ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 2, 2010, pp. 289–301

ORDER STRUCTURE AND ENERGY OF CONFLICTING CHIP FIRING GAME

LE MANH HA AND PHAN THI HA DUONG

Abstract. In this paper, we introduce a variation of the chip-firing game on a directed acyclic graph $G = (V, E)$. Starting from a given chip configuration, we can fire a vertex v by sending one chip along one of its outgoing edges to the corresponding neighbors if v has at least one chip.

Our main result is to give the collection of energies to show the partial order structure of the configuration space of the game. After that, we consider the case when support graph has only one source, we give the characterization of its reachable configurations and of its fixed points.

1. Introduction

The Chip Firing Game (CFG) is a mathematical model which describes the distribution resources used as physics, economics and computer science. A chip firing game [1] is defined over a (directed) multigraph $G = (V, E)$, called the *support* or the *base* of the game. A weight $w(v)$ is associated with each vertex $v \in V$, which can be regarded as the number of *chips* stored at the *site* v. The CFG is then considered as a discrete dynamical system with the following rule, called the *firing rule*: a vertex containing at least as many chips as its outgoing degree (its number of going out edges) transfers one chip along each of its outgoing edges. A configuration of CFG is a composition of n into V where n is the total number of chips which is constant over transfers process of CFG.

We call configuration space, and denote by $CFG(G, n, \mathcal{O})$, the set of all reachable configuration from $\mathcal{O}$. If at a configuration μ there is no firing that is possible then μ is said to be a *fixed point*. We call the CFG a *strongly convergent* game if it has a unique fixed point. Notice that there exist CFGs with no fixed point. It is known from [7] that the configuration space of CFGs which defined on a support graph G which has no close component is a graded lattice.

From the first definition of CFG, many variants of this system were introduced in different domains: the game of cards [4, 5] in the context of distributed system, the rotor-router model [8, 6, 9] in the random walks, the color chip firing game

Received January 6, 2009.
2000 *Mathematics Subject Classification.* Primary 68R05. Secondary 91A46.
Key words and phrases. Discrete dynamical system, Chip Firing Game, poset, fixed point, energies collection.
This work is supported in part by the Vietnamese National Foundation for Science and Technology Development (NAFOSTED).

in lattice theory [10, 11] and there are many in reality CFGs model in which the firing rule is reduced. The vertex v is firable if it contains at least one chip and its firing is carried out by sending one chip along one edge from v to one of its neighbors. Each transition of such a general CFGs performs only one transfers chip along one edge. However, the firing of a chip along one edge may cause a conflict with the one along another edge. Hence we call our model Conflicting Chip Firing Game (CCFG).

The paper is structured as follows. We first recall in Section 2 some basic definitions of directed acyclic graph theory and of partial order set theory. Further, in this new model, by relaxing the condition about the number of chips in a vertex, the evolution rule is much more flexible. In other side, the obtained configuration space has not the lattice structure, and the convergence properties. This situation is illustrated at the end of Section 3. Especially, in this section, we give an important characterization to show the partial order structure of the configuration of this game. This is also our main result. Moreover, we note that finding a support graph which has good properties in CCFG model is more difficult than in CFG model. In Section 4, we examine the configuration space of CCFGs in the relation with the support graph. We also consider a particular but important case of CCFGs, where the support graph is a directed acyclic graph which has one source. In this case, we characterize the reachable configurations and fixed points of the model and the induced games on subgraphs induced of G are also considered.

2. Definitions and notations

We recall here some definitions and basic results.

A *directed acyclic graph* (DAG) is a directed graph without cycles. A *sink* is a vertex with out-degree zero, while a *source* is a vertex with in-degree zero. It is clear to see that a DAG has at least one sink and one source. Throughout this paper, $G = (V, E)$ is a DAG. A *topological sort* of a DAG is an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that for all edge (v_i, v_j) of the graph we have $i < j$. We can see at once that a directed graph G has a topological sort if and only if it is acyclic.

Definition 2.1. [13] A *walk* in a directed graph is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_k, v_k$ such that for each $1 \leq i \leq k, e_i$ goes from v_{i-1} to v_i. A (*directed*) *trail* is a walk without repeated edges, and a (*directed*) *path* is a trail without repeated vertices.

Let x, y be in V. We define a binary relation $\leq$ on V as follows: for all $x, y \in V, y \leq x$ if and only if either $x = y$ or there is a path from x to y.

Next, we recall the notation of partially ordered set and some properties of order ideal and order filter. For more details about order theory, see e.g [3]. Besides, we used these notations in the set of vertices of a DAG.

An *order relation* or *partial order relation* is a binary relation $\leq$ over a set, such that for all x, y and z in this set, $x \leq x$ (reflexivity), $x \leq y$ and $y \leq z$

implies $x \leq z$ (transitivity), and $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry). The set is then called a *partially ordered* set or, for short, a *poset*.

Let P be a poset and let Q be a subset of V. Then Q inherits an order relation from V; given $x, y \in Q, x \leq y$ in Q if and only if $x \leq y$ in P. We say in these circumstances that Q has the order *induced from* P and call it a *subposet* of P.

The following result is straightforward from the definition of relation $\leq$ on the set of vertices of a DAG.

