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SOME MULTIPLIER DOUBLE SEQUENCE SPACES

KULDIP RAJ AND SUNIL K. SHARMA

Abstract. In the present paper we define multiplier double sequence spaces $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ and $\Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$ by using a sequence of Orlicz functions $M = (M_{mn})$ and a multiplier function $u = (u_{mn})$. We also make an effort to study some topological properties and inclusion relations between these spaces.

1. Introduction and preliminaries

The initial work on double sequences is found in Bromwich [4]. Later on it was studied by Hardy [6], Moricz [12], Moricz and Rhoades [13], Tripathy ([28], [29]), Basarir and Sonalcan [2] and many others. Hardy [7] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [31] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Subsequently, Mursaleen [15] and Mursaleen and Edely [18] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{mn})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Basar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [27] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations. By the convergence of a double sequence we mean the convergence of the

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Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{kl} - L| < \epsilon$ for all $k, l > n$ see [20]. We shall write more briefly as P-convergent. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k and l. Let l''_∞ be the space of all bounded double sequences such that $||x_{kl}||_{\infty,2} = \sup_{kl} |x_{kl}| < \infty$.

The notion of difference sequence spaces was introduced by Kizmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Colak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+v}.$$

Taking $m = 1$, we get the spaces which were introduced and studied by Kizmaz [9].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p. If X is a sequence space, we give the following definitions:

(1) $X' =$ the continuous dual of X;

(2) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\}$;

(3) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}, \text{ is convergent, for each } x \in X \right\}$;

(4) $X^{\gamma} = \left\{ a = (a_{mn}) : \sup_{MN \geq 1} \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X \right\}$;

(5) let X be an FK-space$\supset \varphi$; then $X^{f} = \left\{ f(\zeta_{mn}) : f \in X' \right\}$;

(6) $X^{\delta} = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{\frac{1}{mn}} < \infty, \text{ for each } x \in X \right\}$;

X^{α}, X^{β}, X^{γ} and X^{δ} are called α-(or Kothe-Toeplitz) dual of X, β- (or generalized-Kothe-Toeplitz) dual of X, γ- dual of X, δ-dual of X respectively. X^{α} is defined by Kamthan and Gupta [8]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \not\subset X^{\gamma}$ since the sequence of partial sums of a double convergent series need not be bounded.

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar-valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then w^2 is a linear space under the coordinate wise addition and scalar multiplication.

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \ \ (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{\frac{1}{m+n}} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m, n \to \infty$. The vector space of double gai sequences will be denoted by χ^2. By φ, we denote the set of all finite sequences.

Consider a double sequence $x = (x_{mn})$. The $(m, n)^{\text{th}}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=1}^{m,n} x_{ij}\zeta_{ij}$ for all $m, n \in \mathbb{N}$; where ζ_{ij} denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i, j)^{\text{th}}$ place for each $i, j \in \mathbb{N}$.

A double sequence space E is said to be solid if $\alpha_{kl}x_{kl} \in E$ whenever $x_{kl} \in E$ and for all double sequences α_{kl} of scalars with $|\alpha_{kl}| \leq 1$, for all $k, l \in \mathbb{N}$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30], Theorem 10.4.2, p. 183). For more details about sequence spaces (see [11, 14, 16, 19, 21-26]).

The following inequality will be used throughout the paper. Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 \leq p_{mn} \leq \sup p_{mn} = G$, $K = \max(1, 2^{G-1})$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq K\{|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\} \tag{1.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{p_{mn}} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

Let w^2 denote the set of all complex double sequences, M be an Orlicz function and $p = (p_{mn})$ be a bounded sequence of positive real numbers. Then

$$\chi^2_M = \Big\{x \in w^2 : \Big(M\Big(\frac{((m+n)!|x_{mn}|)^{\frac{1}{m+n}}}{\rho}\Big)\Big) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0\Big\}$$

and

$$\Lambda^2_M = \Big\{x \in w^2 : \sup_{m,n \geq 1} \Big(M\Big(\frac{(|x_{mn}|)^{\frac{1}{m+n}}}{\rho}\Big)\Big) < \infty \ \text{ for some } \ \rho > 0\Big\}.$$

