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CONVERGENCE TO COMMON FIXED POINT OF MULTI-STEP ITERATION WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, strong convergence theorems for modified multi-step iteration scheme with errors for a finite family of uniformly ϕ - continuous and asymptotically quasi-nonexpansive mappings are established in the framework of real uniformly convex Banach spaces. The results presented in this paper extend and improve the corresponding results of Khan and Fukhar-ud-din [7], Khan and Takahashi [8], Shahzad and Udomene [18], Xu and Noor [21], Cho et al. [3], Rhoades [14], Schu [15] and some others.

1. Introduction and preliminaries

Let E be a real Banach space, K be a nonempty subset of E. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $F(T) = \{x : Tx = x\}$ the set of fixed points of a mapping T. A mapping $T\colon K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\|,$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. Moreover, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [5, 8, 14, 15, 21] and references therein).

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The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - p\| \leq (1 + r_n) \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

The mapping T is said to be uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

The mapping T is said to be uniformly Holder continuous [13] if there exist positive constants L and α such that

$$\|T^n x - T^n y\| \leq L \|x - y\|^{\alpha}$$

for all $x, y \in K$ and $n \geq 1$.

The mapping T is said to be uniformly ϕ - continuous [3] if, there exists a real function ϕ: $[0, \infty) \to [0, \infty)$ with $\phi(t) \to 0$ as $t \to 0^+$ such that

$$\|T^n x - T^n y\| \leq \phi(\|x - y\|)$$

for all for all $x, y \in K$ and $n \geq 1$.

Remark 1.1. (1) It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian.

(2) If T is uniformly L-Lipschitzian, then it is uniformly Holder continuous with constants $L > 0$ and $\alpha = 1$.

(3) If T is uniformly Holder continuous, then it is uniformly ϕ-continuous, but the converse is not true.

In recent years, Mann iterative scheme [11], Ishikawa iterative scheme [6] and Noor iterative scheme [21] have been studied extensively by many authors. In 1995, Liu [9] introduced iterative schemes with errors as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T x_n + u_n, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{u_n\}$ is a sequence in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ is known as Mann iterative scheme with errors.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n + u_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n + v_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$ is known as Ishikawa iterative scheme with errors.

While it is clear that consideration of error terms in iterative scheme is an important part of the theory, it is also clear that the iterative scheme with errors introduced by Liu [9], as in (1.1), (1.2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1.1), (1.2) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. Xu [23] introduced a more satisfactory error term in the following iterative schemes.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T x_n + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in K, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $\gamma_n = 0$.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \\ y_n &= \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if $\gamma_n = \gamma'_n = 0$. Chidume and Moore [1] and Takahashi and Tamura [20] studied the above schemes, respectively.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} z_n &= \alpha''_n T x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n &= \alpha'_n T z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K, is known as Noor iterative scheme with errors. This scheme reduces to Noor iterative schemes if $\gamma_n = \gamma'_n = \gamma''_n = 0$.

Many authors starting from Das and Debata [4] and including Takahashi and Tamura [20], Khan and Takahashi [8] and Shahzad and Udomene [18] have studied the two mappings case of iterative schemes for different types of mappings.

Motivated by above all and many others, we study in this paper a modified multi-step iteration with errors for a finite family of uniformly ϕ- continuous and asymptotically quasi-nonexpansive mappings in real uniformly convex Banach spaces. The scheme is as follows:

$$\begin{aligned} x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}, \\ x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)}, \\ \ldots &= \ldots \\ \ldots &= \ldots \end{aligned} \tag{1.6}$$

$$
\begin{aligned}
x_n^{(3)} &= \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}, \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}, \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)},
\end{aligned}
$$

where $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, ..., $\{u_n^{(N)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are appropriate sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, N\}$. It is worth mentioning that our scheme can be viewed as an extension of all above schemes.

The purpose of this paper is to establish strong convergence theorems of the above iterative scheme for common fixed point of a finite family of uniformly ϕ-continuous and asymptotically quasi-nonexpansive mappings in real uniformly convex Banach spaces. Our results improve and extend the corresponding results of Khan and Fukhar-ud-din [7], Khan and Takahashi [8], Rhoades [14], Schu [15], Shahzad and Udomene [18], Xu and Noor [21], Cho et al. [3] and some others.