Lemma 2.2. *If $G = (V, E)$ is a DAG, then $(V, \leq)$ is a poset.*

A *chain* is a poset in which two elements are comparable. A subset C of a poset P is call a *chain* if C is a chain when regarded as a subposet of P. The chain C of P is called *saturated* if there does not exist $z \in P \setminus C$ such that $x < z < y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is a chain. The *length* $l(C)$ of a finite chain is defined by $l(C) = |C| - 1$. The *length* or *rank* of a finite poset P is $l(P) := \max\{l(C) : C \text{ is a chain of } P\}$. A *longest chain* from a to b is a chain of greatest length and a *shortest chain* from a to b is a saturated chain of smallest length.

Definition 2.3. Let P and Q be two posets. A map $\phi : P \to Q$ is said to be
(i) *order preserving* (or, alternatively, *monotone*) if $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q;
(ii) an *order-embedding* if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q.

When $\phi : P \to Q$ is an order-embedding we write $\phi : P \hookrightarrow Q$.

Definition 2.4. Let V be a poset, and let $Q \subseteq V$.
(i) Q is an *order ideal* or, for short, *ideal* (alternative terms include *decreasing set* or *down-set*) if, whether $x \in Q, y \in V$ and $y \leq x$, we have $y \in Q$.
(ii) Dually, Q is an *order filter* or, for short, *filter* (alternative terms are *increasing set* or *up-set*) if, whenever $x \in Q, y \in V$ and $y \geq x$, we have $y \in Q$. Given an arbitrary element $x \in V$, we define $Pred(x) \stackrel{\text{def}}{:=} \{y \in V \mid y \geq x\}$ and $Succ(x) \stackrel{\text{def}}{:=} \{y \in V \mid y \leq x\}$.

We denote by $\mathcal{I}(V)$ the set of all ideals of V and by $\mathcal{F}(V)$ the set of all filters of V.

The following properties are straightforward from definition

Property 2.5. *Let V be a finite poset. Then*

- *For all $x \in V$, we have $Pred(x) \in \mathcal{F}(V)$ and $Succ(x) \in \mathcal{I}(V)$.*
- *For all subset $U \subseteq V$, we have $U \in \mathcal{F}(V)$ if and only if $V \setminus U \in \mathcal{I}(V)$.*

Property 2.6. *$\mathcal{F}(V), \mathcal{I}(V)$ contain $\emptyset, V$ and closed under union and intersection.*

Lemma 2.7. *Let $G = (V, E)$ be a DAG and let $B \in \mathcal{F}(V)$. Let $V' = V - B$. Then for all $A \in \mathcal{F}(V')$, we have $A \cup B \in \mathcal{F}(V)$.*

Proof. Let $x \in A \cup B$ be an arbitrary element, y an element of V such that $y \geq x$. We prove that $x \in A \cup B$. If $x \in B$ then $y \in B$ due to $B \in \mathcal{F}(V)$ and $y \geq x$. If $x \in A$ and $y \notin B$, that means $y \in V', y \geq x$ then $y \in A$ due to $A \in \mathcal{F}(V')$. Thus $x \in A \cup B$. $\square$

The following corollary is immediate from Lemma 2.7 and closure properties of $\mathcal{F}(V)$.

Corollary 2.8. *If* $B \in \mathcal{F}(V)$ *then* $\mathcal{F}(B) \subseteq \mathcal{F}(V)$.

Next, to represent configuration of CCFG, we use integer composition, whose explicit notion is given as follows:

Definition 2.9. Let n be a positive integer and let S be a set of k elements. A composition of n into S is an ordered sequence $(a_1, a_2, \ldots, a_k)$ of non negative integers such that $a_1 + a_2 + \ldots + a_k = n$. The integer number a_i is called the weight of i.

It is easy to check that the number of compositions of n into S is $\binom{n+k-1}{n}$.

Definition 2.10. [12] The conflicting chip firing game (CCFG) on a DAG $G = (V, E)$ with n chips, denoted by $CCFG(G, n)$, is a dynamical model defined as follows: each configuration is a composition of n into V; an edge (u, v) of E is firable if u has at least one chip; the evolution rule (firing rule) of this game is the firing of one firable edge (u, v), that means the vertex u gives one chip to the vertex v.

We also denote by $CCFG(G, n)$ the set of all configurations of $CCFG(G, n)$ and call it the *configuration space* of this game. This set is exactly the set of compositions of n into V.

Definition 2.11. Given two configurations a and b of a $CCFG(G, n)$, we say that b is *reachable* from a, denoted by $b \leq a$, if b can be obtained from a by a firing sequence (in the case the firing sequence is empty, $a = b$). In particular, we write $a \to b$ if b is obtained from a by applying once firing rule.

Definition 2.12. Given $CCFG(G, n)$ and let $\mathcal{O}$ be a composition of n into V. We denote by $CCFG(G, n, \mathcal{O})$ the *configuration space of all reachable configurations from* $\mathcal{O}$ and we write $\mathcal{O} \rightsquigarrow a$ if $a \in CCFG(G, n, \mathcal{O})$.

We recall that a *Garden of Eden* configuration in a dynamical system is a configuration which is unreachable from any other configuration. And a *fixed point* is a configuration in which no edge is firable. A CCFG is said to be a *strongly convergent game* if it has a unique fixed point.

3. Order Structure and energies of CCFG

The goal of this section is to give an explicit definition of *energy* of configurations which is an important characterization to show the partial order structure of the configuration space of the game.