A sequence $x \in \Lambda^2$ is said to be almost convergent if all Banach limits of x coincide. Then

$$\hat{c} = \Big\{x = (x_{mn}) : \frac{1}{\mu\gamma} \sum_{m,n=1}^{\mu\gamma} x_{m+s,n+s} \to 0, \ \text{ as } \ \mu, \gamma \to \infty, \ \text{ uniformly in } \ s\Big\}.$$

Let $M = (M_{mn})$ be a sequence of Orlicz functions, $p = (p_{mn})$ be a bounded sequence of positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. Also, let (X, q) be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q and by $w^2(X)$ we denote the space of all X-valued double sequences. Now, we define the following sequence spaces in this paper:

$$\chi^2_M[\hat{c}, \Delta^m, u, p, q] = \Big\{x = (x_{mn}) \in w^2(X) :$$
$$\lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0$$
$$\text{uniformly in } \ s, \ \text{ for some } \ \rho > 0\Big\}$$

and

$$\Lambda^2_M[\hat{c},\Delta^m,u,p,q] = \Big\{x=(x_{mn})\in w^2(X): \sup_{s,\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}=0 \text{ uniformly in } s, \text{ for some } \rho>0\Big\}.$$

If $M_{mn}(x)=x$ for all m,n, we get

$$\chi^2[\hat{c},\Delta^m,u,p,q] = \Big\{x=(x_{mn})\in w^2(X): \lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}u_{mn}\Big[\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}=0 \text{ uniformly in } s, \text{ for some } \rho>0\Big\}$$

and

$$\Lambda^2[\hat{c},\Delta^m,u,p,q] = \Big\{x=(x_{mn})\in w^2(X): \sup_{s,\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}u_{mn}\Big[\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}=0 \text{ uniformly in } s, \text{ for some } \rho>0\Big\}.$$

If $p_{mn}=1$ for all m,n, we get

$$\chi^2_M[\hat{c},\Delta^m,u,q] = \Big\{x=(x_{mn})\in w^2(X): \lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]=0 \text{ uniformly in } s, \text{for some } \rho>0\Big\}$$

and

$$\Lambda^2_M[\hat{c},\Delta^m,u,q] = \Big\{x=(x_{mn})\in w^2(X): \sup_{s,\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]=0 \text{ uniformly in } s, \text{for some } \rho>0\Big\}.$$

The main aim of the present paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $M = (M_{mn})$ be a sequence of Orlicz functions, $p = (p_{mn})$ be a bounded sequence of positive real numbers and $u = (u_{mn})$ be a sequence of strictly positive real numbers. Then the spaces $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ and $\Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$ are linear over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. Let $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1, ρ_2 such that

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_1}\Big)\Big)\Big]^{p_{mn}} = 0$$

and

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_2}\Big)\Big)\Big]^{p_{mn}} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M = (M_{mn})$ is non-decreasing and convex and therefore by using inequality (1.1), we have

$$\begin{aligned}
&\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m (\alpha x_{m+s,n+s} + \beta y_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_3}\Big)\Big)\Big]^{p_{mn}} \\
&\le \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\Big(\frac{((m+n)!|\Delta^m \alpha x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_3}\Big) \\
&\quad + \Big(\frac{((m+n)!|\Delta^m \beta y_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_3}\Big)\Big)\Big)\Big]^{p_{mn}} \\
&\le K \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \frac{1}{2^{mn}} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_1}\Big)\Big)\Big]^{p_{mn}} \\
&\quad + K \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \frac{1}{2^{mn}} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m y_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_2}\Big)\Big)\Big]^{p_{mn}} \\
&\le K \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_1}\Big)\Big)\Big]^{p_{mn}} \\
&\quad + K \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m y_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_2}\Big)\Big)\Big]^{p_{mn}} \\
&= 0.
\end{aligned}$$

