \ldots, x_n) \in S$ implies $(f_1(x_1), \ldots, f_n(x_n)) \in R$ whenever $x_1, \ldots, x_n \in B$.

In [6] J. Šlapal investigated some relations and studied their properties such as the following.

Let H be a set with card $H \geq 2$. A *b-decomposition* of H is a pair $(\{K_i\}_{i=1}^3, \sigma)$ where $\{K_i\}_{i=1}^3$ is a sequence of three sets satisfying

(i) $\bigcup_{i=1}^{3} K_i = H$,

(ii) $K_i \cap K_j = \emptyset$ for all $i,\ j \in \{1, 2, 3\}$, $i \neq j$,

(iii) $0 <$ card $K_1 =$ card K_2 and $\sigma : K_1 \to K_2$ is a bijection.

Let H be a set with card $H \geq 3$. Let $m \in \mathbb{N}$ be such that there exists a cardinal number p with $p \cdot m =$ card H. A *t_m-decomposition* of H is a pair $(\{K_i\}_{i=1}^m, \{\sigma_i\}_{i=1}^{m-1})$ where $\{K_i\}_{i=1}^m$ is a sequence of m sets satisfying

(i) $\bigcup_{i=1}^{m} K_i = H$,

(ii) $K_i \cap K_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$,

(iii) card $K_i = p$ for each $i \in \{1, \ldots, m\}$ and $\{\sigma_i\}_{i=1}^{m-1}$ is a sequence of bijections $\sigma_i : K_i \to K_{i+1}$ for $i = 1, \ldots, m-1$.

For any map $f : H \to A$ and any subset $K \subseteq H$, we denote by $f|_K$ the restriction of f to K.

Let A, H be sets and let $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a *b-decomposition* of H. If $R, S \subseteq A^H$, then we define relations

(i) $E_{\mathcal{K}} \subseteq A^H$, called the *diagonal* with regard to $\mathcal{K}$ by:

$$f \in E_{\mathcal{K}} \Longleftrightarrow f|_{K_1} = f|_{K_2} \circ \sigma;$$

(ii) $R_{\mathcal{K}}^{-1} \subseteq A^H$, called the *inversion of* R with regard to $\mathcal{K}$ by:

$$f \in R_{\mathcal{K}}^{-1} \Longleftrightarrow \exists g \in R :\ f|_{K_1} = g|_{K_2} \circ \sigma,\ f|_{K_2} = g|_{K_1} \circ \sigma^{-1},\ f|_{K_3} = g|_{K_3};$$

(iii) $(RS)_{\mathcal{K}} \subseteq A^H$, called the *composition of* R *and* S with regard to $\mathcal{K}$ by:

$$\begin{aligned} f \in (RS)_{\mathcal{K}} \Longleftrightarrow \exists g \in R,\ h \in S :\ g|_{K_2} &= h|_{K_1} \circ \sigma^{-1},\ g|_{K_3} = h|_{K_3}, \\ f|_{K_1} &= g|_{K_1},\ f|_{K_2 \cup K_3} = h|_{K_2 \cup K_3}. \end{aligned}$$

Let $R \subseteq A^H$, and let $\mathcal{K} = (\{K\}_{i=1}^m, \{\sigma_i\}_{i=1}^{m-1})$ be a t_m-decomposition of H. Then we define a relation ${}^1R_{\mathcal{K}} \subseteq A^H$ by:

$$\begin{aligned} f \in {}^1R_{\mathcal{K}} \Longleftrightarrow \exists g \in R :\ f|_{K_i} &= g|_{K_{i+1}} \circ \sigma_i \text{ for } i = 1, \ldots, m-1, \\ f|_{K_m} &= g|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}. \end{aligned}$$

Definition 1.1. Let $R \subseteq A^H$ with *card* $H \geq 2$ and let $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a b-decomposition of H. Then R is called

(i) *reflexive* (*irreflexive*) with regard to $\mathcal{K}$ if $E_{\mathcal{K}} \subseteq R$ ($R \cap E_{\mathcal{K}} = \emptyset$),

(ii) *symmetric* (*asymmetric*, *antisymmetric*) with regard to $\mathcal{K}$ if $R_{\mathcal{K}}^{-1} \subseteq R$ ($R \cap R_{\mathcal{K}}^{-1} = \emptyset$, $R \cap R_{\mathcal{K}}^{-1} \subseteq E_{\mathcal{K}}$),

(iii) *transitive* (*atransitive*) with regard to $\mathcal{K}$ if $R_{\mathcal{K}}^2 \subseteq R$ ($R \cap R_{\mathcal{K}}^n = \emptyset$ for every $n \in \mathbb{N}, n \geq 2$).

Definition 1.2. Let $R \subseteq A^H$ with card $H \geq 3$ and let $\mathcal{K} = (\{K\}_{i=1}^m, \{\sigma_i\}_{i=1}^{m-1})$ be a t_m-decomposition of H. Then R is called *cyclic* (*acyclic*) with regard to $\mathcal{K}$ if ${}^1R_{\mathcal{K}} \subseteq R$ ($R \cap {}^1R_{\mathcal{K}} = \emptyset$).

Let $\mathbf{A} = (A, R)$ be a relational system. If R has any one of the properties with regard to $\mathcal{K}$ defined above, we say that $\mathbf{A}$ has the same property.

Proposition 1.3. [6] *Let $R, \acute{R} \subseteq A^H$, and let $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a b-decomposition of H. Then the following statements hold.*

(i) *If $R, \acute{R}$ are reflexive (irreflexive, symmetric) with regard to $\mathcal{K}$, then $R \cup \acute{R}, R \cap \acute{R}$ and $R_{\mathcal{K}}^{-1}$ have the same property.*

(ii) *If $R, \acute{R}$ are transitive (asymmetric, antisymmetric, atransitive) with regard to $\mathcal{K}$, then $R \cap \acute{R}$ and $R_{\mathcal{K}}^{-1}$ have the same property.*

Proposition 1.4. [6] *Let $R, \acute{R} \subseteq A^H$, and let $\mathcal{K} = (\{K_i\}_{i=1}^m, \{\sigma_i\}_{i=1}^{m-1})$ be a t_m-decomposition of H. Then the following statements hold.*

(i) *If $R, \acute{R}$ are cyclic with regard to $\mathcal{K}$, then $R \cup \acute{R}, R \cap \acute{R}$ and ${}^1R_{\mathcal{K}}$ have the same property.*

(ii) *If $R, \acute{R}$ are acyclic with regard to $\mathcal{K}$, then $R \cap \acute{R}$ and ${}^1R_{\mathcal{K}}$ have the same property.*

Let $R \subseteq G^H$, and let $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a b-decomposition of H. Let $X = \{x_k; k \in K_3\}$ be a family of elements of G. Then we define a relation $R_{X,\mathcal{K}} \subseteq G^{H-K_3}$ by

$$\begin{aligned} f \in R_{X,\mathcal{K}} \Longleftrightarrow \exists g \in R : f(k) &= g(k) \text{ for each } k \in K_1 \cup K_2 \text{ and,} \\ g(k) &= x_k \text{ for each } k \in K_3. \end{aligned}$$

$R_{X,\mathcal{K}}$ is called the *X-projection* of R with regard to $\mathcal{K}$.

