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PARAMETRIZED SADDLE POINTS

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Abstract. In this paper, one considers the vector saddle point problem with respect to a cone which depends on two parameters. Necessary and sufficient conditions for a point to be a solution of the parameterized cone saddle point problem are given. Then the parameterized cone saddle point problem is related to the parameterized cone vector variational inequality problem. One shows that, under some hypotheses, the problems of parameterized cone vector saddle point and of parameterized cone vector variational inequality have the same solution set. Also, an existence result for a parameterized cone vector saddle point problem to have a solution is given.

1. Introduction

Let A and B be nonempty sets and $F : A \times B \to \mathbf{R}$ be a function. We remember that a point $(a, b) \in A \times B$ is said to be a saddle point of F on $A \times B$ if

$$F(a, y) \leq F(a, b) \leq F(x, b), \quad \text{for all}\,(x, y) \in A \times B. \tag{1.1}$$

The condition (1.1) is equivalent to

$$\max_{y\in B} \min_{x\in A} F(x, y) \;=\; \min_{x\in A} \max_{y\in B} F(x, y). \tag{1.2}$$

Let us consider a two-person zero-sum game G_F generated by the function F. This means that the first player selects a point x from A and the second player selects a point y from B. As a result of this choice, the first player pays the second one the amount $F\,(x, y)$. Then a point $(a, b) \in A \times B$ is a solution of the game G_F if and only if it is a saddle point of F on $A \times B$.

The first saddle point theorem was proved by von Neumann [20]. Von Neumann's theorem can be stated as follows: if A and B are finite dimensional simplices and F is a bilinear function on $A \times B$, then F has a saddle point; i.e. (1.2) holds. M. Shiffman [25] seems to have been the first to have considered convex-concave functions in a saddle point theorem. H. Kneser [19], K. Fan [10], and C. Berge [2] (using induction and the method of separating two disjoint convex sets in an Euclidean space by a hyperplane) got saddle point theorems for

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convex-concave functions that are appropriately semicontinuous in one of the two variables. H. Nikaido [22], on the other hand, using Brouwer's fixed point theorem, proved the existence of a saddle point for a function satisfying the weaker algebraic condition of being quasi-convex-concave, but the stronger topological condition of being continuous in each variable. M. Sion [24] proved a very general saddle point theorem for a function which is quasi-convex and lower semicontinuous in its first variabile and quasi-concave and upper semicontinuous in its second variable in a topological vector space.

Most of the efforts have been spent on relaxing the assumptions on the convex-concavity of F and also on the compactness condition for one of the sets A and B. As examples we can give the papers of K. Fan [11], H. Tuy [29], [30], [31], J. Hartung [16], U. Passy and E. Z. Prisman [23], G. H. Greco and C. D. Horvath [15], S. Simons [26], J. Yu and X.Z. Yuan [33], D.I. Duca and L. Lupşa [7], [9], etc. A little less study was dedicated to the case when the function F is defined on a proper subset M of $A \times B$ (see, for example [8]).

Studies on saddle points of scalar functions have been extended to studies of saddle points, with respect to a cone, of vector valued functions; see, for example: [1], [12], [18], [21], [27], [28]. Necessary and sufficient conditions for cone saddle points have been given in more papers; see, for example, [12], [21], [27]. Existence results for cone saddle points are based on some fixed point theorems or scalar minimax theorems; see, for example [28].

Recently, these problems are solved by a different approach; they are reduced to vector variational inequality problems. The concept of vector variational inequality (VVI) was introduced by F. Giannessi in 1980 [13].

Recently, VVIs have been studied intensively because they can be efficient tools for investigating vector optimization problems and also because they provide a mathematical model for the problem of equilibrium in a mechanical structure when there are several conflicting criteria under consideration, such as weight, cost, resistance, etc.

In [14], some relationships have been spelled out between a solution of a Minty VVI and an efficient solution or a weakly efficient solution of a VOP. Convexity and monotonicity assumptions are used in these results.

In [17], the reduction of the vector saddle point problem to a vector variational inequality problem is treated in a finite dimensional vector space. In [18], one considers its generalization to a vector problem involving the concept of moving cone in the general setting of a normed space; the moving cone depends on one parameter.

In this paper, one considers the vector saddle point problem with respect to a cone which depends on two parameters. Necessary and sufficient conditions for a point to be a solution of the parameterized cone saddle point problem are given. Then the parameterized cone saddle point problem is related to the parameterized cone vector variational inequality problem. One shows that, under some hypotheses, the problems of parameterized cone vector saddle point and of parameterized cone vector variational inequality have the same solution set. Also, an existence result for a parameterized cone vector saddle point problem to have a solution is given.