In the sequel we need the following lemmas and definitions to prove our main results:

Lemma 1.1. (Tan and Xu [19]). *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \le (1 + \beta_n)\alpha_n + r_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 1.2. (Xu [22]) *Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \le \lambda \|x\|^p + (1-\lambda) \|y\|^p - W_p(\lambda)g(\|x-y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \le R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

Recall that a mapping $T\colon K \to K$ where K is a subset of E, is said to satisfy *condition* (A) [17] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \ge f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Senter and Dotson [17] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [10] and Tan and Xu [19] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same Condition (A) which is weaker than the requirement that T is demicompact. We modify this condition for N mappings $T_1, T_2, \dots, T_N\colon K \to K$ as follows.

A finite family $\{T_1, T_2, \dots, T_N\}$ of N self mappings of K where K is a subset of E, is said to satisfy *condition* (B) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1 \|x - T_1x\| + a_2 \|x - T_2x\| + \dots + a_N \|x - T_Nx\| \ge f(d(x, F))$ for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F = \cap_{i=1}^N F(T_i)\}$ and $a_1, a_2, \dots, a_N$ are N nonnegative real numbers such that $a_1 + a_2 + \dots + a_N = 1$.

Remark 1.2. Condition (B) reduces to Condition (A) when $T_1 = T_2 = \cdots = T_N = T$.

2. Main results

In this section, we shall prove the strong convergence theorems of the iterative scheme (1.6) for common fixed point of a finite family of ϕ - continuous and asymptotically quasi-nonexpansive mappings in real uniformly convex Banach spaces. We first prove the following lemmas:

Lemma 2.1. *Let E be a normed space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N \colon K \to K$ be N asymptotically quasi-nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ where $r_n = \max\{r_n^{(i)} : i = 1, 2, \ldots, N\}$. Let $\{x_n\}$ be the sequence as defined in (1.6) with the restriction $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$. If $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. Since $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\ldots$, $\{u_n^{(N)}\}$ are bounded sequences in K. So we can set

$$M = \max\left\{\sup_{n\geq 1}\left\|u_n^{(i)} - p\right\| : i = 1, 2, \ldots, N\right\}.$$

It follows from (1.6) that

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\right\| \\
&\leq \alpha_n^{(1)} \left\|T_1^n x_n - p\right\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leq \alpha_n^{(1)}(1 + r_n) \|x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leq \left(\alpha_n^{(1)} + \beta_n^{(1)}\right)(1 + r_n) \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&= \left(1 - \gamma_n^{(1)}\right)(1 + r_n) \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leq (1 + r_n) \|x_n - p\| + \gamma_n^{(1)} M \\
&\leq (1 + r_n) \|x_n - p\| + t_n^{(1)} \qquad (2.1)
\end{aligned}$$

where $t_n^{(1)} = \gamma_n^{(1)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$. Now using (1.6) and (2.1), we note that

$$\begin{aligned}
\left\|x_n^{(2)} - p\right\| &\leq \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - p\right\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leq \alpha_n^{(2)}(1 + r_n) \left\|x_n^{(1)} - p\right\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leq \alpha_n^{(2)}(1 + r_n)\left[(1 + r_n) \|x_n - p\| + t_n^{(1)}\right] + \beta_n^{(2)} \|x_n - p\|
\end{aligned}$$

$$
\begin{aligned}
& +\gamma_n^{(2)}\left\|u_n^{(2)}-p\right\| \\
\leq\; & \left(\alpha_n^{(2)}+\beta_n^{(2)}\right)(1+r_n)^2\left\|x_n-p\right\|+\alpha_n^{(2)}(1+r_n)t_n^{(1)} \\
& +\gamma_n^{(2)}\left\|u_n^{(2)}-p\right\| \\
=\; & \left(1-\gamma_n^{(2)}\right)(1+r_n)^2\left\|x_n-p\right\|+\alpha_n^{(2)}(1+r_n)t_n^{(1)} \\
& +\gamma_n^{(2)}M \\
\leq\; & (1+r_n)^2\left\|x_n-p\right\|+(1+r_n)t_n^{(1)}+\gamma_n^{(2)}M \\
\leq\; & (1+r_n)^2\left\|x_n-p\right\|+t_n^{(2)} \qquad (2.2)
\end{aligned}
$$