We denote the b-decomposition of $H - K_3$ given by $\tilde{\mathcal{K}} = (\{K_i\}_{i=1}^3, \sigma)$, where $\tilde{K}_i = K_i$ for $i = 1, 2$, and $K_3 = \emptyset$.

Proposition 1.5. [6] *Let $R \subseteq G^H$, and let $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a b-decomposition of H. If R has any one of the properties with regard to $\mathcal{K}$ introduced in Definitions 1.1 and 1.2, then $R_{X,\mathcal{K}}$ has the same property with regard to $\tilde{\mathcal{K}}$ for each family $X = \{x_k; k \in K_3\}$ of elements of G.*

2. Main Results

Lemma 2.1. *Let* $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 2$, $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ *be a b-decomposition of* H, *and let* $f_i \in Hom(\mathbf{B}, \mathbf{A})$ *for* $i = 1, \ldots, n$ *then* $(f_1, \ldots, f_n) \in E_{\mathcal{K}}$ *if and only if* $(f_1(x), \ldots, f_n(x)) \in E_{\mathcal{K}}$ *for each* $x \in B$.

Proof. Suppose $|K_1| = |K_2| = l, |K_3| = m$. Let $\varphi = (f_1, \ldots, f_n)$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$.

($\Longrightarrow$) Let $\varphi \in E_{\mathcal{K}}$. So $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ and $\varphi|_{K_1} = \varphi|_{K_2} \circ \sigma$. Therefore $\varphi|_{K_1}(i) = \varphi|_{K_2} \circ \sigma(i)$ for each $i \in \{1, \ldots, l\}$. Thus we have

$$f_i = f_{\sigma(i)} \text{ for each } i = 1, \ldots, l. \tag{2.1}$$

Let $x \in B$ and $f_x = (f_1(x), \ldots, f_n(x)) \in A^H$. By (2.1), we have $f_i(x) = f_{\sigma(i)}(x)$ for each $i = 1, \ldots, l$. Thus $f_x|_{K_1}(i) = f_x|_{K_2} \circ \sigma(i)$ for each $i \in \{1, \ldots, l\}$, so we get $f_x|_{K_1} = f_x|_{K_2} \circ \sigma$. We can see that $f_x = (f_1(x), \ldots, f_n(x)) \in E_{\mathcal{K}}$.

($\Longleftarrow$) Assume $(f_1(x), \ldots, f_n(x)) \in E_{\mathcal{K}}$ for each $x \in B$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. Let $x \in B$ and $f_x = (f_1(x), \ldots, f_n(x))$. By the assumption, we have $f_x = (f_1(x), \ldots, f_n(x)) \in E_{\mathcal{K}}$. Then $f_x|_{K_1} = f_x|_{K_2} \circ \sigma$. This implies that

$$f_i(x) = f_{\sigma(i)}(x) \text{ for each } i = 1, \ldots, l. \tag{2.2}$$

Suppose $\varphi \notin E_{\mathcal{K}}$. Then $\varphi|_{K_1} \neq \varphi|_{K_2} \circ \sigma$, so there exists $j \in \{1, \ldots, l\}$ such that $\varphi|_{K_1}(j) \neq \varphi|_{K_2} \circ \sigma(j)$. This yields $f_j \neq f_{\sigma(j)}$. Therefore $f_j(x) \neq f_{\sigma(j)}(x)$ for some $x \in B$, which contradicts (2.2). Thus $\varphi \in E_{\mathcal{K}}$. $\square$

Lemma 2.2. *Let* $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 2$, $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ *be a b-decomposition of* H. *Let* $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$ *and let* $\mathbf{B}$ *be reflexive. Then the following statements hold.*

(i) *If* $(f_1, \ldots, f_n) \in r_{\mathcal{K}}^{-1}$, *then* $(f_1(x), \ldots, f_n(x)) \in R_{\mathcal{K}}^{-1}$ *for each* $x \in B$ *where* $f_i \in Hom(\mathbf{B}, \mathbf{A})$ *for* $i = 1, \ldots, n$.

(ii) *If* $(f_1, \ldots, f_n) \in r_{\mathcal{K}}^{k}$, *then* $(f_1(x), \ldots, f_n(x)) \in R_{\mathcal{K}}^{k}$ *for each* $x \in B$ *where* $f_i \in Hom(\mathbf{B}, \mathbf{A})$ *for* $i = 1, \ldots, n$, $k \in \mathbb{N}, k \geq 1$.

Proof. Suppose $|K_1| = |K_2| = l, |K_3| = m$.

(i) Let $\varphi \in r_{\mathcal{K}}^{-1}$. Then $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$ and there exists a mapping $\psi \in r$ such that $\varphi|_{K_1} = \psi|_{K_2} \circ \sigma$, $\varphi|_{K_2} = \psi|_{K_1} \circ \sigma^{-1}$, $\varphi|_{K_3} = \psi|_{K_3}$. Thus $\varphi|_{K_1}(i) = \psi|_{K_2} \circ \sigma(i)$, $\varphi|_{K_2}(l+i) = \psi|_{K_1} \circ \sigma^{-1}(l+i)$ for each $i \in \{1, \ldots, l\}$ and $\varphi|_{K_3}(j) = \psi|_{K_3}(j)$ for each $j \in \{2l+1, \ldots, n\}$. Let $\varphi = (f_1, \ldots, f_n)$ and $\psi = (g_1, \ldots, g_n)$ where $f_i, g_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. Then we have

$$f_i = g_{\sigma(i)} \text{ and } f_{l+i} = g_{\sigma^{-1}(l+i)} \text{ for each } i = 1, \ldots, l, \tag{2.3}$$