First of all, we present here some preliminary definitions.

Definition 3.1. Let $G = (V, E)$ be a DAG and let $a = (a_1, a_2, \ldots, a_{|V|})$ be a composition of n on V. The *energy* $e(A, a)$ of a on a subset $A \subseteq V$ is the quality $e(A, a) = \sum_{i \in A} a_i$, the set $(e(A, a)_{A \in \mathcal{F}(V)})$ is called the *energies collection* of a and the energy $\mathcal{E}(a)$ of a is the quality $\mathcal{E}(a) = \sum_{A \in \mathcal{F}(V)} e(A, a)$.

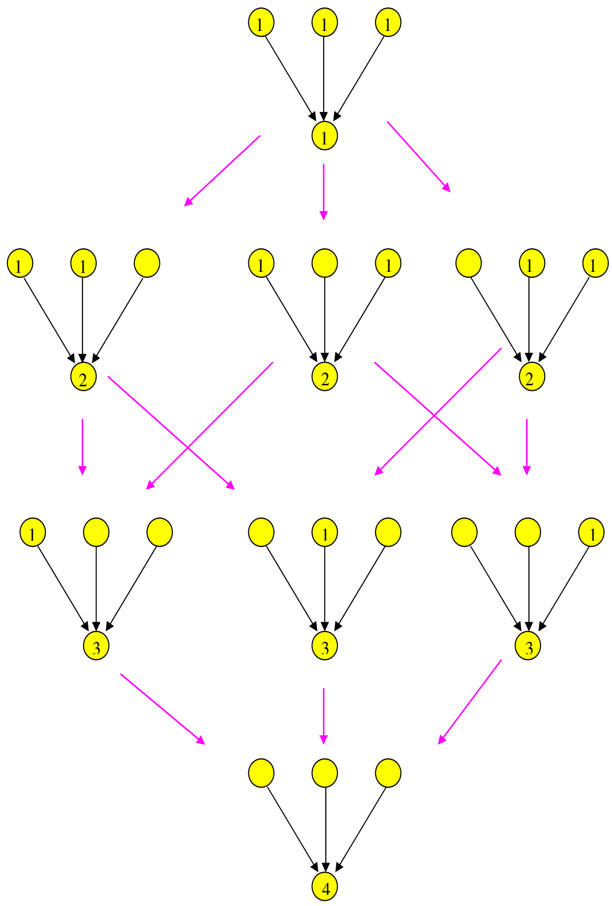

FIGURE 1. The configuration space of a CCFG with 4 chips.

Firstly, the basic relation between the configurations of the game $CCFG(G, n)$ and its energies collection is as follows:

Lemma 3.2. *A configuration a of $CCFG(G, n)$ is totally determined by its energies collection $(e(A, a)_{A \in \mathcal{F}(V)})$. That is, if a and b are two configurations of $CCFG(G, n)$ which have the same energies collection then $a = b$.*

Proof. Let $v \in V$. We prove that $a(v) = b(v)$ by induction on the cardinality of $Pred(v)$.

Basic step: If $|Pred(v)| = 1$ then v is a source and $Pred(v) = \{v\}$ and that $e(Pred(v), a) = e(Pred(v), b)$ is equivalent to $a(v) = b(v)$.

Inductive step: Assume that $|Pred(v)| = k + 1$ and that $a(u) = b(u)$ for all $u \in V$ with $|Pred(u)| \leq k$. Then we have $a(u) = b(u), \forall u \gneqq v$. On the other hand $Pred(v) = \{u \gneqq v\} \cup \{v\}$ and by hypothesis, $e(Pred(v), a) = e(Pred(v), b)$, so we have $a(v) = b(v)$. This completes the induction. $\square$

Now, we show that the configuration space of $CCFG$ has an order structure as the configuration space of many other dynamical systems.

Lemma 3.3. *$(CCFG(G,n), \leq)$ is a poset.*

Proof. Let us first prove that if $b \leq a$ then $e(A,b) \leq e(A,b)$ for all filters $A \in \mathcal{F}(V)$. It is sufficient to prove the statement for the case $a \to b$. Assume that b is obtained from a by transferring one chip from vertex u to v. Then $a(u) - 1 = b(u), a(v) + 1 = b(v)$ and $a(w) = b(w)$ for all $w \neq u, v$, where $a(u)$ is the number of chips of the vertex u at the configuration a. Let $A \in \mathcal{F}(V)$. If $v \in A$ then $u \in A$ due to the fact that $u \geq v$ in $(V, \leq)$ and A is a filter. So $e(A,a) = e(A,b)$. If $v \notin A$ then

$$e(A,a) = \begin{cases} e(A,b) \text{ if } u \notin A \\ e(A,b) + 1 \text{ if } u \in A. \end{cases}$$

Therefore $e(A,a) \geq e(A,b)$ for all filter $A \in \mathcal{F}(V)$.

From this, we have that $b \leq a$ implies $\mathcal{E}(b) \leq \mathcal{E}(a)$. Moreover, if $b < a$ then $\mathcal{E}(b) < \mathcal{E}(a)$ so $(CCFG(G,n), \leq)$ is a poset. □

Actually, the problem to characterize the order relation of a dynamical system is always difficult. Recall that in the classical CFG there are different chains from a to b if $b \leq a$. Nevertheless, all these chains have the same length and involve the same applications of the rule which is represented by shot vector. However, this is not true in the case of CCFG. So we can not use a similar notation to shot vector. We must use a more complicated technique to give a characterization of the order of $CCFG(G,n)$ which is the use of energies collection and this is our main result.

