ACTA MATHEMATICA VIETNAMICA
Volume 37, Number 2, 2012, pp. 231–242

REMARKS ON EKELAND'S VARIATIONAL PRINCIPLE FOR POLYNOMIAL FUNCTIONS

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ABSTRACT. In this paper, we provide a method to determine points $v \in \mathbb{R}^n$ which satisfy Ekeland's variational principle, and also to choose Palais-Smale minimizing sequences that satisfy the second order condition for polynomial functions bounded from below on $\mathbb{R}^n$.

1. Introduction

Let V be a complete metric space, and $f : V \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function, $\not\equiv +\infty$, bounded from below. In weak form, Ekeland's variational principle [3] says that for any $\epsilon > 0$, there is some point $v \in V$ such that

$$f(v) \leq \inf_V f + \epsilon,$$

$$f(z) \geq f(v) - \epsilon d(v, z), \quad \forall z \in V.$$

Let f be a C^2-function that is bounded from below on a Hilbert space H. It is well-known that Ekeland's variational principle yields minimizing sequences $\{v_k\}$ (i.e., $f(v_k) \to \inf_H f$) that are also Palais-Smale sequences (i.e., $f'(v_k) \to 0$) ([3]). Less known is the smooth variational principle of Borwein and Preiss [1] which yields the existence of Palais-Smale minimizing sequences for f that also verify the following second order condition:

$$\liminf_k \langle f''(v_k)\omega, \omega\rangle \geq 0 \quad \text{for all } \omega \in H.$$

Unfortunately, it is not shown how to choose points v (which satisfy Ekeland's variational principle) and Palais-Smale minimizing sequences $\{v_k\}$ (which satisfy the second order condition). The object of this paper is to provide a method to determine such points v and also to choose such sequences $\{v_k\}$ for polynomial functions bounded from below on $\mathbb{R}^n$. We will show that these points can be chosen in the so called tangency curve, and in some cases, in the polar curve.

Received February 21, 2011.
1991 *Mathematics Subject Classification.* 14P15, 58K40.
Key words and phrases. Polynomial functions, Ekeland's variational principle, Palais-Smale minimizing sequences, the second order condition, the tangency curve, the polar curve.

2. The tangency curve

In this section, we briefly recall the notion of the tangency curve. For details, the reader may consult [5] (see also [4]).

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Put

$$X := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n : \operatorname{rank} \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ x_1 - a_1 & \cdots & x_n - a_n \end{pmatrix} \leq 1\}.$$

We shall denote by $\Sigma(f)$ the set of critical points of f. Put

$$\Gamma(a, f) := \{x \in \mathbb{R}^n : x \notin \Sigma(f) \text{ and } (x, a) \in X\}.$$

Lemma 2.1 ([5]). *With the previous notations,*

(i) *$\Gamma(a, f)$ is a nonempty, unbounded and semi-algebraic set;*
(ii) *There exists a proper algebraic set $\Omega \subsetneq \mathbb{R}^n$ such that $\Gamma(a, f)$ is a one-dimensional submanifold of $\mathbb{R}^n$ for each $a \in \mathbb{R}^n \backslash \Omega$.*

Definition 2.2. If $\dim \Gamma(a, f) = 1$, we call it the *tangency curve* of f with respect to $a \in \mathbb{R}^n$.

Let $\Gamma(a, f)$ be a tangency curve of f. For large $r > 0$, the intersection of $\Gamma(a, f)$ with the complement of the closed ball $\mathbb{B}_r$ of radius r centered at the origin has a fixed number of connected components. The germ at infinity of such a connected component will be called a *half-branch at infinity* of $\Gamma(a, f)$. Let $\Gamma_1, \ldots, \Gamma_s$ be the half branches at infinity of $\Gamma(a, f)$. Then there exist $\sigma > 0$ and Nash functions $\rho_i : (0, \sigma) \to \mathbb{R}^n$ such that Γ_i is the germ of the curve $x = \rho_i(\tau)$ as $\tau \to 0$. We may also assume (taking $\sigma > 0$ small enough if necessary) that the function $\|\rho_i\| : (0, \sigma) \to \mathbb{R}$, $\tau \mapsto \|\rho_i(\tau)\|$, is strictly decreasing and the function $f \circ \rho_i : (0, \sigma) \to \mathbb{R}$, $\tau \mapsto f[\rho(\tau)]$, is strictly monotone or constant. Set $t_i := \lim_{\tau \to 0} f[\rho_i(\tau)] \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 2.3. Each value t_i, $i = 1, \ldots, s$, is called a *tangency value* of f with respect to $a \in \mathbb{R}^n$.

Definition 2.4 ([7]). We say that a polynomial f satisfies the *Malgrange condition* at the value t_0 if there are $r \gg 1$, $\delta > 0$, $c > 0$ such that for every $x \in f^{-1}(D_\delta) \backslash \mathbb{B}_r$, we have

$$\|x\| \|f'(x)\| > c,$$

where $D_\delta = \{t \in \mathbb{R} : |t - t_0| < \delta\}$ and $\mathbb{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

Proposition 2.5 ([6]). *Let Γ_i be a half-branch at infinity of $\Gamma(a, f)$. Then*

$$\lim_{x \in \Gamma_i, \|x\| \to \infty} \|x\| \|f'(x)\| = 0.$$

In particular, if t_i is a tangency value of f and $t_i \neq \pm\infty$, then f does not satisfy the Malgrange condition at the value t_i.