and we also have

$$f_j = g_j \text{ for each } j = 2l+1, \ldots, n. \tag{2.4}$$

Let $x \in B$ and $f_x = (f_1(x), \ldots, f_n(x))$, $g_x = (g_1(x), \ldots, g_n(x)) \in A^H$. By (2.3) and (2.4), we have $f_i(x) = g_{\sigma(i)}(x)$ and $f_{l+i}(x) = g_{\sigma^{-1}(l+i)}(x)$ for each $i = 1, \ldots, l$, and $f_j = g_j$ for each $j = 2l+1, \ldots, n$. Thus $f_x|_{K_1}(i) = g_x|_{K_2} \circ \sigma(i)$ and $f_x|_{K_2}(l+i) = g_x|_{K_1} \circ \sigma^{-1}(l+i)$ for each $i \in \{1, \ldots, l\}$, and $f_x|_{K_3}(j) = g_x|_{K_3}(j)$ for each $j \in \{2l+1, \ldots, n\}$. Hence $f_x|_{K_1} = g_x|_{K_2} \circ \sigma$, $f_x|_{K_2} = g_x|_{K_1} \circ \sigma^{-1}$, $f_x|_{K_3} = g_x|_{K_3}$. As $\psi \in r$ and $\mathbf{B}$ is reflexive, we get $g_x \in R$. This implies that $f_x \in R_{\mathcal{K}}^{-1}$.

(ii) By the definition of $\mathbf{A}^{\mathbf{B}}$, the statement holds for $k = 1$. Suppose the statement holds for $k = p$, $p \in \mathbb{N}$, $p \geq 1$. Let $\varphi = (f_1, \ldots, f_n) \in r_{\mathcal{K}}^{p+1}$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. Then there exist mappings $\psi \in r_{\mathcal{K}}^{p}$, $\chi \in r$ such that $\varphi|_{K_1} = \psi|_{K_1}$, $\varphi\ |_{K_2 \cup K_3} = \chi|_{K_2 \cup K_3}$, $\psi|_{K_2} = \chi|_{K_1} \circ \sigma^{-1}, \psi|_{K_3} = \chi|_{K_3}$. Let $\psi = (g_1, \ldots, g_n)$ and $\chi = (h_1, \ldots, h_n)$ where $g_i, h_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. Then $\varphi|_{K_1}(i) = \psi|_{K_1}(i)$, $\psi|_{K_2}(l+i) = \chi|_{K_1} \circ \sigma^{-1}(l+i)$ for each $i = 1, \ldots, l$, $\varphi|_{K_2 \cup K_3}(j) = \chi|_{K_2 \cup K_3}(j)$ for each $j = l+1, \ldots, n$, and $\psi|_{K_3}(q) = \chi|_{K_3}(q)$ for each $q = 2l+1, \ldots, n$. Therefore we have

$$f_i = g_i \text{ and } g_{l+i} = h_{\sigma^{-1}(l+i)} \text{ for each } i = 1, \ldots, l, \tag{2.5}$$

$$f_j = h_j \text{ for each } j = l+1, \ldots, n, \tag{2.6}$$

$$g_q = h_q \text{ for each } q = 2l+1, \ldots, n. \tag{2.7}$$

Let $x \in B$ and

$$f_x = (f_1(x), \ldots, f_n(x)),\ g_x = (g_1(x), \ldots, g_n(x)),\ h_x = (h_1(x), \ldots, h_n(x)) \in A^H.$$

By (2.5), (2.6) and (2.7), we have

$$f_i(x) = g_i(x) \text{ and } g_{l+i} = h_{\sigma^{-1}(l+i)}(x) \text{ for each } i = 1, \ldots, l,$$

$$f_j(x) = h_j(x) \text{ for each } j = l+1, \ldots, n,$$

$$g_q(x) = h_q(x) \text{ for each } q = 2l+1, \ldots, n.$$

Thus $f_x|_{K_1}(i) = g_x|_{K_1}(i)$, $g_x|_{K_2}(l+i) = h_x|_{K_1} \circ \sigma^{-1}(l+i)$ for each $i = 1, \ldots, l$, $f_x|_{K_2 \cup K_3}(j) = h_x|_{K_2 \cup K_3}(j)$ for each $j = l+1, \ldots, n$, and $g_x|_{K_3}(q) = h_x|_{K_3}(q)$ for each $q = 2l+1, \ldots, n$. Hence $f_x|_{K_1} = g_x|_{K_1}$, $g_x|_{K_2} = h_x|_{K_1} \circ \sigma^{-1}$, $f_x|_{K_2 \cup K_3} = h_x|_{K_2 \cup K_3}$ and $g_x|_{K_3} = h_x|_{K_3}$. By the assumption for $k = p$, $\mathbf{B}$ is reflexive and $\psi \in r_{\mathcal{K}}^{p}$, we see that $(g_1(x), \ldots, g_n(x)) \in R_{\mathcal{K}}^{p}$. Because $\mathbf{B}$ is reflexive and $\chi \in r$, $(h_1(x), \ldots, h_n(x)) \in R$. This implies that $f_x = (f_1(x), \ldots, f_n(x)) \in R_{\mathcal{K}}^{p+1}$. Hence $(f_1(x), \ldots, f_n(x)) \in R_{\mathcal{K}}^{k}$ for each $x \in B$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$, $k \in \mathbb{N}, k \geq 1$ if $(f_1, \ldots, f_n) \in r_{\mathcal{K}}^{k}$. □

Theorem 2.3. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$ and let $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ be a b-decomposition of H. Then the following statements hold.*

(i) *If $\mathbf{A}$ is both diagonal and reflexive with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is reflexive with regard to $\mathcal{K}$.*
(ii) *If $\mathbf{B}$ is reflexive and $\mathbf{A}$ is irreflexive with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is irreflexive with regard to $\mathcal{K}$.*

Proof. Suppose $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$.

(i) Assume $\varphi = (f_1, \dots, f_n) \in E_{\mathcal{K}}$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$. Let $(x_1, \dots, x_n) \in S$ where $x_i \in B$ for $i = 1, \dots, n$. By Lemma 2.1, we have $(f_1(x_i), \dots, f_n(x_i)) \in E_{\mathcal{K}}$ for each $i = 1, \dots, n$. Since $\mathbf{A}$ is reflexive with regard to $\mathcal{K}$, we also have

$$(f_1(x_i), \dots, f_n(x_i)) \in R \text{ for each } i = 1, \dots, n. \tag{2.8}$$

As f_i is a homomorphism, we have

$$(f_i(x_1), \dots, f_i(x_n)) \in R \text{ for each } i = 1, \dots, n. \tag{2.9}$$

Because $\mathbf{A}$ is diagonal, we have $(f_1(x_1), \dots, f_n(x_n)) \in R$, and then $\varphi \in r$. Hence $\mathbf{A}^{\mathbf{B}}$ is reflexive with regard to $\mathcal{K}$.