The paper is outlined as follows. In Section 2, one formulates the parameterized vector saddle point problem and parameterized vector variational inequality problem for vector valued functions. In Section 3, some definitions and preliminary results are given. Section 4 contains the main results: Necessary and sufficient conditions for a point to be a solution of the parameterized cone saddle point problem are given. Then, one shows that, under some hypotheses, the problems of parameterized cone vector saddle point and of parameterized cone vector variational inequality have the same solution set. The paper ends with an existence result for a parameterized cone vector saddle point problem.

2. Problem formulation

Definition 2.1. Let X and Z be two normed spaces, Y be a topological linear space, and $C \subseteq Z$ be a pointed (i.e. $C \cap (-C) = \{0\}$) convex cone in Z, with nonempty interior (i.e. C is solid).

Let A and B be two nonempty subsets of X and Y respectively, and $F : A \times B \to Z$ be a function. We say that the point $\left(x^0, y^0\right) \in A \times B$ is a weak $C-$saddle point of F on $A \times B$ if there is no $(x, y) \in A \times B$ such that

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \in \mathrm{int}C,$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \in \mathrm{int}C.$$

Obviously, the point $\left(x^0, y^0\right) \in A \times B$ is a weak $C-$saddle point of F on $A \times B$ if and only if

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \mathrm{int}C, \text{ for all } x \in A,$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \mathrm{int}C, \text{ for all } y \in B.$$

Given two nonempty subsets A and B of X and Y, respectively, and a vector-valued function $F : A \times B \to Z$, the Vector Saddle Point Problem, $(VSPP)$ for short, is to find $x^0 \in A$ and $y^0 \in B$ such that $\left(x^0, y^0\right)$ is a weak $C-$saddle point of F on $A \times B$.

On the other hand, given two nonempty subsets A and B of X and Y, respectively, and a vector valued function $F : A \times B \to Z$, the Vector Variational Inequality Problem, $(VVIP)$ for short, is to find $x^0 \in A$ and $y^0 \in T\left(x^0\right)$ such that

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0 \right\rangle \notin -\mathrm{int}C, \text{ for all } x \in A,$$

where $T : A \to 2^Y$ is the multifunction defined by

$$T(x) = \{y \in B : F(x, v) - F(x, y) \notin \mathrm{int}C, \text{ for all } v \in B\}, \text{ for all } x \in A,$$

and $\bigtriangledown_x F\left(x^0, y^0\right)$ denotes the Fréchet derivative of F with respect to the first argument x at $\left(x^0, y^0\right)$.

We extend problems $(VSPP)$ and $(VVIP)$ by considering a moving cone which depends on two parameters. To begin with, we introduce some parameterized concepts for extension.

Definition 2.2. Let X and Y be two nonempty sets, Z be a topological space, A and B be two nonempty subsets of X and Y respectively, $\left(x^0 y^0\right) \in A \times B$, $F : A \times B \to Z$ be a function, and $C : A \times B \to 2^Z$ be a multifunction. We say that the point $\left(x^0, y^0\right) \in A \times B$ is a parameterized weak $C-$saddle point of F on $A \times B$ if there is no $(x, y) \in A \times B$ such that

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \in \text{int}C\left(x, y^0\right), \tag{2.1}$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \in \text{int}C\left(x^0, y\right). \tag{2.2}$$

Obviously, the point $\left(x^0, y^0\right) \in A \times B$ is a parameterized weak $C-$saddle point of F on $A \times B$ if and only if

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \text{int}C\left(x, y^0\right), \text{ for all } x \in A, \tag{2.3}$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \text{int}C\left(x^0, y\right), \text{ for all } y \in B. \tag{2.4}$$

Given two nonempty subsets A and B of X and Y, respectively, $C : A \times B \to 2^Z$ a multifunction, and a vector-valued function $F : A \times B \to Z$, the Parameterized Vector Saddle Point Problem, $(PVSPP)$ for short, is to find $x^0 \in A$ and $y^0 \in B$ such that $\left(x^0, y^0\right)$ is a parameterized weak $C-$saddle point of F on $A \times B$.

On the other hand, given a nonempty subset A of the normed space X, a nonempty subset B of Y, Z a normed space, $C : A \times B \to 2^Z$ a multifunction, and a vector valued function $F : A \times B \to Z$, Fréchet differentiable with respect to the first argument x at $\left(x^0, y^0\right)$, the Parameterized Vector Variational Inequality Problem, $(PVVIP)$ for short, is to find $x^0 \in A$ and $y^0 \in T\left(x^0\right)$ such that

$$\left\langle \bigtriangledown_x F\left(x^0, y^0\right), x - x^0 \right\rangle \notin -\text{int}C\left(x, y^0\right), \text{ for all } x \in A, \tag{2.5}$$

where $T : A \to 2^Y$ is the multifunction defined by

$$T(x) = \{y \in B : F(x, v) - F(x, y) \notin \text{int}C(x, v), \text{ for all } v \in B\}, \text{ for all } x \in A,$$

and $\bigtriangledown_x F\left(x^0, y^0\right)$ denotes the Fréchet derivative of F with respect to the first argument x at $\left(x^0, y^0\right)$.