Thus $\alpha x + \beta y \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. This proves that $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ is a linear space. Similarly, we can prove that $\Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$ is a linear space. This completes the proof of the theorem. ☐

Theorem 2.2. *Let $M = (M_{mn})$ be a sequence of Orlicz functions, $p = (p_{mn})$ be a bounded sequence of positive real numbers and $u = (u_{mn})$ be a sequence of* *strictly positive real numbers. Then* $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ *is a paranormed space with*

$$g(x) = \inf \Big\{ \rho^{p_{mn}/H} : \sup_{\mu\gamma \geq 1} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho} \Big) \Big) \Big]^{p_{mn}} \leq 1, \ \rho > 0 \Big\},$$

where $H = \max(1, \sup_{mn} p_{mn})$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. Since $M(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$.
(iii) Let $x = (x_{mn})$, $y = (y_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$, then there exist positive numbers ρ_1, $\rho_2 > 0$ such that

$$\lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_1} \Big) \Big) \Big]^{p_{mn}} = 0$$

and

$$\lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m y_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_2} \Big) \Big) \Big]^{p_{mn}} = 0.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned} & u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m (x_{m+s,n+s} + y_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho} \Big) \Big) \Big]^{p_{mn}} \\ &= u_{mn} \Big[M_{mn} \Big(q \frac{((m+n)!|\Delta^m (x_{m+s,n+s} + y_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_1 + \rho_2} \Big) \Big]^{p_{mn}} \\ &\leq u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m (x_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_1 + \rho_2} \Big) \Big) \Big]^{p_{mn}} \\ &\quad + u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m (y_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_1 + \rho_2} \Big) \Big) \Big]^{p_{mn}} \\ &\leq \Big(\frac{\rho_1}{\rho_1 + \rho_2} \Big) u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m (x_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_1} \Big) \Big) \Big]^{p_{mn}} \\ &\quad + \Big(\frac{\rho_1}{\rho_1 + \rho_2} \Big) u_{mn} \Big[M_{mn} \Big(q \Big(\frac{((m+n)!|\Delta^m (y_{m+s,n+s}|))^{\frac{1}{m_s+n_s}}}{\rho_2} \Big) \Big) \Big]^{p_{mn}} \end{aligned}$$

and thus

$$g(x+y) = \inf\Big\{(\rho_1+\rho_2)^{p_{mn}/H} :$$
$$\sup_{\mu\gamma\geq 1}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m(x_{m+s,n+s}+y_{m+s,n+s})|)^{\frac{1}{m_s+n_s}}}{\rho_1+\rho_2}\Big)\Big)\Big]^{p_{mn}}\Big\}$$
$$\leq \inf\Big\{(\rho_1)^{p_{mn}/H} :$$
$$\sup_{\mu\gamma\geq 1}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_1}\Big)\Big)\Big]^{p_{mn}}\Big\}$$
$$+\inf\Big\{(\rho_2)^{p_{mn}/H} :$$
$$\sup_{\mu\gamma\geq 1}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m y_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho_2}\Big)\Big)\Big]^{p_{mn}}\Big\}.$$

Now, let $\lambda \in \mathbb{C}$, then the continuity of the product follows from the following equality

$$g(\lambda x) = \inf\Big\{\rho^{p_{mn}/H} :$$
$$\sup_{\mu\gamma\geq 1}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m \lambda x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}\Big\}$$
$$= \inf\Big\{(|\lambda|r)^{p_{mn}/H} :$$
$$\sup_{\mu\gamma\geq 1}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{r}\Big)\Big)\Big]^{p_{mn}}\Big\},$$

where $r = \frac{\rho}{|\lambda|}$. This completes the proof of the theorem. □

Theorem 2.3. *Let $M = (M_{mn})$ and $T = (T_{mn})$ be two sequences of Orlicz functions. Then*

$$\chi^2_M[\hat{c}, \Delta^m, u, p, q] \cap \chi^2_T[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2_{M+T}[\hat{c}, \Delta^m, u, p, q].$$