where $t_n^{(2)}=(1+r_n)t_n^{(1)}+\gamma_n^{(2)}M$. Since $\sum_{n=1}^{\infty}\gamma_n^{(2)}<\infty$ and $\sum_{n=1}^{\infty}t_n^{(1)}<\infty$, it follows that $\sum_{n=1}^{\infty}t_n^{(2)}<\infty$. Again using (1.6) and (2.2), we note that

$$
\begin{aligned}
\left\|x_n^{(3)}-p\right\| \leq\; & \alpha_n^{(3)}\left\|T_3^n x_n^{(2)}-p\right\|+\beta_n^{(3)}\left\|x_n-p\right\|+\gamma_n^{(3)}\left\|u_n^{(3)}-p\right\| \\
\leq\; & \alpha_n^{(3)}(1+r_n)\left\|x_n^{(2)}-p\right\|+\beta_n^{(3)}\left\|x_n-p\right\|+\gamma_n^{(3)}\left\|u_n^{(3)}-p\right\| \\
\leq\; & \alpha_n^{(3)}k_n\left[(1+r_n)^2\left\|x_n-p\right\|+t_n^{(2)}\right]+\beta_n^{(3)}\left\|x_n-p\right\| \\
& +\gamma_n^{(3)}\left\|u_n^{(3)}-p\right\| \\
\leq\; & \left(\alpha_n^{(3)}+\beta_n^{(3)}\right)(1+r_n)^3\left\|x_n-p\right\|+\alpha_n^{(3)}(1+r_n)t_n^{(2)} \\
& +\gamma_n^{(3)}\left\|u_n^{(3)}-p\right\| \\
=\; & \left(1-\gamma_n^{(3)}\right)(1+r_n)^3\left\|x_n-p\right\|+\alpha_n^{(3)}(1+r_n)t_n^{(2)}+\gamma_n^{(3)}M \\
\leq\; & (1+r_n)^3\left\|x_n-p\right\|+(1+r_n)t_n^{(2)}+\gamma_n^{(3)}M \\
\leq\; & (1+r_n)^3\left\|x_n-p\right\|+t_n^{(3)} \qquad (2.3)
\end{aligned}
$$

where $t_n^{(3)}=(1+r_n)t_n^{(2)}+\gamma_n^{(3)}M$. Since $\sum_{n=1}^{\infty}\gamma_n^{(3)}<\infty$ and $\sum_{n=1}^{\infty}t_n^{(2)}<\infty$, it follows that $\sum_{n=1}^{\infty}t_n^{(3)}<\infty$. Continuing the above process, we get

$$
\begin{aligned}
\left\|x_{n+1}-p\right\| =\; & \left\|x_n^{(N)}-p\right\| \\
\leq\; & (1+r_n)^N\left\|x_n-p\right\|+t_n^{(N)} \qquad (2.4)
\end{aligned}
$$

where $\{t^{(N)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty}t_n^{(N)}<\infty$. Since $\sum_{n=1}^{\infty}r_n<\infty$ and $\sum_{n=1}^{\infty}t_n^{(N)}<\infty$, therefore from Lemma 1.1, we know that $\lim_{n\to\infty}\left\|x_n-p\right\|$ exists. This completes the proof. □

Lemma 2.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N \colon K \to K$ be N asymptotically quasi-nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ where $r_n = \max\{r_n^{(i)} : i = 1, 2, \ldots, N\}$. Let $T_1, T_2, \ldots, T_N$ also be uniformly ϕ - continuous. Let $\{x_n\}$ be the sequence as defined in (1.6) with $\sum\limits_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $1 \le i \le N$ and $\{\alpha_n^{(i)}\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ and for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then $\lim\limits_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $1 \le i \le N$.*