3. Definitions and preliminary results

Definition 3.1. Let X be a linear space, A be a nonempty subset of X and $x^0 \in A$.

We say that A is convex at x^0 if

$$(1-t)\,x^0 + tx \in A, \text{ for all } x \in A \text{ and all } t \in [0,1].$$

We say that A is convex if A is convex at each $x^0 \in A$, that is

$$(1-t)\,x^0 + tx \in A, \text{ for all } x, x^0 \in A \text{ and all } t \in [0,1].$$

Definition 3.2. Let X and Z be two linear spaces, A be a nonemty subset of X, $x^0 \in A$, $f : A \to Z$ be a function and $C : A \to 2^Z$ be a multifunction.

We say that f is convex at x^0 with respect to C if A is convex at x^0 and

$$(1-t)\,f\left(x^0\right) + tf\,(x) - f\left((1-t)\,x^0 + tx\right) \in C\left((1-t)\,x^0 + tx\right),$$

for all $x \in A$ and all $t \in [0,1]$.

We say that f is concave at x^0 with respect to C if $(-f)$ is convex at x^0 with respect to C.

We say that f is convex (respectively concave) on A with respect to C if A is convex and f is convex (respectively concave) at each $x \in A$ with respect to C, that is

$$(1-t)\,f\left(x^0\right) + tf\,(x) - f\left((1-t)\,x^0 + tx\right) \in C\left((1-t)\,x^0 + tx\right),$$

for all $x, x^0 \in A$ and all $t \in [0,1]$.

Definition 3.3. Let X, Y, and Z be three linear spaces, A be a nonempty subset of X, B be a nonempty subset of Y, $\left(x^0, y^0\right) \in A \times B$, $C : A \times B \to 2^Z$ be a multifunction and $F : A \times B \to Z$ be a function. We say that the function F is convex-concave at $\left(x^0, y^0\right)$ with respect to C if A is convex at x^0, B is convex at y^0 and if $F : \left(\cdot, y^0\right) : A \to Z$ is convex at x^0 with respect to $C\left(\cdot, y^0\right)$ and $F\left(x^0, \cdot\right) : B \to Z$ is concave at y^0 with respect to $C\left(x^0, \cdot\right)$.

Definition 3.4. Let X and Z be two topological spaces, and A be a nonempty subset of X. A multifunction $T : A \to 2^Z$ is said to be closed if for each sequence $\left(x^n\right)_{n\in\mathbb{N}}$ of elements from A, convergent to an element x from A, and for each sequence $\left(z^n\right)_{n\in\mathbb{N}}$ of elements $z^n \in T\left(x^n\right)$, $(n \in \mathbb{N})$, convergent to an element z from Z, we have $z \in T\,(x)$.

4. Main results

Theorem 4.1. *Let X, Y and Z be three normed spaces, A be a nonempty subset of X, B be a nonempty subset of Y, $\left(x^0, y^0\right) \in A \times B$, $F : A \times B \to Z$ be a function, and $C : A \times B \to 2^Z$ be a multifunction such that:*

i) *the set A is convex at x^0, and the set B is convex at y^0;*

ii) *the multifunctions $C\left(., y^0\right) : A \to 2^Z$, and $C\left(x^0, .\right) : B \to 2^Z$ are closed;*

iii) *for each $(x, y) \in A \times B$ and $t \in [0,1]$, the sets $C\left(x^0, (1-t)\,y^0 + ty\right)$ and $C\left((1-t)\,x^0 + ty, y^0\right)$ are solid pointed closed convex cones;*

iv) *the function F is Fréchet differentiable in each of the arguments x and y at $\left(x^0, y^0\right)$.*

If the point $\left(x^0,y^0\right)\in A\times B$ *is a parameterized weak* C*-saddle point of the function* F *on* $A\times B$*, then*

$$\left\langle \nabla_x F\left(x^0,y^0\right), x-x^0\right\rangle \notin -\mathrm{int}C\left(x^0,y^0\right), \text{ for all } x\in A \tag{4.1}$$

and

$$\left\langle \nabla_y F\left(x^0,y^0\right), y-y^0\right\rangle \notin \mathrm{int}C\left(x^0,y^0\right), \text{ for all } y\in B. \tag{4.2}$$

Proof. Assume that $\left(x^0,y^0\right)\in A\times B$ is a parameterized weak C-saddle point of F on $A\times B$. Then

$$F\left(x^0,y^0\right)-F\left(x,y^0\right)\notin \mathrm{int}C\left(x,y^0\right), \tag{4.3}$$

for all $x\in A$, and

$$F\left(x^0,y\right)-F\left(x^0,y^0\right)\notin \mathrm{int}C\left(x^0,y\right), \tag{4.4}$$

for all $y\in B$.