Proof. It is easy to prove, so we omit the details. □

Proposition 2.4. *Let $M = (M_{mn})$ and $T = (T_{mn})$ be two sequences of Orlicz functions and let q_1 and q_2 be two seminorms on X, we have*

(i) $\chi^2_M[\hat{c}, \Delta^m, u, p, q_1] \cap \chi^2_M[\hat{c}, \Delta^m, u, p, q_2] \subseteq \chi^2_M[\hat{c}, \Delta^m, u, p, q_1+q_2]$.

(ii) *If q_1 is stronger than q_2 then* $\chi^2_M[\hat{c}, \Delta^m, u, p, q_1] \subseteq \chi^2_M[\hat{c}, \Delta^m, u, p, q_2]$.

(iii) *If q_1 is equivalent to q_2 then* $\chi^2_M[\hat{c}, \Delta^m, u, p, q_1] = \chi^2_M[\hat{c}, \Delta^m, u, p, q_2]$.

Proof. It is trivial, so we omit it. □

Theorem 2.5. (i) *Let $0 \leq p_{mn} \leq r_{mn}$ and $\Big\{\frac{r_{mn}}{p_{mn}}\Big\}$ be bounded. Then*

$$\chi^2_M[\hat{c}, \Delta^m, u, r, q] \subset \chi^2_M[\hat{c}, \Delta^m, u, p, q].$$

(ii) $u_1 \leq u_2$ *implies* $\chi^2_M[\hat{c}, \Delta^m, u_1, p, q] \subset \chi^2_M[\hat{c}, \Delta^m, u_2, p, q]$.

Proof. (i) Let $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, r, q]$. Then

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}} = 0.$$

Let

$$t_{mn} = \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}}$$

and $\lambda_{mn} = \frac{p_{mn}}{r_{mn}}$. Since $p_{mn} \le r_{mn}$, we have $0 \le \lambda_{mn} \le 1$. Take $0 < \lambda < \lambda_{mn}$. Define $u_{mn} = t_{mn}(t_{mn} \ge 1)$; $u_{mn} = 0\ (t_{mn} < 1)$; and $v_{mn} = 0\ (t_{mn} \ge 1)$; $v_{mn} = t_{mn}\ (t_{mn} < 1)$; $t_{mn} = u_{mn} + v_{mn}$; $t^{\lambda_{mn}}_{mn} \le v^{\lambda_{mn}}_{mn}$.
Now it follows that

$$u^{\lambda_{mn}}_{mn} \le t_{mn} \text{ and } v^{\lambda_{mn}}_{mn} \le v^{\lambda}_{mn}. \tag{2.1}$$

i.e. $t^{\lambda_{mn}}_{mn} = u^{\lambda_{mn}}_{mn} + v^{\lambda_{mn}}_{mn}$; $t^{\lambda_{mn}}_{mn} \le t_{mn} + v^{\lambda}_{mn}$ by (2.1). Thus

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)^{r_{mn}}\Big]^{\lambda_{mn}}$$
$$\le \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}}.$$

This implies

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)^{r_{mn}}\Big]^{\frac{p_{mn}}{r_{mn}}}$$
$$\le \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}}.$$

This implies

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}$$
$$\le \lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}}.$$

But $\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{r_{mn}} = 0,$
we have

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0.$$

Hence $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. This proves that $\chi^2_M[\hat{c}, \Delta^m, u, r, q] \subset \chi^2_M[\hat{c}, \Delta^m, u, p, q]$.