We state now the main result of this paper.

Theorem 3.4. *Let a and b be two configurations of $CCFG(G,n)$. Then $a \geq b$ in $CCFG(G,n)$ if and only if $e(A,a) \geq e(A,b)$, for all filters $A \in \mathcal{F}(V)$.*

Proof. The necessary condition is obtained by Lemma 3.3.

We prove the sufficient condition for showing that there exists a firing sequence from a to b. We prove by induction on $\mathcal{E}(a) - \mathcal{E}(b)$.

Basic case: If $\mathcal{E}(a) - \mathcal{E}(b) = 0$ then $e(A,a) = e(A,b)$ for all filters A. It follows that $a = b$ by Lemma 3.2.

Inductive case: Assume that $\mathcal{E}(a) - \mathcal{E}(b) > 0$. We will prove that there exists a configuration $c \neq b$ such that $e(A,a) \geq e(A,c) \geq e(A,b), \forall A \in \mathcal{F}(V)$ and that $c \geq b$. The existence of such a configuration c is sufficient for our proof because in this case by induction hypothesis we have $a \geq c$, which implies that $a \geq b$.

By assumption $E(a) - E(b) > 0$, so there exist filters A such that $e(A,a) > e(A,b)$. Let A_0 be a maximal element among these filters. Then for all $C \in \mathcal{F}(V)$ satisfying $A_0 \subsetneq C$ we have $e(C,a) = e(C,b)$. Because of $e(V,a) = e(V,b) = n$, so there exists $v \notin A_0$ such that $a(v) < b(v)$. Let us first prove that such an element v is unique. Suppose that there are $v_1, v_2 \notin A_0$ such that $a(v_1) < b(v_1), a(v_2) < b(v_2)$ and $v_1 \neq v_2$, without loss of generality we can assume $v_2 \notin$

$Pred(v_1)$. Set $Q_2 = Pred(v_2) \setminus \{v_2\}$, we have $Q_2 \in \mathcal{F}(V)$. Also, we have $A_0 \cup Pred(v_1) \cup Pred(v_2) \in \mathcal{F}(V)$ so by assumption,

$$e(A_0 \cup Pred(v_1) \cup Pred(v_2), a) \geq e(A_0 \cup Pred(v_1) \cup Pred(v_2), b)$$

or equivalently

$$e(A_0 \cup Pred(v_1) \cup Q_2, a) + a(v_2) \geq e(A_0 \cup Pred(v_1) \cup Q_2, b) + b(v_2).$$

From this and $a(v_2) < b(v_2)$, we obtain $e(A_0 \cup Pred(v_1) \cup Q_2, a) > e(A_0 \cup Pred(v_1) \cup Q_2, b)$. But $A_0 \cup Pred(v_1) \cup Q_2 \in \mathcal{F}(V)$ actually contains A_0 since $v_1 \notin A_0$. This contradicts our assumption that A_0 is a maximal element. We conclude that there is a unique $v \notin A_0$ such that $a(v) < b(v)$.

Define $\mathcal{B} = \mathcal{B}(b) := \{U \in \mathcal{F}(V), \emptyset \neq U \subseteq A_0 \mid e(U, a) = e(U, b)\}$. The proof will be divided into two cases:

- Case 1: $\mathcal{B} = \emptyset$

  In this case we will point out a configuration c satisfying $c \to b$ and $e(A, a) \geq e(A, c) \geq e(A, b)$, for all filters $A \in \mathcal{F}(V)$. Let us first show that if $A \in \mathcal{F}(V)$ and $v \notin A$, then $e(A, a) > e(A, b)$. By assumption $A_0 \in \mathcal{F}(V)$ a maximal element satisfying $e(A_0, a) > e(A_0, b)$ and $\mathcal{B} = \{U \in \mathcal{F}(V), \emptyset \neq U \subseteq A \mid e(U, a) = e(U, b)\} = \emptyset$, that is, for all $U \in \mathcal{F}(V), \emptyset \neq U \subseteq A$ then $e(U, a) > e(U, b)$.

  Now given $B \in \mathcal{F}(V), v \notin B$ in which $e(B, a) = e(B, b)$. Then B is not contained in A_0 and $A_0 \subsetneq A_0 \cup B$. By the maximality of A_0 we have $e(A_0 \cup B, a) = e(A_0 \cup B, b)$. On the other hand: $B \cup A_0 = (B \cap \overline{A_0}) \cup A_0$. So

$$e(B \cup A_0, a) = e(B \cap \overline{A_0}, a) + e(A_0, a)$$
$$e(B \cup A_0, b) = e(B \cap \overline{A_0}, b) + e(A_0, b).$$

  Since $e(A_0, a) > e(A_0, b)$, there is an element $u \in B \cap \overline{A_0}$ such that $a(u) < b(u)$ (otherwise $e(B \cup A_0, a) > e(B \cup A_0, b)$, which is impossible). By the unique existence of v we have $u = v$. But this contradicts the fact that B does not contain v. So, $e(A, a) > e(A, b)$ for all filters $A \in \mathcal{F}(V)$ which does not contain v.