3. Polynomial functions on $\mathbb{R}^n$

Assume that $\Gamma(f) := \Gamma(f, 0)$ is a tangency curve and t_1 is the smallest tangency value of f.

Proposition 3.1 ([5], [4]). *A polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is bounded from below if and only if $t_1 > -\infty$.*

From now on we assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^n$. Then it is easily seen that $t_1 = f^* := \inf_{\mathbb{R}^n} f$. Moreover, we may assume that the half-branch Γ_1 is contained in the set

$$\{x \in \mathbb{R}^n : f(x) = \min\{f(y) : \|y\| = \|x\|, y \in \mathbb{R}^n\}\}.$$

Lemma 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^n$. Let Γ_1 be parameterized by a Nash function $\rho : (0, \sigma) \to \mathbb{R}^n$. Then there is $\lambda : (0, \sigma) \to \mathbb{R}$ such that $f'[\rho(\tau)] = \lambda(\tau)\rho(\tau)$ for all $\tau \in (0, \sigma)$ and $\lambda(\tau) < 0$ for τ sufficiently small.*

Proof. Set $g(\tau) := f \circ \rho(\tau)$. We have

$$\frac{d}{d\tau} g(\tau) = \langle f'[\rho(\tau)], \rho'(\tau)\rangle = \lambda(\tau)\langle \rho(\tau), \rho'(\tau)\rangle.$$

Let

$$\rho(\tau) = a\tau^\alpha + \text{ higher order terms in } \tau, \quad \text{with } a \neq 0,$$

$$\lambda(\tau) = b\tau^\gamma + \text{ higher order terms in } \tau, \quad \text{with } b \neq 0.$$

By assumption, we see that g is strictly increasing in $(0, \sigma)$. Hence

$$\frac{d}{d\tau} g(\tau) = b\alpha\|a\|^2\tau^{\gamma+2\alpha-1} + \cdots > 0.$$

It follows that $b\alpha > 0$. Since $\|\rho(\tau)\| \to +\infty$ as $\tau \to 0$, $\alpha < 0$. Thus $b < 0$ and $\lambda(\tau) < 0$ for τ small enough. □

Theorem 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^n$. Then for any $\epsilon > 0$, there is some $v \in \Gamma_1$ such that*

$$f(v) \leq f^* + \epsilon,$$

$$f(z) \geq f(v) - \epsilon d(v, z) \quad \textit{for all } z \in \mathbb{R}^n.$$

Let Γ_1 be parameterized by $\rho : (0, \sigma) \to \mathbb{R}^n$, where

$$\rho(\tau) = a\tau^\alpha + \text{ higher order terms in } \tau, \quad \text{with } a \neq 0.$$

We begin by proving the following.

Claim 3.4. *For $\tau_0 \in (0, \sigma)$ small enough, there is some $A \geq 1$ such that*

$$d(\rho(\tau_0), \rho(\tau)) \leq A(\|\rho(\tau)\| - \|\rho(\tau_0)\|) \quad \textit{for all } \tau \in (0, \tau_0).$$

Proof. We have

$$(\|\rho(\tau)\|)' = \alpha\|a\|\tau^{(\alpha-1)} + \text{ higher order terms in } \tau.$$

By the proof of Lemma 3.2, $\alpha < 0$. Hence $(\|\rho(\tau)\|)' < 0$ for all $\tau < \tau_0$, with τ_0 small enough. It implies that $\|\rho(\tau)\|$ is strictly decreasing in $(0, \tau_0)$. Hence

$$\|\rho(\tau)\| - \|\rho(\tau_0)\| > 0 \quad \text{for all } \tau \in (0, \tau_0).$$

Moreover, we have $\rho'(\tau) = a\alpha\tau^{\alpha-1} +$ higher order terms in τ. Hence

$$\begin{aligned}\|\rho'(\tau)\| &= |\alpha|\|a\|\tau^{\alpha-1} + \text{ higher order terms in } \tau \\ &= -\alpha\|a\|\tau^{\alpha-1} + \text{ higher order terms in } \tau \qquad (\text{since } \alpha < 0).\end{aligned}$$

It implies that $\lim_{\tau\to 0} \dfrac{\|\rho'(\tau)\|}{-(\|\rho(\tau)\|)'} = 1$. Thus there is some $A \geq 1$ such that

$$\|\rho'(\tau)\| \leq -A(\|\rho(\tau)\|)' \quad \text{for all } \tau \in (0, \tau_0),$$

with τ_0 small enough. Therefore

$$d(\rho(\tau_0), \rho(\tau)) \leq \int_\tau^{\tau_0} \|\rho'(t)\|dt \leq -A\int_\tau^{\tau_0} (\|\rho(t)\|)'dt = A(\|\rho(\tau)\| - \|\rho(\tau_0)\|).$$

This completes the proof of the claim. □

Proof of Theorem 3.3. We now choose $\tau_0 \in (0, \sigma)$ which satisfies Claim 3.4 and the following conditions:

(a) $\lambda(\tau) < 0$ for all $\tau \in (0, \tau_0)$, where $f'[\rho(\tau)] = \lambda(\tau)\rho(\tau)$ (This follows from Lemma 3.2);
(b) $\|f'[\rho(\tau)]\| \leq \dfrac{\epsilon}{A}$ for all $\tau \in (0, \tau_0)$ (This follows from Proposition 2.5);
(c) $\langle \rho(t), \rho'(t) \rangle = \|a\|^2\alpha\tau^{2\alpha-1} + \cdots < 0$ for all $\tau \in (0, \tau_0)$, since $\alpha < 0$;
(d) $f[\rho(\tau_0)] \leq \min_{\mathbb{B}_r} f$, where $r = \|\rho(\sigma)\|$ and $\mathbb{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$, since $\min_{\mathbb{B}_r} f > f^* = t_1 = \lim_{\tau\to 0} f[\rho(\tau)]$.

The proof of Theorem 3.3 will be divided into 3 steps.
Step 1. We prove that

$$f[\rho(\tau)] \geq f[\rho(\tau_0)] - \epsilon d(\rho(\tau_0), \rho(\tau)) \quad \text{for all } \tau \in (0, \sigma).$$

In fact,

- If $\tau \in [\tau_0, \sigma)$, since $f \circ \rho$ is increasing in $[\tau_0, \sigma)$, we have

$$f[\rho(\tau_0)] - f[\rho(\tau)] \leq 0 \leq \epsilon d(\rho(\tau_0), \rho(\tau)).$$

- If $\tau \in (0, \tau_0)$, we have

$$\begin{aligned}
f[\rho(\tau_0)] - f[\rho(\tau)] &= \int_\tau^{\tau_0} \langle f'[\rho(t)], \rho'(t)\rangle dt \\
&= \int_\tau^{\tau_0} \lambda(t)\langle \rho(t), \rho'(t)\rangle dt = \int_\tau^{\tau_0} \lambda(t)\|\rho(t)\| \frac{\langle \rho(t), \rho'(t)\rangle}{\|\rho(t)\|} dt \\
&= \int_\tau^{\tau_0} \|f'[\rho(t)]\| \frac{-\langle \rho(t), \rho'(t)\rangle}{\|\rho(t)\|} dt \qquad \text{(by (a))} \\
&\leq \frac{\epsilon}{A} \int_\tau^{\tau_0} \frac{-\langle \rho(t), \rho'(t)\rangle}{\|\rho(t)\|} dt \qquad \text{(by (b) and (c))} \\
&= -\frac{\epsilon}{A} \int_\tau^{\tau_0} (\|\rho(t)\|)' dt = \frac{\epsilon}{A}(\|\rho(\tau)\| - \|\rho(\tau_0)\|) \leq \frac{\epsilon}{A} d(\rho(\tau_0), \rho(\tau)) \\
&\leq \epsilon d(\rho(\tau_0), \rho(\tau)) \qquad \text{(since } A \geq 1).
\end{aligned}$$

Step 2. We show that

$$f(z) \geq f[\rho(\tau_0)] - \epsilon d(\rho(\tau_0), z) \quad \text{for all } z \in \mathbb{R}^n \backslash \mathbb{B}_r.$$

In fact, assume that $\|z\| = \|\rho(\tau)\|$ for some $\tau \in (0, \sigma)$. Then $f(z) \geq f[\rho(\tau)]$, and hence

- If $\tau \in [\tau_0, \sigma)$, since $f \circ \rho$ is increasing in $[\tau_0, \sigma)$, we have

$$f[\rho(\tau_0)] - f(z) \leq f[\rho(\tau_0)] - f[\rho(\tau)] \leq 0 \leq \epsilon d(\rho(\tau_0), \rho(\tau)).$$

- If $\tau \in (0, \tau_0)$, we have

$$\begin{aligned}
f[\rho(\tau_0)] - f(z) &\leq f[\rho(\tau_0)] - f[\rho(\tau)] \\
&\leq \frac{\epsilon}{A} d(\rho(\tau_0), \rho(\tau)) \quad \text{(by Step 1)} \\
&\leq \frac{\epsilon}{A} A(\|\rho(\tau)\| - \|\rho(\tau_0)\|) \quad \text{(by Claim 3.4)} \\
&= \epsilon(\|z\| - \|\rho(\tau_0)\|) \leq \epsilon\|z - \rho(\tau_0)\| = \epsilon d(\rho(\tau_0), z)).
\end{aligned}$$

Step 3. We claim that

$$f(z) \geq f[\rho(\tau_0)] - \epsilon d(\rho(\tau_0), z) \quad \text{for all } z \in \mathbb{B}_r.$$

In fact, by (d), for all $z \in \mathbb{B}_r$ we have $f(z) \geq f[\rho(\tau_0)]$. It follows that

$$f[\rho(\tau_0)] - f(z) \leq 0 \leq \epsilon d(\rho(\tau_0), z).$$

Take $v := \rho(\tau_0)$, the proof of Theorem 3.3 is complete. □

Theorem 3.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^n$. Then*

(i) $\lim_{x \in \Gamma_1, \|x\| \to \infty} f(x) = f^*$,

(ii) $\lim_{x \in \Gamma_1, \|x\| \to \infty} \|f'(x)\| = 0$,

(iii) $\lim_{x \in \Gamma_1, \|x\| \to \infty} \langle f''(x)\omega, \omega\rangle \geq 0$ *for all $\omega \in \mathbb{R}^n$.*