(ii) The proof is easy, so omitted. □

Theorem 2.6. *Let $M = (M_{mn})$ be a sequence of Orlicz functions, $p = (p_{mn})$ be a bounded sequence of positive real numbers and $u = (u_{mn})$ be a sequence of strictly positive real numbers. Then the following statements are equivalent:*
(i) $\Lambda^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$
(ii) $\chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$
(iii) $\sup_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} < \infty.$

Proof. (i) ⇒ (ii) is obvious.
(ii) ⇒ (iii) Let $\chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$. Suppose that (iii) is not satisfied. Then for some $\rho > 0$

$$\sup_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} = \infty$$

and therefore there is a sequence $(\mu_i\gamma_i)$ of positive integers such that

$$\frac{1}{\mu_i\gamma_i} \sum_{mn=1}^{\mu_i\gamma_i} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{i^{-1}}{\rho}\Big)\Big)\Big]^{p_{mn}} > i, \quad i = 1, 2, \ldots.$$

Define $x = (x_{mn})$ by

$$\Big((m+n)!x_{mn}\Big)^{\frac{1}{m+n}} = \begin{cases} i^{-1}, & \text{if } 1 \le m, n \le \mu_i\gamma_i, \quad i = 1, 2, \ldots \\ 0, & \text{if } m > \mu_i, \quad n > \gamma_i. \end{cases} \tag{2.2}$$

Then $x \in \chi^2[\hat{c}, \Delta^m, u, p, q]$ but by (2.2),$x \notin \Lambda^2_M[\hat{c}, \Delta^m, u, p, q]$ which contradicts (ii). Hence (iii) must hold.
(iii) ⇒ (i) Let (iii) be satisfied and $x = (x_{mn}) \in \Lambda^2[\hat{c}, \Delta^m, u, p, q]$. Suppose that $x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q]$. Then

$$\sup_{s,(\mu\gamma)} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} = \infty. \tag{2.3}$$

Let $t = \big((m+n)!|\Delta^m x_{m+s,n+s}|\big)^{\frac{1}{m_s+n_s}}$ for each m, n and fixed s, then by (2.3)

$$\sup_{s,(\mu\gamma)} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t}{\rho}\Big)\Big)\Big]^{p_{mn}} = \infty,$$

which contradicts (iii). Hence (i) must hold. This completes the proof. □

Theorem 2.7. *Let $1 \le p_{mn} \le \sup_{mn} p_{mn}$. Let $M = (M_{mn})$ be a sequence of Orlicz functions, $p = (p_{mn})$ be a bounded sequence of positive real numbers and $u = (u_{mn})$ be a sequence of strictly positive real numbers. Then the following statements are equivalent:*
(i) $\chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q]$;

(ii) $\chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2[\hat{c}, \Delta^m, u, p, q]$;

(iii) $\inf_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t}{\rho}\Big)\Big)\Big]^{p_{mn}} > 0,\ (t, \rho > 0)$,

where $t = \big((m+n)!|\Delta^m x_{m+s,n+s}|\big)^{\frac{1}{m_s+n_s}}$.

Proof. (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (iii) Let $\chi^2[\hat{c}, \Delta^m, u, p, q] \subseteq \Lambda^2[\hat{c}, \Delta^m, u, p, q]$.
Suppose that (iii) is not satisfied. Then for some $\rho > 0$

$$\inf_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0 \ \ (t, \rho > 0). \tag{2.4}$$

We can choose an index sequence $(\mu_i\gamma_i)$ of positive integers such that

$$\frac{1}{\mu_i\gamma_i} \sum_{mn=1}^{\mu_i\gamma_i} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{i}{\rho}\Big)\Big)\Big]^{p_{mn}} > i^{-1},\ \ i = 1, 2, \ldots.$$