Proof. Let $p \in F = \cap_{i=1}^{N} F(T_i)$, it follows from Lemma 2.1 that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists. Let $\lim\limits_{n\to\infty} \|x_n - p\| = a$ for some $a \ge 0$. If $a = 0$, there is nothing to prove. Assume that $a > 0$. Firstly, we are going to show that $\lim\limits_{n\to\infty} \|T_N^n x_n - x_n\| = 0$. Since $\{x_n\}$ and $\{u_n^{(i)}\}$ are bounded for all $i = 1, 2, \ldots, N$, there exists $R > 0$ such that $x_n - p + \gamma_n^{(i)}(u_n^{(i)} - x_n), T_i^n x_n^{(i-1)} - p + \gamma_n^{(i)}(u_n^{(i)} - x_n) \in B_R(0)$ for all $n \ge 1$ and for all $i = 1, 2, \ldots, N$. Using Lemma 1.2, we have

$$\begin{aligned}
\left\|x_n^{(N)} - p\right\|^2 =& \|\alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} - p\|^2 \\
=& \|\alpha_n^{(N)} (T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)}(u_n^{(N)} - x_n)) \\
&+ (1-\alpha_n^{(N)})(x_n - p + \gamma_n^{(N)}(u_n^{(N)} - x_n))\|^2 \\
\le& \alpha_n^{(N)} \|T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)}(u_n^{(N)} - x_n)\|^2 \\
&+ (1-\alpha_n^{(N)})\|x_n - p + \gamma_n^{(N)}(u_n^{(N)} - x_n)\|^2 \\
&- W_2(\alpha_n^{(N)}) g\Big(\|T_N^n x_n^{(N-1)} - x_n\|\Big) \\
\le& \alpha_n^{(N)} \Big[\|T_N^n x_n^{(N-1)} - p\| + \gamma_n^{(N)} \|u_n^{(N)} - x_n\|\Big]^2 \\
&+ (1-\alpha_n^{(N)})\Big[\|x_n - p\| + \gamma_n^{(N)}\|u_n^{(N)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N)}) g\Big(\|T_N^n x_n^{(N-1)} - x_n\|\Big) \\
\le& \alpha_n^{(N)} \Big[(1+r_n)\|x_n^{(N-1)} - p\| + \gamma_n^{(N)} \|u_n^{(N)} - x_n\|\Big]^2 \\
&+ (1-\alpha_n^{(N)})\Big[\|x_n - p\| + \gamma_n^{(N)}\|u_n^{(N)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N)}) g\Big(\|T_N^n x_n^{(N-1)} - x_n\|\Big) \\
\le& \alpha_n^{(N)} \Big[(1+r_n)((1+r_n)^{N-1}\|x_n - p\| + t_n^{(N-1)}) \\
&+ \gamma_n^{(N)} \|u_n^{(N)} - x_n\|\Big]^2 + (1-\alpha_n^{(N)})\Big[\|x_n - p\| + \gamma_n^{(N)}\|u_n^{(N)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N)}) g\Big(\|T_N^n x_n^{(N-1)} - x_n\|\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n^{(N)}\Big[(1+r_n)^N\|x_n-p\|+(1+r_n)t_n^{(N-1)}+\gamma_n^{(N)}\|u_n^{(N)}-x_n\|\Big]^2\\
&\quad+(1-\alpha_n^{(N)})\Big[(1+r_n)^N\|x_n-p\|+(1+r_n)t_n^{(N-1)}\\
&\quad+\gamma_n^{(N)}\|u_n^{(N)}-x_n\|\Big]^2-W_2(\alpha_n^{(N)})g\Big(\|T_N^n x_n^{(N-1)}-x_n\|\Big)\\
&\le\Big[(1+r_n)^N\|x_n-p\|+(1+r_n)t_n^{(N-1)}+\gamma_n^{(N)}\|u_n^{(N)}-x_n\|\Big]^2\\
&\quad-W_2(\alpha_n^{(N)})g\Big(\|T_N^n x_n^{(N-1)}-x_n\|\Big)\\
&\le\Big[\|x_n-p\|+\theta_n^{(N-1)}\Big]^2-W_2(\alpha_n^{(N)})g\Big(\|T_N^n x_n^{(N-1)}-x_n\|\Big) \qquad (2.5)
\end{aligned}
$$