Proof. (i) The first assertion follows from the fact that $t_1 = f^*$.
(ii) The second assertion follows immediately from Proposition 2.5.
(iii) Let Γ_1 be parameterized by $\rho : (0, \sigma) \to \mathbb{R}^n$, where

$$\rho(\tau) = a\tau^\alpha + \text{ higher order terms in } \tau, \quad \text{with } a \neq 0.$$

Then there is $\lambda : (0, \sigma) \to \mathbb{R}^n$ such that $f'[\rho(\tau)] = \lambda(\tau)\rho(\tau)$. We first prove the following claims.

Claim 3.6. *For τ sufficiently small, $\rho'(\tau)$ does not belong to $T_{\rho(\tau)}\mathbb{S}^{n-1}_{\|\rho(\tau)\|}$.*

Proof. We have

$$\langle \rho(\tau), \rho'(\tau)\rangle = \alpha\|a\|^2\tau^{2\alpha-1} + \cdots .$$

Since $a \neq 0$ and $\alpha < 0$, we have $\langle \rho(\tau), \rho'(\tau)\rangle \neq 0$. This implies that $\rho'(\tau) \notin T_{\rho(\tau)}\mathbb{S}^{n-1}_{\|\rho(\tau)\|}$ for τ small enough. □

Claim 3.7. *For τ small enough, we have*

(a) $\langle f''[\rho(\tau)]\rho'(\tau), \rho'(\tau)\rangle > 0$,
(b) $\langle f''[\rho(\tau)]\rho'(\tau), h\rangle = \lambda(\tau)\langle \rho'(\tau), h\rangle$ *for all* $h \in T_{\rho(\tau)}\mathbb{S}^{n-1}_{\|\rho(\tau)\|}$,
(c) $\langle f''[\rho(\tau)]h, h\rangle \geq \lambda(\tau)\|h\|^2$ *for all* $h \in T_{\rho(\tau)}\mathbb{S}^{n-1}_{\|\rho(\tau)\|}$.

Proof. Let $\tau_0 \in (0, \sigma)$.
(a) Set $g(\tau) := \langle f'[\rho(\tau)], \rho'(\tau_0)\rangle$. We have

$$g'(\tau) = \langle f''[\rho(\tau)]\rho'(\tau), \rho'(\tau_0)\rangle.$$

Moreover, $g(\tau) = \lambda(\tau)\langle \rho(\tau), \rho'(\tau_0)\rangle$. Hence

$$g'(\tau) = \lambda'(\tau)\langle \rho(\tau), \rho'(\tau_0)\rangle + \lambda(\tau)\langle \rho'(\tau), \rho'(\tau_0)\rangle.$$

Therefore, $\langle f''[\rho(\tau_0)]\rho'(\tau_0), \rho'(\tau_0)\rangle = \lambda'(\tau_0)\langle \rho(\tau_0), \rho'(\tau_0)\rangle + \lambda(\tau_0)\|\rho'(\tau_0)\|^2$. Let

$$\lambda(\tau) = b\tau^\gamma + \text{ higher order terms in } \tau, \quad \text{with } b \neq 0.$$

Then

$$\langle f''[\rho(\tau_0)]\rho'(\tau_0), \rho'(\tau_0)\rangle = \|a\|^2 b\alpha(\gamma + \alpha)\tau_0^{\gamma+2\alpha-2} + \cdots .$$

Since $\|f'[\rho(\tau)]\| = |\lambda(\tau)|\|\rho(\tau)\| = |b|\|a\|\tau^{\gamma+\alpha} + \cdots$ and (ii), $\gamma + \alpha > 0$. Moreover, it follows from Lemma 3.2 that $\alpha < 0$ and $b < 0$. Thus $\|a\|^2 b\alpha(\gamma + \alpha) > 0$. Therefore,

$$\langle f''[\rho(\tau_0)]\rho'(\tau_0), \rho'(\tau_0)\rangle > 0 \quad \text{ for } \tau_0 \text{ small enough.}$$

(b) For every $h \in T_{\rho(\tau_0)}\mathbb{S}^{n-1}_{\|\rho(\tau_0)\|}$, set $k(\tau) := \langle f'[\rho(\tau)], h)\rangle$. We have

$$k'(\tau) = \langle f''[\rho(\tau)]\rho'(\tau), h\rangle.$$

Moreover, $k(\tau) = \lambda(\tau)\langle \rho(\tau), h\rangle$. Hence

$$k'(\tau) = \lambda'(\tau)\langle \rho(\tau), h\rangle + \lambda(\tau)\langle \rho'(\tau), h\rangle.$$

Therefore

$$\langle f''[\rho(\tau_0)]\rho'(\tau_0), h\rangle = \lambda'(\tau_0)\langle \rho(\tau_0), h\rangle + \lambda(\tau_0)\langle \rho'(\tau_0), h\rangle = \lambda(\tau_0)\langle \rho'(\tau_0), h\rangle.$$