  Let u be a neighbor of v such that $(u, v) \in E$. Let c be a configuration defined by $c(u) = b(u) + 1, c(v) = b(v) - 1$ and $c(w) = b(w), \forall w \neq u, v$. It is easy to see that $c \to b$.

  It remains to prove that $e(A, a) \geq e(A, c), \forall A \in \mathcal{F}(V)$. Let $A \in \mathcal{F}(V)$ be an arbitrary filter, we need only consider two cases:

  + If $v \in A$ then $u \in A$, due to $u \geq v$ in $(V, \leq)$ and $A \in \mathcal{F}(V)$. Hence, $e(A, a) \geq e(A, b) = e(A, c)$.

  + If $v \notin A$ then

$$e(A, c) = \begin{cases} e(A, b), & \text{if } u \notin A \\ e(A, b) + 1, & \text{otherwise.} \end{cases}$$

  Therefore, $e(A, c) \leq e(A, b) + 1 \leq e(A, a)$ (due to $e(A, a) > e(A, b)$).

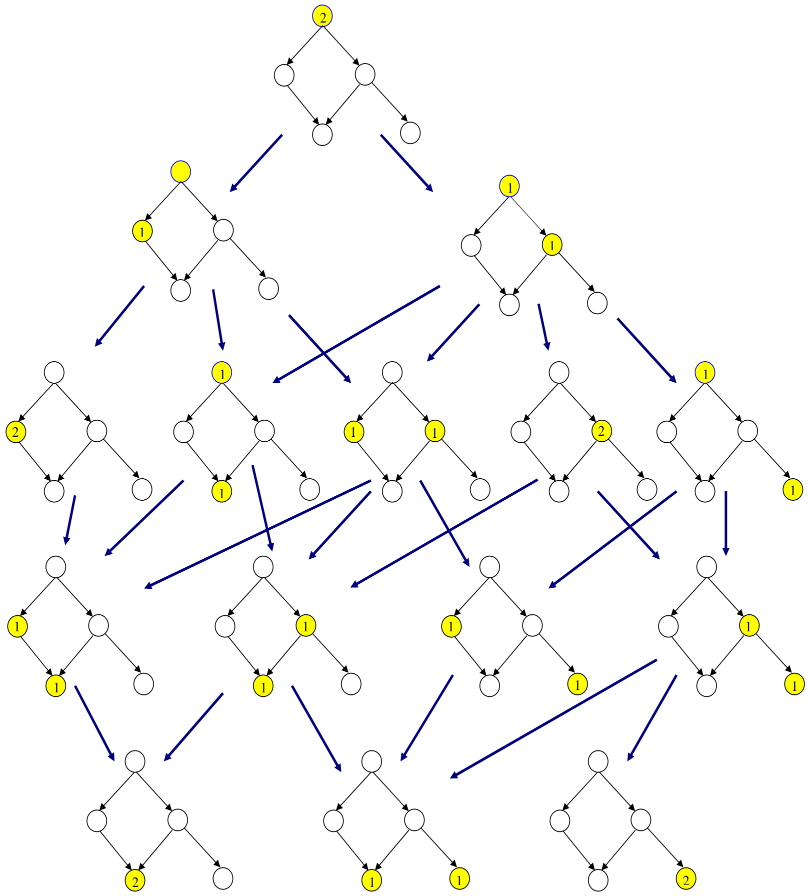

FIGURE 2. The configuration space of a $CCFG$ with 2 chips.

- Case 2: $\mathcal{B} \neq \emptyset$

In this case we will indicate a configuration $c \neq b$ such that $e(A, a) \geq e(A, c) \geq e(A, b), \forall A \in \mathcal{F}(V)$ and then the proof is completed by showing $c \geq b$. Let $B \in \mathcal{B}$ and let c be a configuration defined as follows:

$$c(u) = \begin{cases} b(u), & \text{if } u \in B \\ a(u), & \text{otherwise.} \end{cases}$$

Clearly $c(v) = a(v) < b(v)$ so $c \neq b$. Let $G_1 = G - B = (V_1, B_1)$ be the induced subgraph by its set $V_1 = V \setminus B$ of vertices.

For all $A \in \mathcal{F}(V)$, we have:

$$e(A, c) = e(A \cap B, c) + e(A \cap V_1, c) = e(A \cap B, b) + e(A \cap V_1, a).$$

As $A, B \in \mathcal{F}(V)$, it follows that $A \cap B \in \mathcal{F}(V)$ and hence, $e(A \cap B, b) \leq e(A \cap B, a)$.

Therefore, $e(A, c) \leq e(A \cap B, a) + e(A \cap V_1, a) = e(A, a)$.

Also, we have $e(A,b) = e(A \cap B, b) + e(A \cap V_1, b)$. As $A, B \in \mathcal{F}(V)$, this implies that $(A \cap V_1) \cup B = A \cup B \in \mathcal{F}(V)$ and consequently $e(A \cup B, a) \geq e(A \cup B, b)$, equivalently $e(A \cup V_1, a) + e(B, a) \geq e(A \cup V_1, b) + e(B, b)$. It follows that $e(A \cup V_1, a) \geq e(A \cup V_1, b)$ (note that $e(B,a) = e(B,b)$). Thus,

$$e(A,c) = e(A \cap B, b) + e(A \cap V_1, a) \geq e(A \cap B, b) + e(A \cup V_1, b) = e(A, b).$$

We conclude that for all filters $A \in \mathcal{F}(V)$, $e(A,a) \geq e(A,c) \geq e(A,b)$.