Define $x = (x_{mn})$ by

$$\Big((m+n)!x_{mn}\Big)^{\frac{1}{m+n}} = \begin{cases} i, & \text{if } 1 \le m, n \le \mu_i\gamma_i,\ \ i = 1, 2, \ldots \\ 0, & \text{if } m, n > \mu_i, \gamma_i. \end{cases}$$

Thus by (2.4) $x \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$ but $x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q]$ which contradicts (ii). Hence (iii) must hold.
(iii) $\Rightarrow$ (i) Let (iii) be satisfied and $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$, then

$$\lim_{\mu\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0 \tag{2.5}$$

uniformly in s.
Suppose that $x \notin \chi^2[\hat{c}, \Delta^m, u, p, q]$. Then for some number $\epsilon_0 > 0$ and index $\mu_0\gamma_0$, we have $(m_s+n_s)!|\Delta^m x_{m_s+n_s}|^{\frac{1}{m+s,n+s}} \ge \epsilon_0$, for some $s > s'$ and $1 \le m, n \le \mu_0\gamma_0$. Therefore,

$$u_{mn}\Big[M_{mn}\Big(q\Big(\frac{\epsilon_0}{\rho}\Big)\Big)\Big]^{p_{mn}} \le u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m_s+n_s)|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}$$

and consequently by (2.5). Hence

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{\epsilon_0}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0$$

which contradicts (iii). Hence $\chi^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q]$. This completes the proof. ☐

Theorem 2.8. *Let* $1 \le p_{mn} \le \sup_{mn} p_{mn} < \infty$. *The inclusion*

$$\Lambda^2_M[\hat{c}, \Delta^m, u, p, q] \subseteq \chi^2[\hat{c}, \Delta^m, u, p, q] \ \ \textit{holds if and only if}$$

$$\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t}{\rho}\Big)\Big)\Big]^{p_{mn}} = \infty \quad (t,\rho>0). \tag{2.6}$$

Proof. Let $\Lambda^2_M[\hat{c},\Delta^m,u,p,q] \subseteq \chi^2[\hat{c},\Delta^m,u,p,q]$. Suppose that (2.6) does not hold. Therefore there is a number $t_0>0$ and an index sequence $(\mu_i\gamma_i)$ such that

$$\frac{1}{\mu_i\gamma_i}\sum_{mn=1}^{\mu_i\gamma_i} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t_0}{\rho}\Big)\Big)\Big]^{p_{mn}} \le N < \infty \;\; i=1,2,\ldots. \tag{2.7}$$

Define the sequence $x=(x_{mn})$ by

$$x_{mn} = \begin{cases} t_0, & \text{if } 1 \le mn \le \mu_i\gamma_i, \;\; i=1,2,\ldots \\ 0, & \text{if } mn > \mu_i\gamma_i. \end{cases}$$

Thus by (2.7), $x \in \Lambda^2_M[\hat{c},\Delta^m,u,p,q]$, but $x \notin \chi^2[\hat{c},\Delta^m,u,p,q]$. Hence (2.6) must hold.

Conversely, let (2.6) hold. If $x \in \Lambda^2_M[\hat{c},\Delta^m,u,p,q]$, then for each s and $\mu\gamma$

$$\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} \le N < \infty. \tag{2.8}$$

Suppose that $x \notin \chi^2[\hat{c},\Delta^m,u,p,q]$. Then for some number $\epsilon_0>0$ there is a number s_0 and an index $\mu_0\gamma_0$ such that

$$\big((m+n)!|\Delta^m x_{m+s,n+s}|\big)^{\frac{1}{m_s+n_s}} \ge \epsilon_0 \;\text{ for }\; s \ge s_0.$$

Therefore

$$u_{mn}\Big[M_{mn}\Big(q\Big(\frac{\epsilon_0}{\rho}\Big)\Big)\Big]^{p_{mn}} \le u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}$$

and hence for each m,n and s, we get

$$\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{\epsilon_0}{\rho}\Big)\Big)\Big]^{p_{mn}} \le N < \infty,$$

for some $N>0$, clearly (2.8) contradicts (2.6). Hence $\Lambda^2_M[\hat{c},\Delta^m,u,p,q] \subseteq \chi^2[\hat{c},\Delta^m,u,p,q]$. This completes the proof. ☐