where $\theta_n^{(N-1)}=(1+r_n)t_n^{(N-1)}+\gamma_n^{(N)}\|u_n^{(N)}-x_n\|$. Observe that $\varepsilon^3\le W_2(\alpha_n^{(N)})$ now (2.5) implies that

$$\varepsilon^3 g\Big(\Big\|T_N^n x_n^{(N-1)}-x_n\Big\|\Big)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\delta_n^{(N-1)},$$

where $\delta_n^{(N-1)}:=2\theta_n^{(N-1)}\|x_n-p\|+(\theta_n^{(N-1)})^2$. Since $\sum_{n=1}^{\infty}t_n^{(N-1)}<\infty$ and $\sum_{n=1}^{\infty}\theta_n^{(N-1)}<\infty$, we get $\sum_{n=1}^{\infty}\delta_n^{(N-1)}<\infty$. This implies that

$$\lim_{n\to\infty}g\Big(\Big\|T_N^n x_n^{(N-1)}-x_n\Big\|\Big)=0.$$

Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty}\Big\|T_N^n x_n^{(N-1)}-x_n\Big\|=0.$$

Again note that

$$
\begin{aligned}
\|x_n-p\| &\le \Big\|x_n-T_N^n x_n^{(N-1)}\Big\|+\Big\|T_N^n x_n^{(N-1)}-p\Big\|\\
&\le \Big\|x_n-T_N^n x_n^{(N-1)}\Big\|+(1+r_n)\Big\|x_n^{(N-1)}-p\Big\|, \qquad (2.6)
\end{aligned}
$$

for all $n\ge 1$. Thus

$$a=\lim_{n\to\infty}\|x_n-p\|\le\liminf_{n\to\infty}\Big\|x_n^{(N-1)}-p\Big\|\le\limsup_{n\to\infty}\Big\|x_n^{(N-1)}-p\Big\|\le a,$$

and therefore $\lim_{n\to\infty}\Big\|x_n^{(N-1)}-p\Big\|=a$. Using the same argument as in the proof above, we have

$$
\begin{aligned}
\Big\|x_n^{(N-1)}-p\Big\|^2 &\le\alpha_n^{(N-1)}\|T_{N-1}^n x_n^{(N-2)}-p+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\|^2\\
&\quad+(1-\alpha_n^{(N-1)})\|x_n-p+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\|^2\\
&\quad-W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)}-x_n\|\Big)\\
&\le\alpha_n^{(N-1)}\Big[\|T_{N-1}^n x_n^{(N-2)}-p\|+\gamma_n^{(N-1)}\|u_n^{(N-1)}-x_n\|\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\alpha_n^{(N-1)})\Big[\|x_n - p\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big) \\
\le&\alpha_n^{(N-1)}\Big[(1+r_n)\|x_n^{(N-2)} - p\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&+ (1-\alpha_n^{(N-1)})\Big[\|x_n - p\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big) \\
\le&\alpha_n^{(N-1)}\Big[(1+r_n)((1+r_n)^{N-2}\|x_n - p\| \\
&+ t_n^{(N-2)}) + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&+ (1-\alpha_n^{(N-1)})\Big[\|x_n - p\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big) \\
\le&\alpha_n^{(N-1)}\Big[(1+r_n)^{N-1}\|x_n - p\| + (1+r_n)t_n^{(N-2)} \\
&+ \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 + (1-\alpha_n^{(N-1)})\Big[(1+r_n)^{N-1}\|x_n - p\| \\
&+ (1+r_n)t_n^{(N-2)} + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big) \\
\le&\Big[(1+r_n)^{N-1}\|x_n - p\| + (1+r_n)t_n^{(N-2)} + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\Big]^2 \\
&- W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big)
\end{aligned}
$$

$$
\le\Big[\|x_n - p\| + \theta_n^{(N-2)}\Big]^2 - W_2(\alpha_n^{(N-1)})g\Big(\|T_{N-1}^n x_n^{(N-2)} - x_n\|\Big) \tag{2.7}
$$

where $\theta_n^{(N-2)} = (1+r_n)t_n^{(N-2)} + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|$. This implies that