We now complete the proof by showing $c \geq b$ by induction on $|V|$. We observe that $c \geq b$ in $(\mathcal{CCFG}(G,n), \leq)$ is equivalent to $a \geq b$ in $(CCFG(G_1, n_1), \leq)$ where $n_1 = n - \sum_{u \in B} b(u)$. Since $B \neq \emptyset$, we have $V_1 = V \setminus B \subsetneq V$. By induction on the cardinality of V, we only need to show that $e(A', a) \geq e(A', b)$ for all filters $A' \in \mathcal{F}(V_1)$. Indeed, by Lemma 2.7 we have $A' \cup B \in \mathcal{F}(V)$ and $e(A' \cup B, a) \geq e(A' \cup B, b)$ by hypothesis. But $A' \cap B = \emptyset$ and $e(B,a) = e(B,b)$, we conclude that $e(A', a) \geq e(A', b)$. □

4. Configuration space of CCFG

Our aim is now to study the configuration space of the conflicting chip firing game on a DAG. Moreover, in the case the support graph has only one source, we show a characterization for reachable configurations and for fixed points of this game. This allows us to describe the complexity of the game by giving the cardinality of its configuration space. We first give the following lemma which is straightforward from definition.

Lemma 4.1. *A Garden of Eden configuration in $CCFG(G,n)$ is a composition of n into the set of sources of G. A fixed point of $CCFG(G,n)$ is a composition of n into the set of sinks of G.*

It is evident that a Garden of Eden configuration is a maximal element of $CCFG(G,n)$ and a fixed point is a minimal element of $CCFG(G,n)$.

Denote by $GE(G,n)$ the set of all Garden of Eden configurations of $CCFG(G,n)$. It is easy to see that

$$CCFG(G,n) = \bigcup_{\mathcal{O} \in GE(G,n)} CCFG(G,n,\mathcal{O}).$$

Now, let $(P, \leq)$ be a poset. We define the *dual poset* $(P^\partial, \leq)$ of P as follows: for all $x, y \in P$, $x \leq y$ in P^∂ if and only if $y \leq x$ in P. For a DAG $G = (V, E)$, we obtain the dual poset $(V^\partial, \leq)$ of $(V, \leq)$ by reversing direction of arcs. On the other hand, for a given graph G, we define the *reverse* of G, and write $G^\partial = (V^\partial, \overleftarrow{E})$, the graph obtained from G by reversing direction of arcs. That is, $(u,v) \in E$ if and only if $(v,u) \in \overleftarrow{E}$.

We give now the duality of configuration space of CCFG.

Proposition 4.2. *Let G be a DAG and let n be an integer. Then*

$$CCFG(G^\partial, n) = (CCFG(G,n))^\partial.$$

Proof. Let a and b be two configurations satisfying $a \leq b$ in $(CCFG(G,n))^{\partial}$. This is equivalent to $b \leq a$ in $CCFG(G,n)$. There is no loss of generality in assuming b is obtained from a by firing edge $(u,v) \in E$. Then $a(u) = b(u)+1, a(v) = b(v)-1$ and $a(w) = b(w)$ for all $w \neq u, v$. This is also nothing but $b \to a$ in $CCFG(G^{\partial}, n)$ by firing edge $(v,u) \in \overleftarrow{E}$.

∎

The relations among induced posets by the game $CCFG(G,n)$ are described by the Figure 3.

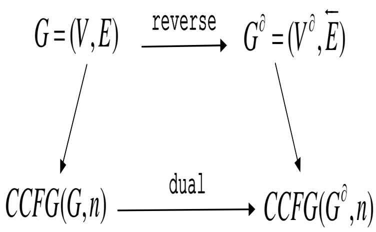

FIGURE 3. The diagram describing the relations among posets.

Let us next consider the maximum and minimum convergence time of CCFG. That is nothing but the length of longest and shortest chains in the configuration space of the game.

Denote by $l_{\max}$ the length of a longest directed paths in G and by $l_{\min}$ the length of a shortest from a source to a sink of G. Let P be the longest path in the graph, and suppose that it goes from s_0 to s_i. Then s_0 is a source while s_i is a sink of G. Denote by a_v the configuration of $CCFG(G,n)$ in which all n chips are centered at v. That is,

$$a_v(w) = \begin{cases} n, & \text{if } w = v \\ 0, & \text{otherwise.} \end{cases}$$

We can check at once that a_{s_0} is a Garden of Eden configuration while a_{s_i} is a fixed point of $CCFG(G,n)$ and the longest chain from a_{s_0} to a_{s_i} is exactly a longest chain in $CCFG(G,n)$. The following proposition is immediate.

Proposition 4.3. (i) *The lenght of longest chains in $CCFG(G,n)$ is $n.l_{max}$.*
(ii) *Dually, the length of shortest chains from a Garden of Eden configuration to a fixed point in $CCFG(G,n)$ is $n.l_{min}$.*

Let $\mathcal{O}$ be a configuration of $CCFG(G,n)$. We already know in the previous section that $CCFG(G,n)$ is a poset and it is easily seen that $CCFG(G,n,\mathcal{O})$ is an ideal of $CCFG(G,n)$. The following proposition gives the behavior of $CCFG(G,n,\mathcal{O})$ in the $CCFG(G,n)$ which is related to the set $\mathcal{I}(V)$ of all ideals of V.