(ii) Assume $\varphi = (f_1, \dots, f_n) \in r \cap E_{\mathcal{K}}$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$. Since $\varphi \in E_{\mathcal{K}}$ and Lemma 2.1, we have

$$(f_1(x), \dots, f_n(x)) \in E_{\mathcal{K}} \text{ for each } x \in B. \tag{2.10}$$

As $\varphi \in r$ and $\mathbf{B}$ is reflexive, we have

$$(f_1(x), \dots, f_n(x)) \in R \text{ for each } x \in B. \tag{2.11}$$

From (2.10) and (2.11), we have $(f_1(x), \dots, f_n(x)) \in R \cap E_{\mathcal{K}}$ for each $x \in B$, which contradicts the assumption that $\mathbf{A}$ is irreflexive with regard to $\mathcal{K}$. Hence $\mathbf{A}^{\mathbf{B}}$ is irreflexive with regard to $\mathcal{K}$. □

Theorem 2.4. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$ and let $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ be a b-decomposition of H. If $\mathbf{B}$ is reflexive then the following statements hold.*

(i) *If $\mathbf{A}$ is both diagonal and symmetric (transitive) with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is symmetric (transitive) with regard to $\mathcal{K}$.*
(ii) *If $\mathbf{A}$ is asymmetric (atransitive) with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is asymmetric (atransitive) with regard to $\mathcal{K}$.*
(iii) *If $\mathbf{A}$ is antisymmetric with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is antisymmetric with regard to $\mathcal{K}$.*

Proof. Since $\mathbf{B}$ is reflexive, $\mathbf{A}$ is diagonal and symmetric (transitive) with regard to $\mathcal{K}$ by using Lemma 2.2, so we can prove in the same manner of Theorem 2.3(i) that $\mathbf{A}^{\mathbf{B}}$ is symmetric (transitive) with regard to $\mathcal{K}$. The proof of (ii) is similar to that of Theorem 2.3(ii) by using Lemma 2.2.

(iii) Let $\varphi = (f_1, \dots, f_n)$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \dots, n$ and $\varphi \in r \cap r_{\mathcal{K}}^{-1}$. By using the same argument as that used in the proof of Theorem 2.3(ii), we have $(f_1(x), \dots, f_n(x)) \in R \cap R_{\mathcal{K}}^{-1}$. Since $\mathbf{A}$ is antisymmetric with regard to $\mathcal{K}$, we get $(f_1(x), \dots, f_n(x)) \in E_{\mathcal{K}}$ for each $x \in B$. By Lemma 2.1, we can see that $\varphi \in E_{\mathcal{K}}$. Hence $\mathbf{A}^{\mathbf{B}}$ is antisymmetric with regard to $\mathcal{K}$. □

Lemma 2.5. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = (\{K_i\}_{i=1}^{m}, \{\sigma_i\}_{i=1}^{m-1})$ be a t_m-decomposition of H. Let $\mathbf{A}^{\mathbf{B}} = (Hom(\mathbf{B}, \mathbf{A}), r)$, and let $\mathbf{B}$ be reflexive. If $(f_1, \dots, f_n) \in {}^1r_{\mathcal{K}}$, then $(f_1(x), \dots, f_n(x)) \in {}^1R_{\mathcal{K}}$ for each $x \in B$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \dots, n$.*

Proof. Assume $|K_1| = |K_2| = \cdots = |K_m| = l$.

Let $\varphi = (f_1, \ldots, f_n) \in {}^1r_{\mathcal{K}}$ where $f_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. Then $\varphi \in (Hom(\mathbf{B}, \mathbf{A}))^H$, and there exists a mapping $\psi \in r$ such that $\varphi|_{K_i} = \psi|_{K_{i+1}} \circ \sigma_i$ for $i = 1, \ldots, m-1$, and $\varphi|_{K_m} = \psi|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \ldots \circ \sigma_{m-1}^{-1}$. Therefore we have $\varphi|_{K_i}((i-1)l + j) = \psi|_{K_{i+1}} \circ \sigma_i((i-1)l + j)$ and $\varphi|_{K_n}((m-1)l + j) = \psi|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l + j)$ for $i = 1, \ldots, m-1$, and for $j = 1, \ldots, l$, i.e. for each $j = 1, \ldots, l$, we get

$$\begin{aligned} \varphi|_{K_1}(j) &= \psi|_{K_2} \circ \sigma_1(j), \\ \varphi|_{K_2}(l + j) &= \psi|_{K_3} \circ \sigma_2(l + j), \\ &\cdots \\ \varphi|_{K_{m-1}}((m-2)l + j) &= \psi|_{K_m} \circ \sigma_{m-1}((m-2)l + j), \\ \text{and } \varphi|_{K_m}((m-1)l + j) &= \psi|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l + j). \end{aligned}$$

Let $\psi = (g_1, \ldots, g_n)$ where $g_i \in Hom(\mathbf{B}, \mathbf{A})$ for $i = 1, \ldots, n$. So we see that $f_{((i-1)l+j)} = g_{\sigma_i((i-1)l+j)}$ and $f_{((m-1)l+j)} = g_{\sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l+j)}$ for $i = 1, \ldots, m-1$ and for $j = 1, \ldots, l$. Therefore we have

$$\begin{aligned} f_j &= g_{\sigma_1(j)}, \\ f_{l+j} &= g_{\sigma_2(l+j)}, \\ &\cdots \\ f_{(m-2)l+j} &= g_{\sigma_{m-1}((m-2)l+j)}, \\ \text{and } f_{(m-1)l+j} &= g_{\sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l+j)}. \end{aligned}$$

Let $x \in B$. Then we have $f_{((i-1)l+j)}(x) = g_{\sigma_i((i-1)l+j)}(x)$ and

$$f_{((m-1)l+j)}(x) = g_{\sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l+j)}(x)$$

for $i = 1, \ldots, m-1$ and for $j = 1, \ldots, l$. Suppose

$$f_x = (f_1(x), \ldots, f_n(x)),\ g_x = (g_1(x), \ldots, g_n(x)).$$

Then we have $f_{((i-1)l+j)}(x) = g_{\sigma_i((i-1)l+j)}(x)$ and

$$f_{((m-1)l+j)}(x) = g_{\sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l+j)}(x)$$

for $i = 1, \ldots, m-1$ and for $j = 1, \ldots, l$. Therefore we have

$$f_x|_{K_i}((i-1)l + j) = g_x|_{K_{i+1}} \circ \sigma_i((i-1)l + j)$$

and $f_x|_{K_m}((m-1)l + j) = g_x|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}((m-1)l + j)$ for $i = 1, \ldots, m-1$ and for $j = 1, \ldots, l$. Thus $f_x|_{K_i} = g_x|_{K_{i+1}} \circ \sigma_i$ for $i = 1, \ldots, m-1$, and $f_x|_{K_m} = g_x|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}$. So we see that $f_x \in {}^1R_{\mathcal{K}}$. □