$$
\varepsilon^3 g\Big(\Big\|T_{N-1}^n x_n^{(N-2)} - x_n\Big\|\Big) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n^{(N-2)},
$$

where $\delta_n^{(N-2)} := 2\theta_n^{(N-2)}\,\|x_n - p\| + (\theta_n^{(N-2)})^2$.
Since $\sum_{n=1}^{\infty} t_n^{(N-2)} < \infty$ and $\sum_{n=1}^{\infty} \theta_n^{(N-2)} < \infty$, we get $\sum_{n=1}^{\infty} \delta_n^{(N-2)} < \infty$. This implies that $\lim_{n\to\infty} g\Big(\Big\|T_{N-1}^n x_n^{(N-2)} - x_n\Big\|\Big) = 0$. Since g is strictly increasing and continuous at 0, it follows that $\lim_{n\to\infty}\Big\|T_{N-1}^n x_n^{(N-2)} - x_n\Big\| = 0$.

Now, observe that

$$
\Big\|x_n^{(N-1)} - x_n\Big\| \;=\; \Big\|\alpha_n^{(N-1)}T_{N-1}x_n^{(N-2)} + \beta_n^{(N-1)}x_n + \gamma_n^{(N-1)}u_n^{(N-1)} - x_n\Big\|
$$

$$\leq \alpha_n^{(N-1)} \left\| T_{N-1} x_n^{(N-2)} - x_n \right\| + \gamma_n^{(N-1)} \left\| u_n^{(N-1)} - x_n \right\|$$

$$\to 0 \text{ as } n \to \infty, \tag{2.8}$$

and hence, $\phi\Big(\left\| x_n^{(N-1)} - x_n \right\| \Big) \to 0$ as $n \to \infty$. Again observe that

$$\begin{aligned} \|x_n - T_N^n x_n\| &\leq \left\| x_n - T_N^n x_n^{(N-1)} \right\| + \left\| T_N^n x_n^{(N-1)} - T_N^n x_n \right\| \\ &\leq \left\| x_n - T_N^n x_n^{(N-1)} \right\| + \phi\Big(\left\| x_n^{(N-1)} - x_n \right\| \Big) \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \|x_{n+1} - x_n\| &= \left\| \alpha_n^{(N)} T_N x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} - x_n \right\| \\ &\leq \alpha_n^{(N)} \left\| T_N x_n^{(N-1)} - x_n \right\| + \gamma_n^{(N)} \left\| u_n^{(N)} - x_n \right\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.10}$$

and hence, $\phi\big(\|x_{n+1} - x_n\| \big) \to 0$ as $n \to \infty$. Now, we see that

$$\begin{aligned} \|T_N^n x_{n+1} - x_{n+1}\| &\leq \|T_N^n x_{n+1} - T_N^n x_n\| + \|T_N^n x_n - x_n\| + \|x_n - x_{n+1}\| \\ &\leq \phi\big(\|x_{n+1} - x_n\| \big) + \|T_N^n x_n - x_n\| + \|x_n - x_{n+1}\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.11}$$

and hence, $\phi\big(\|T_N^n x_{n+1} - x_{n+1}\| \big) \to 0$ as $n \to \infty$. Therefore, it follows from (2.9) and (2.11) that

$$\begin{aligned} \|x_{n+1} - T_N x_{n+1}\| &\leq \left\| x_{n+1} - T_N^{n+1} x_{n+1} \right\| + \left\| T_N^{n+1} x_{n+1} - T_N x_{n+1} \right\| \\ &\leq \left\| x_{n+1} - T_N^{n+1} x_{n+1} \right\| + \phi\big(\|T_N^n x_{n+1} - x_{n+1}\| \big) \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.12}$$

which implies $\lim_{n\to\infty} \|x_n - T_N x_n\| = 0$. Similarly, by using the same argument as in the proof above, we have

$$\begin{aligned} \lim_{n\to\infty} \left\| x_n - T_{N-2}^n x_n^{(N-3)} \right\| &= \lim_{n\to\infty} \left\| x_n - T_{N-3}^n x_n^{(N-4)} \right\| \\ &= \cdots = \lim_{n\to\infty} \left\| x_n - T_2^n x_n^{(1)} \right\| = 0 \end{aligned}$$

and

$$\lim_{n\to\infty} \left\| x_n - T_{N-1}^n x_n \right\| = \lim_{n\to\infty} \left\| x_n - T_{N-2}^n x_n \right\| = \cdots = \lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0.$$

This implies that

$$\lim_{n\to\infty} \|x_n - T_{N-1} x_n\| = \lim_{n\to\infty} \|x_n - T_{N-2} x_n\| = \cdots = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0.$$

It remains to show that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.