Theorem 4.4. *Let $G = (V, E)$ be a DAG and let n be an integer. Let $\mathcal{O}$ be a configuration of $CCFG(G, n)$. Then the set $I(\mathcal{O}) = \{i \in V \mid \exists a \in CCFG(G, n, \mathcal{O}), a(i) \neq 0\}$ is an ideal of V and the map $\varphi : CCFG(G, n) \to \mathcal{I}(V)$, defined by $\varphi(\mathcal{O}) = I(\mathcal{O})$ for all $\mathcal{O} \in CCFG(G, n)$ is order-preserving.*

Proof. We first prove that $I(\mathcal{O}) \in \mathcal{I}(V)$. Let $u \in I(\mathcal{O})$ be an arbitrary element and let v be an element of V such that $v \leq u$. We prove that $v \in I(\mathcal{O})$. Since $u \in I(\mathcal{O})$, by definition of $\mathcal{O}$, there exists $a \in CCFG(G, n, \mathcal{O})$ such that $a(u) \neq 0$. Let b be a configuration defined as follows:

$$b(w) = \begin{cases} 0, \text{ if } w = u \\ a(u) + a(v), \text{ if } w = v \\ a(w), \text{ otherwise.} \end{cases}$$

We claim that $b \leq a$ (and hence $b \in CCFG(G, n, \mathcal{O})$). Indeed, let $A \in \mathcal{F}(V)$ be an arbitrary filter, we need only consider two cases:
+ If $v \in A$ then $u \in A$ due to $u \geq v$ in $(V, \leq)$ and $A \in \mathcal{F}(V)$. Hence, $e(A, b) = e(A, a)$.
+ If $v \notin A$ then

$$e(A, b) = \begin{cases} e(A, a), \text{ if } u \notin A \\ e(A, a) - a(u), \text{ if } u \in A. \end{cases}$$

Therefore, $e(A, b) \leq e(A, b)$ and this implies that $b \leq a$ by Theorem 3.4. Moreover, $b(v) \neq 0$, so $v \in I(\mathcal{O})$ by definition of $I(\mathcal{O})$.

Notice that for all $\mathcal{O}, \mathcal{O}' \in CCFG(G, n)$ we have $\mathcal{O} \leq \mathcal{O}'$ if and only if $CCFG(G, n, \mathcal{O}) \subseteq CCFG(G, n, \mathcal{O}')$. This implies the monotony of φ and we complete the proof. □

Now, we consider a special case which appears in many dynamical systems in reality, that is when the support graph has only one source.

We first notice that for two arbitrary elements a and b of a $CCFG(G, n)$, it is easy to check that $CCFG(G, n, b) \subseteq CCFG(G, n, a)$ if and only if $b \leq a$ in $CCFG(G, n)$.

From now on, we consider the $CCFG(G, n)$ on the DAG $G = (V, E)$ which has a unique source v_1. The unique Garden of Eden configuration of this game is $\mathcal{O}$ in which all n chips are centered at the source v_1. Hence, $CCFG(G, n) = CCFG(G, n, \mathcal{O})$. Denote by $\mathcal{F}(CCFG(G, n, \mathcal{O}))$ the set of all filters of $CCFG(G, n, \mathcal{O})$. In this case, $\mathcal{F}(CCFG(G, n, \mathcal{O}))$ is exactly $\mathcal{F}(CCFG(G, n))$.

Recall that the vertex set V of G is a poset and $\mathcal{F}(V)$ the set of all filters of V. We know that $< \mathcal{F}(V), \subseteq >$ is a complete lattice in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

Given $A \in \mathcal{F}(V)$. Denote by $G[A]$ the subgraph of G induced by the vertex set A. Since A always contains the source v_1, the game $CCFG(G[A], n, \mathcal{O})$ is well-defined and is exactly the game $CCFG(G[A], n)$.

The following proposition gives the relation between two lattices $\mathcal{F}(V)$ and $\mathcal{F}(CCFG(G, n))$.

Theorem 4.5. *Given $CCFG(G,n)$ on a DAG G which has one source. Then $CCFG(G[A],n)$ is a filter of the poset $CCFG(G,n)$ and the map $f : \mathcal{F}(V) \to \mathcal{F}(CCFG(G,n)), A \mapsto f(A) = CCFG(G[A],n)$ is order-preserving.*

Proof. Fix $A \in \mathcal{F}(V)$. We begin by proving that $CCFG(G[A],n)$ is a filter of $CCFG(G,n)$. Let $b \in CCFG(G[A],n)$ and let $a \in CCFG(G,n)$ such that $a \geq b$. We prove that $a \in CCFG(G[A],n)$. It is sufficient to prove the statement for the case $a \to b$. Assume that b is obtained from a by transferring one chip along the edge (u,v). Then $a(u) = b(u)+1, a(v) = b(v)-1$ and $a(w) = b(w)$ for all $w \neq u,v$. We have $\mathcal{O} \rightsquigarrow a \to b$. We now prove that $a \in CCFG(G[A],n)$ by induction on the length of the transition $\mathcal{O} \rightsquigarrow a$.