Theorem 2.6. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = (\{K_i\}_{i=1}^m, \{\sigma_i\}_{i=1}^{m-1})$ be a t_m-decomposition of H. If $\mathbf{B}$ is reflexive then the following statements hold.*

(i) *If $\mathbf{A}$ is both diagonal and cyclic with regard to $\mathcal{K}$, then $\mathbf{A}^{\mathbf{B}}$ is cyclic with regard to $\mathcal{K}$.*

(ii) *If* $\mathbf{A}$ *is acyclic with regard to* $\mathcal{K}$ *, then* $\mathbf{A}^{\mathbf{B}}$ *is acyclic with regard to* $\mathcal{K}$.

Proof. Since $\mathbf{B}$ is reflexive, $\mathbf{A}$ is diagonal and cyclic with regard to $\mathcal{K}$ by using Lemma 2.5, so we can prove, similarly to Theorem 2.3(i), that $\mathbf{A}^{\mathbf{B}}$ is cyclic with regard to $\mathcal{K}$. The proof of (ii) is similar to that of Theorem 2.3(ii) by using Lemma 2.5, we conclude (ii). □

Let $\mathbf{A} = (A, R), \acute{\mathbf{A}} = (A, \acute{R})$ be n-ary relational systems of type H with the same carrier set. Then we set

$$\begin{aligned}\mathbf{A} \cap \acute{\mathbf{A}} &= (A, R \cap \acute{R}),\\ \mathbf{A} \cup \acute{\mathbf{A}} &= (A, R \cup \acute{R}).\end{aligned}$$

If card $H = n, n \geq 2$ and $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ be a b-decomposition of H then we set

$$\mathbf{A}_{\mathcal{K}}^{-1} = (A, R_{\mathcal{K}}^{-1}).$$

□

Lemma 2.7. *Let* $\mathbf{A} = (A, R), \acute{\mathbf{A}} = (A, \acute{R})$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 2$, $\mathcal{K} = (\{K_i\}_{i=1}^{3}, \sigma)$ *be a* b*-decomposition of* H*. If* $\mathbf{A}$ *and* $\acute{\mathbf{A}}$ *is diagonal then*

(i) *an* n*-ary relational system of type* H $\mathbf{A} \cap \acute{\mathbf{A}}$ *has the same property,*
(ii) *an* n*-ary relational system of type* H $\mathbf{A}_{\mathcal{K}}^{-1}$ *has the same property.*

Proof. Suppose $|K_1| = |K_2| = l, |K_3| = m$.

(i) Let $(x_i^1, \ldots, x_i^n) \in R \cap \acute{R}$ and $(x_1^i, \ldots, x_n^i) \in R \cap \acute{R}$ where $x_i^j \in A$ for $i, j = 1, \ldots, n$. Because $(x_i^1, \ldots, x_i^n), (x_1^i, \ldots, x_n^i) \in R$ and $\mathbf{A}$ is diagonal, we see that $(x_1^1, \ldots, x_n^n) \in R$. Similarly, for $(x_i^1, \ldots, x_i^n) \in \acute{R}$ and $(x_1^i, \ldots, x_n^i) \in \acute{R}$, because $\acute{\mathbf{A}}$ is diagonal, we get $(x_1^1, \ldots, x_n^n) \in \acute{R}$. Therefore $(x_1^1, \ldots, x_n^n) \in R \cap \acute{R}$. Thus $\mathbf{A} \cap \acute{\mathbf{A}}$ is diagonal.

(ii) Let $r_i = (x_i^1, \ldots, x_i^n) \in R_{\mathcal{K}}^{-1}$ and $c_i = (x_1^i, \ldots, x_n^i) \in R_{\mathcal{K}}^{-1}$ where $x_i^j \in A$ for $i, j = 1, \ldots, n$. Since $r_i \in R_{\mathcal{K}}^{-1}$, there exists a mapping $\acute{r}_i \in R$ such that $r_i|_{K_1} = \acute{r}_i|_{K_2} \circ \sigma$, $r_i|_{K_2} = \acute{r}_i|_{K_1} \circ \sigma^{-1}$, $r_i|_{K_3} = \acute{r}_i|_{K_3}$ for $i = 1, \ldots, n$. Then $r_i|_{K_1} \circ \sigma^{-1} = \acute{r}_i|_{K_2}$, $r_i|_{K_2} \circ \sigma = \acute{r}_i|_{K_1}$, $r_i|_{K_3} = \acute{r}_i|_{K_3}$ for $i = 1, \ldots, n$. So we have for each $i = 1, \ldots, n$,

$$\acute{r}_i = (x_i^{\sigma(1)}, \ldots, x_i^{\sigma(l)}, x_i^{\sigma^{-1}(l+1)}, \ldots, x_i^{\sigma^{-1}(2l)}, x_i^{2l+1}, \ldots, x_i^n). \tag{2.12}$$

And we also have for each $i = 1, \ldots, l$,

$$\acute{r}_{\sigma(i)} = (x_{\sigma(i)}^{\sigma(1)}, \ldots, x_{\sigma(i)}^{\sigma(l)}, x_{\sigma(i)}^{\sigma^{-1}(l+1)}, \ldots, x_{\sigma(i)}^{\sigma^{-1}(2l)}, x_{\sigma(i)}^{2l+1}, \ldots, x_{\sigma(i)}^{n}) \in R, \tag{2.13}$$

$$\begin{aligned}\acute{r}_{\sigma^{-1}(l+i)} =&(x_{\sigma^{-1}(l+i)}^{\sigma(1)}, \ldots, x_{\sigma^{-1}(l+i)}^{\sigma(l)}, x_{\sigma^{-1}(l+i)}^{\sigma^{-1}(l+1)}, \ldots, x_{\sigma^{-1}(l+i)}^{\sigma^{-1}(2l)}, x_{\sigma^{-1}(l+i)}^{2l+1},\\ &\ldots, x_{\sigma^{-1}(l+i)}^{n}) \in R.\end{aligned}$$