Theorem 2.9. *Let* $1 \le p_{mn} \le \sup_{mn} p_{mn} < \infty$. *The inclusion*

$$\Lambda^2[\hat{c},\Delta^m,u,p,q] \subseteq \chi^2_M[\hat{c},\Delta^m,u,p,q] \;\textit{ holds if and only if}$$

$$\lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t}{\rho}\Big)\Big)\Big]^{p_{mn}} = 0 \quad (t,\rho>0). \tag{2.9}$$

Proof. Let $\Lambda^2[\hat{c},\Delta^m,u,p,q] \subseteq \chi^2_M[\hat{c},\Delta^m,u,p,q]$. Suppose that (2.9) does not hold. Therefore there is a number $t_0>0$ such that

$$\lim_{\mu\gamma\to\infty}\frac{1}{\mu_i\gamma_i}\sum_{mn=1}^{\mu_i\gamma_i} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{t_0}{\rho}\Big)\Big)\Big]^{p_{mn}} = L \ne 0. \tag{2.10}$$

Define the sequence $x = (x_{mn})$ by

$$\big((m+n)!x_{mn}\big)^{\frac{1}{m+n}} = t_o \sum_{v=0}^{m,n-\eta} (-1)^n \begin{pmatrix} \gamma + (m,n) - v - 1 \\ (m,n) - v \end{pmatrix}$$

for $m, n = 1, 2, \ldots$. Thus by (2.10), $x \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$, but $x \notin \Lambda^2[\hat{c}, \Delta^m, u, p, q]$. Hence (2.9) must hold.
Conversely, let (2.9) hold and $x \in \Lambda^2[\hat{c}, \Delta^m, u, p, q]$, then for every m, n and s

$$\big((m+n)!|\Delta^m x_{m+s,n+s}|\big)^{\frac{1}{m_s+n_s}} \leq N < \infty.$$

Therefore

$$u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} \leq u_{mn}\Big[M_{mn}\Big(\frac{N}{\rho}\Big)\Big]^{p_{mn}}$$

and

$$\begin{aligned}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} \\ \leq \frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(\frac{N}{\rho}\Big)\Big]^{p_{mn}} \\ = 0 \text{ by (2.9).}\end{aligned}$$

Hence $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. This completes the proof. □

Theorem 2.10. *The space $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ is solid.*

Proof. Let $x = (x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$ and (α_{mn}) be a sequence of scalars such that $|\alpha_{mn}|^{\frac{1}{m_s+n_s}} \leq 1$ for all $m, n \in \mathbb{N}$. Then

$$\begin{aligned}\lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m \alpha_{mn} x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}} \\ \leq \lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma} u_{mn}\Big[M_{mn}\Big(q\Big(\frac{((m+n)!|\Delta^m x_{m+s,n+s}|)^{\frac{1}{m_s+n_s}}}{\rho}\Big)\Big)\Big]^{p_{mn}}\end{aligned}$$

for all $m, n \in \mathbb{N}$. Hence $(\alpha_{mn} x_{mn}) \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$ for all sequences of scalars α_{mn} with $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N}$ whenever $x_{mn} \in \chi^2_M[\hat{c}, \Delta^m, u, p, q]$. □

Theorem 2.11. *The space $\chi^2_M[\hat{c}, \Delta^m, u, p, q]$ is monotone.*

Proof. It is obvious. □

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School of Mathematics Shri Mata Vaishno Devi University
Katra-182320, J & K, India
E-mail address: kuldipraj68@gmail.com

School of Mathematics Shri Mata Vaishno Devi University
Katra-182320, J & K, India
E-mail address: sunilksharma42@yahoo.co.in