Note that

$$\|x_n^{(1)} - p\|^2 \leq \alpha_n^{(1)} \Big[\|T_1^n x_n - p\| + \gamma_n^{(1)} \left\| u_n^{(1)} - p \right\| \Big]^2$$

$$
\begin{aligned}
&+ (1-\alpha_n^{(1)})\Big[\, \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&- W_2(\alpha_n^{(1)})g(\|T_1^n x_n - x_n\|) \\
\le&\alpha_n^{(1)}\Big[(1+r_n)\,\|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&+ (1-\alpha_n^{(1)})\Big[\, \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&- W_2(\alpha_n^{(1)})g(\|T_1^n x_n - x_n\|) \\
\le&\alpha_n^{(1)}\Big[(1+r_n)\,\|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&+ (1-\alpha_n^{(1)})\Big[(1+r_n)\,\|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&- W_2(\alpha_n^{(1)})g(\|T_1^n x_n - x_n\|) \\
\le&\Big[(1+r_n)\,\|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&- W_2(\alpha_n^{(1)})g(\|T_1^n x_n - x_n\|) \\
\le&\Big[\, \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 \\
&- W_2(\alpha_n^{(1)})g(\|T_1^n x_n - x_n\|). \qquad (2.13)
\end{aligned}
$$

Thus, we have $\varepsilon^3 g(\|T_1^n x_n - x_n\|) \le \Big[\, \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \Big]^2 - \left\|x_n^{(1)} - p\right\|^2$ and therefore, $\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0$.

Since

$$
\begin{aligned}
\|x_n - T_2^n x_n\| &\le \left\|x_n - T_2^n x_n^{(1)}\right\| + \left\|T_2^n x_n^{(1)} - T_2^n x_n\right\| \\
&\le \left\|x_n - T_2^n x_n^{(1)}\right\| + (1+r_n)\left\|x_n^{(1)} - x_n\right\| \\
&\le \left\|x_n - T_2^n x_n^{(1)}\right\| + (1+r_n)\Big[\alpha_n^{(1)}\,\|T_1^n x_n - x_n\| \\
&\quad +\gamma_n^{(1)}\left\|u_n^{(1)} - x_n\right\|\Big] \\
&\to 0 \text{ as } n \to \infty, \qquad (2.14)
\end{aligned}
$$

this implies that $\lim_{n\to\infty} \|T_2^n x_n - x_n\| = 0$. Thus, we have

$$
\begin{aligned}
\|x_n - T_2 x_n\| &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_2^{n+1} x_{n+1}\right\| \\
&\quad + \left\|T_2^{n+1} x_{n+1} - T_2^{n+1} x_n\right\| + \left\|T_2^{n+1} x_n - T_2 x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_2^{n+1} x_{n+1}\right\| \\
&\quad +\phi(\|x_{n+1} - x_n\|) + \phi(\|T_2^n x_n - x_n\|) \\
&\to 0 \text{ as } n \to \infty. \qquad (2.15)
\end{aligned}
$$

This implies that $\lim_{n\to\infty} \|T_2x_n - x_n\| = 0$. Similarly, we can show that

$$\lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$$

Hence, $\lim_{n\to\infty} \|T_ix_n - x_n\| = 0$ for all $i = 1, 2, \dots, N$. This completes the proof. □

Theorem 2.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N \colon K \to K$ be N asymptotically quasi-nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ where $r_n = \max\{r_n^{(i)} : i = 1, 2, \dots, N\}$. Let $T_1, T_2, \dots, T_N$ also be uniformly ϕ - continuous. Let $\{x_n\}$ be the sequence as defined in* (1.6) *with $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $1 \le i \le N$ and $\{\alpha_n^{(i)}\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \dots, N$ and for some $\varepsilon \in (0, 1)$. If $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N\}$.*