Similarly, for $c_i = (x_1^i, \ldots, x_n^i) \in R_{\mathcal{K}}^{-1}$, we have $\acute{c}_i \in R$ so that for each $i = 1, \ldots, n$,

$$\acute{c}_i = (x_{\sigma(1)}^i, \ldots, x_{\sigma(l)}^i, x_{\sigma^{-1}(l+1)}^i, \ldots, x_{\sigma^{-1}(2l)}^i, x_{2l+1}^i, \ldots, x_n^i). \tag{2.14}$$

And we also have for each $i = 1, \ldots, l$,

$$(2.15) \quad \acute{c}_{\sigma(i)} = (x^{\sigma(i)}_{\sigma(1)}, \ldots, x^{\sigma(i)}_{\sigma(l)}, x^{\sigma(i)}_{\sigma^{-1}(l+1)}, \ldots, x^{\sigma(i)}_{\sigma^{-1}(2l)}, x^{\sigma(i)}_{2l+1}, \ldots, x^{\sigma(i)}_{n}) \in R,$$

$$\acute{c}_{\sigma^{-1}(l+i)} = (x^{\sigma^{-1}(l+i)}_{\sigma(1)}, \ldots, x^{\sigma^{-1}(l+i)}_{\sigma(l)}, x^{\sigma^{-1}(l+i)}_{\sigma^{-1}(l+1)}, \ldots, x^{\sigma^{-1}(l+i)}_{\sigma^{-1}(2l)}, x^{\sigma^{-1}(l+i)}_{2l+1}, \ldots, x^{\sigma^{-1}(l+i)}_{n}) \in R.$$

Since $\mathbf{A}$ is diagonal, we see that

$$(x^{\sigma(1)}_{\sigma(1)}, \ldots, x^{\sigma(l)}_{\sigma(l)}, x^{\sigma^{-1}(l+1)}_{\sigma^{-1}(l+1)}, \ldots, x^{\sigma^{-1}(2l)}_{\sigma^{-1}(2l)}, x^{2l+1}_{2l+1}, \ldots, x^{n}_{n}) \in R.$$

Let

$$g = (x^{\sigma(1)}_{\sigma(1)}, \ldots, x^{\sigma(l)}_{\sigma(l)}, x^{\sigma^{-1}(l+1)}_{\sigma^{-1}(l+1)}, \ldots, x^{\sigma^{-1}(2l)}_{\sigma^{-1}(2l)}, x^{2l+1}_{2l+1}, \ldots, x^{n}_{n})$$

and $f = (x^1_1, \ldots, x^n_n)$. Then we have $g|_{K_1} = f|_{K_2} \circ \sigma$, $g|_{K_2} = f|_{K_1} \circ \sigma^{-1}$, so $f|_{K_1} = g|_{K_2} \circ \sigma$, $f|_{K_2} = g|_{K_1} \circ \sigma^{-1}$, $f|_{K_3} = g|_{K_3}$. Hence $f \in R^{-1}_{\mathcal{K}}$, this implies that $\mathbf{A}^{-1}_{\mathcal{K}}$ is diagonal. □

Theorem 2.8. *Let $\mathbf{A} = (A, R), \acute{\mathbf{A}} = (A, \acute{R}), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 2$, and let $\mathcal{K} = (\{K_i\}^3_{i=1}, \sigma)$ be a b-decomposition of H. Then the following statements hold.*

(i) *If $\mathbf{A}, \acute{\mathbf{A}}$ are diagonal and reflexive with regard to $\mathcal{K}$, then $(\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ and $(\mathbf{A}^{-1}_{\mathcal{K}})^{\mathbf{B}}$ have the same property.*

(ii) *If $\mathbf{A}, \acute{\mathbf{A}}$ are irreflexive with regard to $\mathcal{K}$, then $(\mathbf{A} \cup \acute{\mathbf{A}})^{\mathbf{B}}, (\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ and $(\mathbf{A}^{-1}_{\mathcal{K}})^{\mathbf{B}}$ have the same property.*

(iii) *If $\mathbf{B}$ is reflexive, $\mathbf{A}, \acute{\mathbf{A}}$ are diagonal and symmetric (transitive) with regard to $\mathcal{K}$, then $(\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ and $(\mathbf{A}^{-1}_{\mathcal{K}})^{\mathbf{B}}$ have the same property.*

(iv) *If $\mathbf{B}$ is reflexive, and $\mathbf{A}, \acute{\mathbf{A}}$ are asymmetric (antisymmetric, atransitive) with regard to $\mathcal{K}$, then $(\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ and $(\mathbf{A}^{-1}_{\mathcal{K}})^{\mathbf{B}}$ have the same property.*

Proof. The assertions (i)-(iv) follow from Proposition 1.3, Theorem 2.3, Theorem 2.4 and Lemma 2.7 . □

Let $\mathbf{A} = (A, R)$ be an n-ary relational system of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = (\{K_i\}^m_{i=1}, \{\sigma_i\}^{m-1}_{i=1})$ be a t_m-decomposition of H. Then we set

$$^1\mathbf{A}_{\mathcal{K}} = (A, \ ^1\mathbf{R}_{\mathcal{K}}).$$

Lemma 2.9. *Let $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ be n-ary relational systems of type H with card $H = n, n \geq 3$ and let $\mathcal{K} = (\{K_i\}^m_{i=1}, \{\sigma_i\}^{m-1}_{i=1})$ be a t_m-decomposition of H. If $\mathbf{A}$ is diagonal then $^1\mathbf{A}_{\mathcal{K}}$ has the same property.*

Proof. Assume $|K_1| = |K_2| = |K_3| = \cdots = |K_m| = l$. Let $r_i = (x^1_i, \ldots, x^n_i) \in \ ^1R_{\mathcal{K}}$ and $c_i = (x^i_1, \ldots, x^i_n) \in \ ^1R_{\mathcal{K}}$ where $x^j_i \in A$ for $i, j = 1, \ldots, n$. Since $r_i \in \ ^1R_{\mathcal{K}}$, there exists a mapping $\acute{r}_i \in R$ such that for each $i = 1, \ldots, n$, we get $r_i|_{K_j} = \acute{r}_i|_{K_{j+1}} \circ \sigma_j$ for $j = 1, \ldots, m-1$, and $r_i|_{K_m} = \acute{r}_i|_{K_1} \circ \sigma^{-1}_1 \circ \sigma^{-1}_2 \circ \cdots \circ \sigma^{-1}_{m-1}$. Then for each $i = 1, \ldots, n$ we have $\acute{r}_i|_{K_1} = r_i|_{K_m} \circ \sigma_{m-1} \circ \sigma_{m-2} \circ \cdots \circ \sigma_1$ and