(c) Assume that $r'(0) = h \in T_{\rho(\tau_0)}\mathbb{S}^{n-1}_{\|\rho(\tau_0)\|}$, where $s \mapsto r(s) \in \mathbb{S}^{n-1}_{\|\rho(\tau_0)\|}$. We have

$$(f \circ r)'(s) = \langle f'[r(s)], r'(s)\rangle,$$

$$(f \circ r)''(s) = \langle f''[r(s)]r'(s), r'(s)\rangle + \langle f'[r(s)], r''(s)\rangle.$$

Hence

$$\begin{aligned}(f \circ r)''(0) &= \langle f''[r(0)]r'(0), r'(0)\rangle + \langle f'[r(0)], r''(0)\rangle \\ &= \langle f''[\rho(\tau_0)]h, h\rangle + \lambda(\tau_0)\langle \rho(\tau_0), r''(0)\rangle.\end{aligned}$$

Moreover, since $\|r(s)\|^2 = \|\rho(\tau_0)\|^2$, $(\|r(s)\|^2)' = 2\langle r(s), r'(s)\rangle = 0$. Hence

$$(\|r(s)\|^2)'' = 2\|r'(s)\|^2 + 2\langle r(s), r''(s)\rangle = 0.$$

Thus $\|r'(0)\|^2 + \langle r(0), r''(0)\rangle = 0$, and so $\langle \rho(\tau_0), r''(0)\rangle = -\|h\|^2$. Therefore

$$\langle f''[\rho(\tau_0)]h, h\rangle = (f \circ r)''(0) + \lambda(\tau_0)\|h\|^2.$$

Since the restriction of f to $\mathbb{S}^{n-1}_{\|\rho(\tau_0)\|}$ attains its minimum value at $\rho(\tau_0) = r(0)$, we have $(f \circ r)''(0) \geq 0$. Hence

$$\langle f''[\rho(\tau_0)]h, h\rangle \geq \lambda(\tau_0)\|h\|^2.$$

The proof of Claim 3.7 is complete. □

Proof of (iii): Let $\omega \in \mathbb{R}^n$. It follows from Claim 3.6 that for τ small enough, we can write

$$\omega = u(\tau)\rho'(\tau) + v(\tau)h(\tau),$$

where $h(\tau) \in T_{\rho(\tau)}\mathbb{S}^{n-1}_{\|\rho(\tau)\|}$ and $\|h(\tau)\| = 1$. It is easily seen that

$$u(\tau) = \frac{\langle \omega, \rho(\tau)\rangle}{\langle \rho'(\tau), \rho(\tau)\rangle}, \quad v(\tau) = \|\omega - u(\tau)\rho'(\tau)\|, \quad \text{and}$$

$$h(\tau) = \frac{\omega - u(\tau)\rho'(\tau)}{\|\omega - u(\tau)\rho'(\tau)\|} \quad \text{if } v(\tau) = \|\omega - u(\tau)\rho'(\tau)\| \neq 0.$$

We see that

$$\begin{aligned}\langle f''[\rho(\tau)]\omega, \omega\rangle &= u(\tau)^2\langle f''[\rho(\tau)]\rho'(\tau), \rho'(\tau)\rangle + 2u(\tau)v(\tau)\langle f''[\rho(\tau)]\rho'(\tau), h(\tau)\rangle \\ &\qquad + v(\tau)^2\langle f''[\rho(\tau)]h(\tau), h(\tau)\rangle \\ &\geq \lambda(\tau)\big[2u(\tau)v(\tau)\langle \rho'(\tau), h(\tau)\rangle + v(\tau)^2\big] \qquad \text{(by Claim 3.7)}.\end{aligned}$$

Set

$$k(\tau) := \frac{\rho'(\tau) - \langle \rho'(\tau), h(\tau)\rangle h(\tau)}{\|\rho'(\tau) - \langle \rho'(\tau), h(\tau)\rangle h(\tau)\|} = \frac{\rho'(\tau) - \langle \rho'(\tau), h(\tau)\rangle h(\tau)}{\big(\|\rho'(\tau)\|^2 - \langle \rho'(\tau), h(\tau)\rangle^2\big)^{\frac{1}{2}}}.$$

It is clear that $\|k(\tau)\| = \|h(\tau)\| = 1$ and $\langle k(\tau), h(\tau)\rangle = 0$. Hence

$$\omega = \widetilde{u}(\tau)k(\tau) + \widetilde{v}(\tau)h(\tau),$$

where $\begin{cases} u(\tau) = \dfrac{1}{\big(\|\rho'(\tau)\|^2 - \langle \rho'(\tau), h(\tau)\rangle^2\big)^{\frac{1}{2}}} \widetilde{u}(\tau), \\ v(\tau) = \dfrac{-\langle \rho'(\tau), h(\tau)\rangle}{\big(\|\rho'(\tau)\|^2 - \langle \rho'(\tau), h(\tau)\rangle^2\big)^{\frac{1}{2}}} \widetilde{u}(\tau) + \widetilde{v}(\tau). \end{cases}$