Basic step: Assume that a is obtained from $\mathcal{O}$ by transferring one chip along the edge (x,y). Then $\mathcal{O}(x) = a(x)+1, \mathcal{O}(y) = a(y)-1$ and $\mathcal{O}(z) = a(z)$ for all $z \neq x,y$. Since $b \in CCFG(G[A],n)$, we have $\mathcal{O}(w) = b(w), \forall w \notin A$. We only need to show that $y \in A$. If $y \notin A$ then $b(y) = \mathcal{O}(y) = a(y)-1$ and hence $y \equiv u$. Because $u \equiv y$ is not in A, then $v \notin A$. Therefore, $\mathcal{O}(v) = b(v) = a(v)+1$. It follows that $x \equiv v$ which is impossible (due to G is a DAG). Thus $y \in A$.

Inductive step: Assume that $\mathcal{O} \rightsquigarrow c \to a \to b$ and $c \in CCFG(G[A],n)$. Using the above similar argument, we have $a \in CCFG(G[A],n)$. This finishes the induction.

Considering the map

$$f : \mathcal{F}(V) \to \mathcal{F}(CCFG(G,n)), A \mapsto f(A) = CCFG(G[A],n),$$

we will prove that if $f(A) \subseteq f(B)$ then $A \subseteq B$. Given any $v \in A$, then $Pred(v) \subseteq A$. Let $a_v \in CCFG(G[A],n)$ be the configuration defined as follows:

$$a_v(u) = \begin{cases} n, & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $a_v \in CCFG(G[A],n)$, so a_v also belongs to $CCFG(G[B],n)$. This implies that $v \in B$.

It is easy to check that f is order-preserving and therefore f is an order-embedding.

□

As a particular case of CCFGs on a general DAG, we also compute the convergence time of CCFGs which has one source.

Definition 4.6. Let $G = (V,E)$ be a DAG which has a unique source v_1 and let $v \in V$. We denote by $d(v)$ the length of shortest directed paths from v_1 to v and by $l(v)$ the length of longest directed paths from v_1 to v.

Then the following results are straightforward:

Corollary 4.7. *The fixed points of $CCFG(G,n)$ are compositions of n into the set of sinks of G. Consequently, the number of fixed points of $CCFG(G,n)$ is $\binom{n+s-1}{n}$, where s is the number of sinks of G. In particular, if V has a unique sink then the $CCFG(G,n)$ is a strongly convergent game.*

Corollary 4.8. *Let $G = (V, E)$ be a DAG which has one source and let $S = \{v_{n-s+1}, \ldots, v_n\}$ be the set of sinks of V. Then, in the $CCFG(G, n)$ we have:*
(i) *The length of longest chains from the initial configuration to one fixed point is $n.max\ \{l(v_i)|i \geq n - s + 1\}$*
(ii) *The length of shortest chains from the initial configuration to one fixed point is $n.min\ \{l(v_i)|i \geq n - s + 1\}$*

Acknowledgements

Our sincere thanks to Professor Eric Goles and Pham Van Trung for useful discussions.

References

[1] A. Bjorner, L. Lovasz and W. Shor, Chip-firing game on graphs, *Eur. J. Combin.* **12** (1991), 283–291.
[2] A. Bjorner and L. Lovasz, Chip firing games on directed graphs, *J. Algebraic Combin.* **1** (1992), 305–328.
[3] B.A. Davey and H.A. Priestley., *Introduction to Lattices and Order,* (1990), Cambridge University Press.
[4] E. Goles and M. Morvan and H. D. Phan, Lattice structure and convergence of a Game of Cards, *Ann. of Combin.***6** (2002), 327-335.
[5] S.-T. Huang, Leader election in uniform rings, *ACM Trans. Program. Languages Systems* **15** (3) (1993), 563-573.
[6] Alexander E. Holroyd, Lionel Levine, Karola Meszaros, Yuval Peres, James Propp and David B. Wilson, Chip-Firing and Rotor-Routing on Directed Graphs, *In and out of equilibrium 2., Progr. Probab.* **60** (3) (2008), 331-364.
[7] M. Latapy and H. D. Phan, The Lattice structure of Chip Firing Games, *Physica D* **115** (2001), 69–82.
[8] L. Levine and Y. Peres, The rotor-router shape is spherical, *Math. Intelligencer***27** (3) (2005), 9–11.
[9] L. Levine and Y. Peres, Spherical asymptotics for the rotor-router model in $\mathbb{Z}^d$, *Indiana Univ. Math. Journal, to appear, http://arxiv.org/abs/math/0503251,* (2007).
[10] C. Magnien and H. D. Phan and L. Vuillon, Characterization of Lattices Induced by (extended) Chip Firing Games, *Discrete Math. Theoret. Comput. Sci.* **AA** (2001), 229–244.
[11] Clmence Magnien, Classes of Lattices Induced by Chip Firing (and Sandpile) Dynamics, *Eur. J. Combin.* **24** (6) (2003), 665–683.
[12] Tra An Pham, Thi Ha Duong Phan and Thi Thu Huong Tran, Conflicting Chip Firing Games on directed graphs and on tree, *VNU Journal of Science. Natural Sciences and Technology* **24** (2007), 103–109.
[13] Reinhard Diestel, *Graph Theory*, Electronic Edition, 2005.

Hue University's College of Education
E-mail address: `lemanhha@dhsphue.edu.vn`

Institute of Mathematics
18 Hoang Quoc Viet, Hanoi, Vietnam
E-mail address: `phan@math.ac.vn`