$\acute{r}_i|_{K_{j+1}} = r_i|_{K_j}\sigma_j^{-1}$for $j = 1, \ldots, m-1$. Put $\sigma_{m-1} \circ \sigma_{m-2} \circ \cdots \circ \sigma_1 = \alpha$ and $m - 1 = k$. So for each $i = 1, \ldots, n$, we have

$$\begin{aligned}\acute{r}_i = (x_i^{\alpha(1)} \quad &, \ldots, \quad x_i^{\alpha(l)}, x_i^{\sigma_1^{-1}(l+1)}, \ldots, x_i^{\sigma_1^{-1}(2l)}, x_i^{\sigma_2^{-1}(2l+1)}, \ldots, x_i^{\sigma_2^{-1}(3l)} \\ &, \ldots, \quad x_i^{\sigma_k^{-1}(kl+1)}, \ldots, x_i^{\sigma_k^{-1}(n)}).\end{aligned}$$

Then we have for each $i = 1, \ldots, l$,

$$\begin{aligned}\acute{r}_{\alpha(i)} = (x_{\alpha(i)}^{\alpha(1)} \quad &, \ldots, \quad x_{\alpha(i)}^{\alpha(l)}, x_{\alpha(i)}^{\sigma_1^{-1}(l+1)}, \ldots, x_{\alpha(i)}^{\sigma_1^{-1}(2l)}, x_{\alpha(i)}^{\sigma_2^{-1}(2l+1)}, \ldots, x_{\alpha(i)}^{\sigma_2^{-1}(3l)} \\ &, \ldots, \quad x_{\alpha(i)}^{\sigma_k^{-1}(kl+1)}, \ldots, x_{\alpha(i)}^{\sigma_k^{-1}(n)}) \in R, \end{aligned} \tag{2.16}$$

and for each $j = 1, \ldots, k$,

$$\begin{aligned}\acute{r}_{\sigma_j^{-1}(jl+i)} = (&x_{\sigma_j^{-1}(jl+i)}^{\alpha(1)}, \ldots, x_{\sigma_j^{-1}(jl+i)}^{\alpha(l)}, x_{\sigma_j^{-1}(jl+i)}^{\sigma_1^{-1}(l+1)}, \ldots, x_{\sigma_j^{-1}(jl+i)}^{\sigma_1^{-1}(2l)}, x_{\sigma_j^{-1}(jl+i)}^{\sigma_2^{-1}(2l+1)}, \\ &\ldots, x_{\sigma_j^{-1}(jl+i)}^{\sigma_2^{-1}(3l)}, \ldots, x_{\sigma_j^{-1}(jl+i)}^{\sigma_k^{-1}(kl+1)}, \ldots, x_{\sigma_j^{-1}(jl+i)}^{\sigma_k^{-1}(n)}) \in R.\end{aligned} \tag{2.17}$$

Similarly, for $c_i = (x_1^i, \ldots, x_n^i) \in {}^1R_{\mathcal{K}}$, so we have $\acute{c}_i \in R$ such that for each $i = 1, \ldots, n$

$$\begin{aligned}\acute{c}_i = (x_{\alpha(1)}^i \quad &, \ldots, \quad x_{\alpha(l)}^i, x_{\sigma_1^{-1}(l+1)}^i, \ldots, x_{\sigma_1^{-1}(2l)}^i, x_{\sigma_2^{-1}(2l+1)}^i, \ldots, x_{\sigma_2^{-1}(3l)}^i \\ &, \ldots, \quad x_{\sigma_k^{-1}(kl+1)}^i, \ldots, x_{\sigma_k^{-1}(n)}^i).\end{aligned}$$

Therefore we have for each $i = 1, \ldots, l$,

$$\begin{aligned}\acute{c}_{\alpha(i)} = (x_{\alpha(1)}^{\alpha(i)} \quad &, \ldots, \quad x_{\alpha(l)}^{\alpha(i)}, x_{\sigma_1^{-1}(l+1)}^{\alpha(i)}, \ldots, x_{\sigma_1^{-1}(2l)}^{\alpha(i)}, x_{\sigma_2^{-1}(2l+1)}^{\alpha(i)}, \ldots, x_{\sigma_2^{-1}(3l)}^{\alpha(i)} \\ &, \ldots, \quad x_{\sigma_k^{-1}(kl+1)}^{\alpha(i)}, \ldots, x_{\sigma_k^{-1}(n)}^{\alpha(i)}) \in R,\end{aligned} \tag{2.18}$$

and for each $j = 1, \ldots, k$,

$$\begin{aligned}\acute{c}_{\sigma_j^{-1}(jl+i)} = (&x_{\alpha(j)}^{\sigma_j^{-1}(jl+i)}, \ldots, x_{\alpha(l)}^{\sigma_j^{-1}(jl+i)}, x_{\sigma_1^{-1}(l+1)}^{\sigma_j^{-1}(jl+i)}, \ldots, x_{\sigma_1^{-1}(2l)}^{\sigma_j^{-1}(jl+i)}, x_{\sigma_2^{-1}(2l+1)}^{\sigma_j^{-1}(jl+i)}, \ldots, \\ &x_{\sigma_2^{-1}(3l)}^{\sigma_j^{-1}(jl+i)}, \ldots, x_{\sigma_k^{-1}(kl+1)}^{\sigma_j^{-1}(jl+i)}, \ldots, x_{\sigma_k^{-1}(n)}^{\sigma_j^{-1}(jl+i)}) \in R,\end{aligned} \tag{2.19}$$

Because $\mathbf{A}$ is diagonal, we see that

$$\begin{aligned}(&x_{\alpha(1)}^{\alpha(1)}, \ldots, x_{\alpha(l)}^{\alpha(l)}, x_{\sigma_1^{-1}(l+1)}^{\sigma_1^{-1}(l+1)}, \ldots, x_{\sigma_1^{-1}(2l)}^{\sigma_1^{-1}(2l)}, x_{\sigma_2^{-1}(2l+1)}^{\sigma_2^{-1}(2l+1)}, \ldots, \\ &\qquad x_{\sigma_2^{-1}(3l)}^{\sigma_2^{-1}(3l)}, \ldots, x_{\sigma_k^{-1}(kl+1)}^{\sigma_k^{-1}(kl+1)}, \ldots, x_{\sigma_k^{-1}(n)}^{\sigma_k^{-1}(n)}) \in R.\end{aligned}$$