Now we see that

$\langle f''[\rho(\tau)]\omega, \omega\rangle \geq$

$$\lambda\Bigg[\frac{2\langle \rho', h\rangle \widetilde{u}}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}}\Big(\frac{-\langle \rho', h\rangle}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}}\widetilde{u} + \widetilde{v}\Big) + \Big(\frac{-\langle \rho', h\rangle}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}}\widetilde{u} + \widetilde{v}\Big)^2\Bigg].$$

Since $\|\omega\| = (\widetilde{u}(\tau)^2 + \widetilde{v}(\tau)^2)^{\frac{1}{2}}$ and, by Lemma 3.2, $\lambda < 0$, we can continue this inequality and get $\langle f''[\rho(\tau)]\omega, \omega\rangle \geq \lambda\|\omega\|^2 A$, where

$$A = \frac{2|\langle \rho', h\rangle|}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}}\Big(\frac{|\langle \rho', h\rangle|}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}} + 1\Big) + \Big(\frac{|\langle \rho', h\rangle|}{\big(\|\rho'\|^2 - \langle \rho', h\rangle^2\big)^{\frac{1}{2}}} + 1\Big)^2.$$

Since $\|h\| = 1$, $h(\tau) = e +$ higher order terms in τ, with some constant vector $e \in \mathbb{R}^n \backslash \{0\}$. Hence

$$\frac{\langle \rho', h\rangle^2}{\|\rho'\|^2 - \langle \rho', h\rangle^2} = \frac{\alpha^2\langle a, e\rangle^2 \tau^{2(\alpha-1)} + \cdots}{\alpha^2(\|a\|^2 - \langle a, e\rangle^2)\tau^{2(\alpha-1)} + \cdots}.$$

We see that $\langle a, e\rangle = 0$, since $\langle \rho, h\rangle = \langle a, e\rangle \tau^\alpha + \cdots \equiv 0$. It follows that $\lim\limits_{\tau\to 0} \dfrac{\langle \rho', h\rangle^2}{\|\rho'\|^2 - \langle \rho', h\rangle^2} = 0$. Moreover, since $f'[\rho(\tau)] = \lambda(\tau)\rho(\tau)$ and $\lim\limits_{\tau\to 0} \|\rho(\tau)\| = +\infty$, (ii) shows that $\lim\limits_{\tau\to 0} \lambda(\tau) = 0$. Therefore

$$\lim_{\tau\to 0} \langle f''[\rho(\tau)]\omega, \omega\rangle \geq 0.$$

□

Remark 3.8. If f is a C^2-function that is bounded from below on a Hilbert space H, in [1], Borwein and Preiss obtained a little weaker result. Namely, instead of (iii) of Theorem 3.5, they proved that $\liminf\limits_{k} \langle f''(v_k)\omega, \omega\rangle \geq 0$ for all $\omega \in H$. Moreover, it is not shown how to choose the sequence $\{v_k\}$.

Corollary 3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^n$. Let $\mu_1(x), \ldots, \mu_n(x)$ be eigenvalues of $f''(x)$. Then $\lim\limits_{x\in\Gamma_1, \|x\|\to\infty} \mu_i(x) \geq 0$ for all $i = 1, \ldots, n$, where, as before, Γ_1 is the half-branch of the tangency curve, corresponding to the smallest tangency value t_1.*

4. Polynomial functions on $\mathbb{R}^2$

We will receive in this section a result sharper than Theorem 3.5 for polynomials of two variables. We first recall some notions from [2].

Definition 4.1. A value $t_0 \in \mathbb{R}$ is called a *typical value at infinity* of a given polynomial f if there are $r \gg 1$, $\delta > 0$ such that the restriction function

$$f : f^{-1}(D_\delta)\backslash \mathbb{B}_r \to D_\delta := \{t \in \mathbb{R} : |t - t_0| < \delta\}$$

is a C^∞-trivial fibration, where $\mathbb{B}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. Otherwise, it is called an *atypical value at infinity* of f.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. Assume that f is monic of positive degree m in y. Then, no half-branch of $f_y^{-1}(0)$ is asymptotic to a vertical line. Let C be a half-branch of $f_y^{-1}(0)$. Then there exists a Nash function $g : (M, +\infty) \to \mathbb{R}$ such that C is the germ at infinity of the curve $(x = t, y = g(t))$ (resp., $(x = -t, y = g(t))$) and we say that C is a right half-branch (resp., a left half-branch). We also say that f changes sign along C if $f_y(t, g(t)+\varepsilon)f_y(t, g(t)-\varepsilon) < 0$ (resp., $f_y(-t, g(t) + \varepsilon)f_y(-t, g(t) - \varepsilon) < 0$) with $\varepsilon > 0$ small enough.

If $M > 0$ is large enough, there are Nash functions $g_1 < \ldots < g_p : (M, +\infty) \to \mathbb{R}$ and $h_1 < \ldots < h_q : (M, +\infty) \to \mathbb{R}$ such that the right half-branches $C_1, \ldots, C_p$ (resp., the left half-branches $D_1, \ldots, D_q$) of $f_y^{-1}(0)$ along which f_y changes sign are the germs at infinity of the curves $(x = t, y = g_i(t))$ for $i = 1, \ldots, p$ (resp., $(x = -t, y = h_j(t))$ for $j = 1, \ldots, q$). In this way, we put an order $C_1 < \cdots < C_p$ (resp., $D_1 < \cdots < D_p$).