Proof. By Lemma 2.1, we see that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \ge 0$. If $a = 0$, there is nothing to prove. Assume that $a > 0$, as proved in Lemma 2.1, we have

$$\|x_{n+1} - p\| = \left\|x_n^{(N)} - p\right\| \le (1 + r_n)^N \|x_n - p\| + t_n^{(N)}, \quad \forall n \ge 1,$$

where $t_n^{(N)} = (1 + r_n)t_n^{(N-1)} + \gamma_n^{(N)} M$ such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$. This gives that

$$d(x_{n+1}, F) \le (1 + r_n)^N d(x_n, F) + t_n^{(N)}, \quad \forall n \ge 1,$$

since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. Also by Lemma 2.2, $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$ for all $i = 1, 2, \dots, N$. Since $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, given any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\varepsilon}{3}$ for all $n \ge n_0$. So, we can find $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. For all $n \ge n_0$ and $m \ge 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\le \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\le \|x_{n_0} - p^*\| + \|x_{n_0} - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_1 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \ge n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a natural number $n_2 \ge n_1$ such that for all $n \ge n_2$ we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular we have

$d(x_{n_2}, F) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$. For any $i \in I$ and $n \geq n_2$, we have

$$\begin{aligned} \|T_i q^* - q^*\| &\leq \|T_i q^* - w^*\| + \|w^* - q^*\| \\ &\leq 2\|q^* - w^*\| \\ &\leq 2\Big[\|q^* - x_{n_2}\| + \|x_{n_2} - w^*\|\Big] \\ &< 2\Big[\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big] \\ &< \varepsilon_1. \end{aligned}$$

This implies that $T_i q^* = q^*$. Hence, $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in F = \cap_{i=1}^N F(T_i)$. Thus, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. □

For our next result, we shall need the following definition:

Definition 2.2. Let K be a nonempty closed subset of a Banach space E. A mapping $T\colon K \to K$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n\to\infty} x_{n_j} = x \in K$.

Theorem 2.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N\colon K \to K$ be N asymptotically quasi-nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^\infty r_n < \infty$ where $r_n = \max\{r_n^{(i)} : i = 1, 2, \ldots, N\}$. Let $T_1, T_2, \ldots, T_N$ also be uniformly ϕ -continuous. Let $\{x_n\}$ be the sequence as defined in (1.6) with $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $1 \leq i \leq N$ and $\{\alpha_n^{(i)}\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ and for some $\varepsilon \in (0,1)$. If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.*

Proof. Suppose T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0. \tag{2.16}$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim\limits_{j\to\infty} x_{n_j} = x^* \in K$. Now Lemma 2.2 guarantees that $\lim\limits_{n_j\to\infty} \left\|x_{n_j} - T_i x_{n_j}\right\| = 0$ for all $i = 1, 2, \ldots, N$ and so $\|x^* - T_i x^*\| = 0$ for all $i = 1, 2, \ldots, N$. This implies that $x^* \in F = \cap_{i=1}^N F(T_i)$. Since $\lim\limits_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 2.1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. □

Remark 2.1. Theorem 2.1 extends Theorems 2 and 3 of Rhoades [14], Theorem 1.5 of Schu [15], corresponding result of Khan and Fukhar-ud-din [7] and Khan and Takahashi [8] to the case of finite family of more general class of nonexpansive and asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.2. Theorem 2.1 also extends the corresponding results of Xu and Noor [21] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.3. Theorem 2.1 also extends the corresponding result of Shahzad and Udomene [18] to the case of finite family of uniformly ϕ - continuous asymptotically quasi-nonexpansive mappings and multi-step iteration scheme with errors considered here.

Remark 2.4. Theorem 2.1 also extends the corresponding result of Cho et al. [3] to the case of finite family of mappings and multi-step iteration scheme with errors considered here.

Remark 2.5. Theorem 2.3 extends Theorem 2 of Osilike and Aniagbosor [12] and Theorem 2.2 of Schu [16] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.6. Theorem 2.1 also extends the corresponding result of Cho et al. [2] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here.

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