Let $g = (x_{\alpha(1)}^{\alpha(1)}, \ldots, x_{\alpha(l)}^{\alpha(l)}, x_{\sigma_1^{-1}(l+1)}^{\sigma_1^{-1}(l+1)}, \ldots, x_{\sigma_1^{-1}(2l)}^{\sigma_1^{-1}(2l)}, \ldots, x_{\sigma_k^{-1}(kl+1)}^{\sigma_k^{-1}(kl+1)}, \ldots, x_{\sigma_k^{-1}(n)}^{\sigma_k^{-1}(n)})$ and $f = (x_1^1, \ldots, x_n^n)$. Then we have $g|_{K_1} = f|_{K_m} \circ \sigma_{m-1} \circ \sigma_{m-2} \circ \cdots \circ \sigma_1$ and $g|_{K_{j+1}} = f|_{K_j}\sigma_j^{-1}$, so we get $f|_{K_j} = g|_{K_{j+1}} \circ \sigma_j$ for $j = 1, \ldots, m-1$, and $f|_{K_m} = g|_{K_1} \circ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{m-1}^{-1}$. Thus $f \in {}^1R_{\mathcal{K}}$, which yields that ${}^1\mathbf{A}_{\mathcal{K}}$ is diagonal. $\square$

Theorem 2.10. *Let* $\mathbf{A} = (A, R), \acute{\mathbf{A}} = (A, \acute{R}), \mathbf{B} = (B, S)$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 3$*, and let* $\mathcal{K} = (\{K_i\}_{i=1}^m, \{\sigma_i\}_i^{m-1})$ *be a* t_m*-decomposition of* H*. If* $\mathbf{B}$ *is reflexive then the following statements hold.*

(i) *If* $\mathbf{A}, \acute{\mathbf{A}}$ *are both diagonal and cyclic with regard to* $\mathcal{K}$*, then* $(\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ *and* $({}^1\mathbf{A}_{\mathcal{K}})^{\mathbf{B}}$ *have the same property.*

(ii) *If* $\mathbf{A}, \acute{\mathbf{A}}$ *are acyclic with regard to* $\mathcal{K}$*, then* $(\mathbf{A} \cap \acute{\mathbf{A}})^{\mathbf{B}}$ *and* $({}^1\mathbf{A}_{\mathcal{K}})^{\mathbf{B}}$ *have the same property.*

Proof. The assertions (i)-(ii) follow from Proposition 1.4, Theorem 2.6 and Lemma 2.9. □

Let $\mathbf{A} = (A, R)$ be an n-ary relational system of type H with card $H = n, n \geq 2$ and $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ be a b-decomposition of H. Then we set

$$\mathbf{A}_{X,\mathcal{K}} = (A, R_{X,\mathcal{K}}).$$

Lemma 2.11. *Let* $\mathbf{A} = (A, R), \acute{\mathbf{A}} = (A, \acute{R})$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 2$*,* $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ *be a* b*-decomposition of* H*, if* $\mathbf{A}$ *is diagonal then* $\mathbf{A}_{X,\mathcal{K}}$ *has the same property.*

Proof. Suppose $|K_1| = |K_2| = l, |K_3| = m$. Let $r_i = (x_i^1, \ldots, x_i^{2l}) \in R_{X,\mathcal{K}}$ and $c_i = (x_1^i, \ldots, x_{2l}^i) \in R_{X,\mathcal{K}}$ where $x_i^j \in A$ for $i, j = 1, \ldots, n$. Since $r_i \in R_{X,\mathcal{K}}$, there exists a mapping $\acute{r}_i \in R$ such that $\acute{r}_i(k) = r_i(k)$ for each $k \in K_1 \cup K_2$ and $\acute{r}_i = x_i^k$ for each $k \in K_3$. Thus we get

$$\acute{r}_i = (x_i^1, \ldots, x_i^{2l}, x_i^{2l+1}, \ldots, x_i^n) \text{ for each } i = 1, \ldots, n. \tag{2.20}$$

Similarly, as $c_i \in R_{X,\mathcal{K}}$, there exists a mapping

$$\acute{c}_i = (x_1^i, \ldots, x_{2l}^i, x_{2l+1}^i, \ldots, x_n^i) \in R \text{ for each } i = 1, \ldots, n. \tag{2.21}$$

Because $\mathbf{A}$ is diagonal, we have

$$(x_1^1, \ldots, x_{2l}^{2l}, x_{2l+1}^{2l+1}, \ldots, x_n^n) \in R.$$

Let $g = (x_1^1, \ldots, x_{2l}^{2l}, x_{2l+1}^{2l+1}, \ldots, x_n^n)$. Then there exists a mapping $f = (x_1^1, \ldots, x_{2l}^{2l})$ such that $f(k) = g(k)$ for each $k \in K_1 \cup K_2$ and $g(k) = x_k^k$ for each $k \in K_3$, this implies that $f \in R_{X,\mathcal{K}}$. Hence $\mathbf{A}_{X,\mathcal{K}}$ is diagonal. □

Theorem 2.12. *Let* $\mathbf{A} = (A, R), \mathbf{B} = (B, S)$ *be* n*-ary relational systems of type* H *with card* $H = n, n \geq 2$*,* $\mathcal{K} = (\{K_i\}_{i=1}^3, \sigma)$ *be a* b*-decomposition of* H *and let* $\mathbf{A}_{X,\mathcal{K}} = (A, R_{X,\mathcal{K}})$*. Then the following statements hold.*

(i) *If* $\mathbf{A}$ *is diagonal and reflexive with regard to* $\mathcal{K}$*, then* $(\mathbf{A}_{X,\mathcal{K}})^{\mathbf{B}}$ *is reflexive with regard to* $\tilde{\mathcal{K}}$*.*

(ii) *If* $\mathbf{A}$ *is irreflexive with regard to* $\mathcal{K}$*, then* $(\mathbf{A}_{X,\mathcal{K}})^{\mathbf{B}}$ *is irreflexive with regard to* $\tilde{\mathcal{K}}$*.*

(iii) *If* $\mathbf{B}$ *is reflexive,* $\mathbf{A}$ *is diagonal and symmetric (transitive) with regard to* $\mathcal{K}$*, then* $(\mathbf{A}_{X,\mathcal{K}})^{\mathbf{B}}$ *is symmetric (transitive) with regard to* $\tilde{\mathcal{K}}$*.*

(iv) *If* $\mathbf{B}$ *is reflexive and* $\mathbf{A}$ *is asymmetric (antisymmetric, atransitive) with regard to* $\mathcal{K}$*, then* $(\mathbf{A}_{X,\mathcal{K}})^{\mathbf{B}}$ *is asymmetric (antisymmetric, atransitive) with regard to* $\tilde{\mathcal{K}}$*.*

Proof. The assertions (i)-(iv) follow from Proposition 1.5, Theorem 2.3, Theorem 2.4 and Lemma 2.11. □

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