Definition 4.2 ([2])**.** Let $C_1 < \cdots < C_p$ be the right half-branches at infinity of $f_y^{-1}(0)$ along which f_y changes sign. A sequence of consecutive half-branches $C_k < \cdots < C_l$ is said to be a *right critical cluster* belonging to $\lambda \in \mathbb{R}$ if there is a symbol $\succ$ in $\{\nearrow, \searrow, =\}$ such that:

(i) for every $i = k, \ldots, l$, one has $f \succ_{C_i} \lambda$,
(ii) $f \succ_{C_{k-1}} \lambda$ does not hold (or $k = 1$),
(iii) $f \succ_{C_{l+1}} \lambda$ does not hold (or $l = p$).

The left critical clusters are defined in the same way.

Theorem 4.3 ([2])**.** *The real number λ is an atypical value at infinity of f if and only if there exists a critical cluster belonging to λ consisting of an odd number of half-branches of $f_y^{-1}(0)$ along which f_y changes sign.*

Assume that the polynomial function f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^2$. Then $f^* := \inf_{\mathbb{R}^2} f$ is an atypical value at infinity of f. By Theorem 4.3, there is a critical cluster $C_k < \cdots < C_l$ belonging to f^* consisting of an odd number of half-branches of $f_y^{-1}(0)$ along which f_y changes sign. Notice that every connected component of $f^{-1}(f^* + \varepsilon)$ is vanishing at infinity as ε tends to 0 with $\varepsilon > 0$. Hence, every point of the half-branch C_k is a local minimum point of the restriction of f to some vertical line.

Theorem 4.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^2$. Let $C_k < \cdots < C_l$ be a critical cluster belonging to f^* consisting of an odd number of half-branches of $f_y^{-1}(0)$ along which f_y changes sign. Then for any $\epsilon > 0$, there* *is some $v \in C_k$ such that*

$$f(v) \leq f^* + \epsilon,$$
$$f(z) \geq f(v) - \epsilon d(v,z) \quad \text{for all } z \in \mathbb{R}^2.$$

Proof. The proof goes essentially in the same lines as in the proof of Theorem 3.3. □

Theorem 4.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^2$. Let $C_k < \cdots < C_l$ be a critical cluster belonging to f^* consisting of an odd number of half-branches of $f_y^{-1}(0)$ along which f_y changes sign. Then*

(i) $\lim\limits_{x \in C_k, \|x\| \to \infty} f(x) = f^*,$

(ii) $\lim\limits_{x \in C_k, \|x\| \to \infty} \|f'(x)\| = 0,$

(iii) *For $x \in C_k$, $\|x\|$ large enough, we have $\langle f''(x)\omega, \omega\rangle \geq 0$ for all $\omega \in \mathbb{R}^2$.*

Proof. (i) This assertion is clear.

(ii) Let C_k be parameterized by $\rho : (M, +\infty) \to \mathbb{R}^2$, $t \mapsto \rho(t)$, where

$$\rho(t) = (x = t, y = at^\alpha + \text{ lower order terms in } t).$$

We first observe that $\alpha \leq 1$. Indeed, by contradiction, assume that $a \neq 0$ and $\alpha > 1$. Since

$$f(x,y) = y^m + f_{m-1}(x)y^{m-1} + \cdots + f_0,$$
$$f'_y \circ \rho(t) = ma^{m-1}t^{(m-1)\alpha} + \text{ lower order terms in } t.$$

Since $m > 0$ and $a \neq 0$, we have $f' \circ \rho(t) \not\equiv 0$, which is a contradiction.

We now prove (ii): Since $f'_y[\rho(t)] \equiv 0$,$f'[\rho(t)] = (f'_x[\rho(t)], 0)$. Hence

$$\frac{d}{dt}(f \circ \rho)(t) = \langle f'[\rho(t)], \rho'(t)\rangle = f'_x[\rho(t)].$$

By assumption, we can write

$$(f \circ \rho)(t) = f^* + bt^\beta + \text{ lower order terms in } t, \text{with } b \neq 0 \text{ and } \beta < 0.$$

Therefore

$$\|f'[\rho(t)]\| = |f'_x[\rho(t)]| = |\frac{d}{dt}(f \circ \rho)(t)| = |b\beta t^{\beta-1} + \cdots|.$$

Since $\beta - 1 < 0$, we have $\lim\limits_{t \to 0} \|f'[\rho(t)]\| = 0.$

(iii) Let $\{e_1 = (1,0), e_2 = (0,1)\} \subset \mathbb{R}^2$. We first prove the following two claims.

Claim 4.6. *For every $t \in (M, +\infty)$, two vectors $\rho'(t)$ and e_2 are linearly independent.*

Proof. This claim follows immediately from the fact that $\langle \rho'(t), e_1\rangle = 1$ and $\langle e_1, e_2\rangle = 0$. □

Claim 4.7. *For t sufficiently large, we have*

(a) $\langle f''[\rho(t)]\rho'(t), \rho'(t)\rangle > 0,$

(b) $\langle f''[\rho(t)]\rho'(\tau), e_2\rangle = 0$,
(c) $\langle f''[\rho(\tau)]e_2, e_2\rangle \geq 0$.

Proof. Let $t_0 \in (M, +\infty)$.
(a) Set $h(t) := \langle f'[\rho(t)], \rho'(t_0)\rangle$. We have

$$h'(t) = \langle f''[\rho(t)]\rho'(t), \rho'(t_0)\rangle.$$

Moreover, $h(t) = f'_x[\rho(t)] = \dfrac{d}{dt}(f \circ \rho)(t) = b\beta t^{\beta-1} + \cdots$. Hence

$$h'(t) = b\beta(\beta-1)t^{\beta-2} + \cdots.$$

Therefore

$$\langle f''[\rho(t_0)]\rho'(t_0), \rho'(t_0)\rangle = b\beta(\beta-1)t_0^{\beta-2} + \cdots.$$

By assumption, we see that $f \circ \rho$ is strictly decreasing in $(M, +\infty)$. Hence $\dfrac{d}{dt}(f \circ \rho)(t) = b\beta t^{\beta-1} + \cdots < 0$, and so $b\beta < 0$. Since $\beta < 0$, we have $b > 0$ and $\beta - 1 < 0$. Thus

$$\langle f''[\rho(t_0)]\rho'(t_0), \rho'(t_0)\rangle > 0 \quad \text{for } t_0 \text{ large enough.}$$

(b) Set $k(t) := \langle f'[\rho(t)], e_2\rangle$. We have

$$k'(t) = \langle f''[\rho(t)]\rho'(t), e_2\rangle.$$

Moreover, since grad $f[\rho(t)] = (f'_x[\rho(t)], 0)$, $k(t) = 0$. Therefore

$$\langle f''[\rho(\tau)]\rho'(t), e_2\rangle = k'(t) = 0.$$

(c) Let $s \mapsto r(s) = \rho(t_0) + se_2$. We have $r'(s) = e_2$. Hence

$$(f \circ r)'(s) = \langle f'[r(s)], r'(s)\rangle = \langle f'[r(s)], e_2\rangle,$$
$$(f \circ r)''(s) = \langle f''[r(s)]r'(s), e_2\rangle = \langle f''[r(s)]e_2, e_2\rangle.$$

Thus

$$(f \circ r)''(0) = \langle f''[\rho(t_0)]e_2, e_2\rangle.$$

Since $f \circ r$ attains some local minimum value at $s = 0$, $(f \circ r)''(0) \geq 0$. Therefore $\langle f''[\rho(t_0)]e_2, e_2\rangle \geq 0$. □

Proof of (iii): Let $\omega \in \mathbb{R}^2$. By Claim 4.6, we can write $\omega = u(t)\rho'(t) + v(t)e_2$. Then $\langle f''[\rho(t)]\omega, \omega\rangle =$

$$u(t)^2\langle f''[\rho(t)]\rho'(t), \rho'(t)\rangle + 2u(t)v(t)\langle f''[\rho(t)]\rho'(t), e_2\rangle + v(t)^2\langle f''[\rho(t)]e_2, e_2\rangle.$$

By Claim 4.7, we have $\langle f''[\rho(t)]\omega, \omega\rangle \geq 0$ for t sufficiently large. □

Corollary 4.8. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. Assume that f is bounded from below and that f does not attain the minimum value in $\mathbb{R}^2$. Let $C_k < \cdots < C_l$ be a critical cluster belonging to f^* consisting of an odd number of half-branches of $f_y^{-1}(0)$ along which f_y changes sign, and let $\mu_1(x), \mu_2(x)$ be eigenvalues of $f''(x)$. Then for $x \in C_k$ and $\|x\|$ sufficiently large, we have $\mu_1(x) > 0$ and $\mu_2(x) \geq 0$.*

Proof. The corollary follows immediately from Claim 4.7. □

5. Remarks

We recall here some notions from [4].

Definition 5.1 ([4])**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. A value $y_0 \in \mathbb{R}$ is called a *local infimum value* of f if the following two conditions hold

- there exist some $\delta > 0$, $r > 0$ such that

$$\|x\| \geq r \text{ and } |f(x) - y_0| < \delta \Rightarrow f(x) \geq y_0.$$

- there exists a sequence $x^k \to \infty$ such that $f(x^k) \to y_0$.

Additionally, if $\delta > 0$ and $r > 0$ can be chosen such that

$$\|x\| \geq r \text{ and } |f(x) - y_0| < \delta \Rightarrow f(x) > y_0,$$

then y_0 is called an *isolated infimum value* of f.

Remark 5.2. There is at most only one local infimum value of f. The problem of characterization of the local (or, isolated) infimum value of f is solved in [4].

Remark 5.3. 1. It is easily seen that if f is bounded from below and f does not attain the minimum value then f has the isolated infimum value.
2. The results obtained still hold if we replace "f is bounded from below and f does not attain the minimum value" with "f has the isolated infimum value" and "$f^* := \inf_{\mathbb{R}^n} f$" with "the isolated infimum value".

Acknowledgments

The author wishes to thank Prof. Ha Huy Vui for many valuable suggestions and useful discussions.

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