In order to prove that (4.1) and (4.2) hold, let $x\in A$ and $y\in B$. Since A is convex at x^0 and B is convex at y^0, we have

$$(1-t)\,x^0+tx\in A, \text{ for all } t\in[0,1],$$

and

$$(1-t)\,y^0+ty\in B, \text{ for all } t\in[0,1].$$

Conditions (4.3) and (4.4) imply

$$F\left(x^0,y^0\right)-F\left((1-t)\,x^0+tx,y^0\right)\notin \mathrm{int}C\left((1-t)\,x^0+tx,y^0\right), \tag{4.5}$$

for all $t\in[0,1]$, and

$$F\left(x^0,(1-t)\,y^0+ty\right)-F\left(x^0,y^0\right)\notin \mathrm{int}C\left(x^0,(1-t)\,y^0+ty\right), \tag{4.6}$$

for all $t\in[0,1]$.

Since for each $t\in[0,1]$, the sets

$$C\left((1-t)\,x^0+tx,y^0\right) \quad\text{and}\quad C\left(x^0,(1-t)\,y^0+ty\right)$$

are cones, from (4.5) and (4.6) it follows that

$$\frac{1}{t}\left[F\left(x^0+t(x-x^0),y^0\right)-F\left(x^0,y^0\right)\right]\notin -\mathrm{int}C\left((1-t)\,x^0+tx,y^0\right),$$

for all $t\in]0,1]$, and

$$\frac{1}{t}\left[F\left(x^0,y^0+t\left(y-y^0\right)\right)-F\left(x^0,y^0\right)\right]\notin \mathrm{int}C\left(x^0,(1-t)\,y^0+ty\right),$$

for all $t\in]0,1]$.

Since for each $t\in[0,1]$, the sets

$$Z\setminus\left(-\mathrm{int}C\left((1-t)\,x^0+tx,y^0\right)\right) \quad\text{and}\quad Z\setminus \mathrm{int}C\left(x^0,(1-t)\,y^0+ty\right)$$

are closed, and F is Fréchet differentiable in each of the arguments x and y at $\left(x^0,y^0\right)$, it follows that

$$\left\langle \nabla_x F\left(x^0,y^0\right), x-x^0\right\rangle \notin -\mathrm{int}C\left(x^0,y^0\right)$$

and

$$\left\langle \nabla_y F\left(x^0, y^0\right), y - y^0\right\rangle \notin \operatorname{int} C\left(x^0, y^0\right).$$

The theorem is proved. □

Theorem 4.2. *Let X, Y and Z be three normed spaces, A be a nonempty subset of X, B be a nonempty subset of Y, $\left(x^0, y^0\right) \in A \times B$, $C : A \times B \to 2^Z$ be a multifunction, and $F : A \times B \to Z$ be a function such that:*

i) *the set A is convex at x^0, and the set B is convex at y^0;*

ii) *the multifunctions $C\left(x^0, \cdot\right) : B \to 2^Z$ and $C\left(\cdot, y^0\right) : A \to 2^Z$ are closed;*

iii) *for each $(x, y) \in A \times B$ and $t \in [0, 1]$, the sets $C\left((1-t)x^0 + tx, y^0\right)$ and $C\left(x^0, (1-t)y^0 + ty\right)$ are solid pointed closed convex cones;*

iv) *the function F is convex-concave at $\left(x^0, y^0\right)$ with respect to C, and Fréchet differentiable in each of the arguments x and y at $\left(x^0, y^0\right)$.*

If

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle \notin -\operatorname{int} C\left(x^0, y^0\right), \text{ for all } x \in A \tag{4.7}$$

and

$$\left\langle \nabla_y F\left(x^0, y^0\right), y - y^0\right\rangle \notin \operatorname{int} C\left(x^0, y^0\right), \text{ for all } y \in B, \tag{4.8}$$

then $\left(x^0, y^0\right)$ is a weak $C\left(x^0, y^0\right)$-saddle point of the function F on $A \times B$.

If, in addition,

$$C\left(x, y^0\right) \subseteq C\left(x^0, y^0\right), \text{ for all } x \in A, \tag{4.9}$$

and

$$C\left(x^0, y\right) \subseteq C\left(x^0, y^0\right), \text{ for all } y \in B, \tag{4.10}$$

then $\left(x^0, y^0\right)$ is a solution of $(PVSPP)$.

Proof. Assume that $\left(x^0, y^0\right) \in A \times B$ satisfies the conditions (4.7) and (4.8). In order to prove that $\left(x^0, y^0\right)$ is a weak $C\left(x^0, y^0\right)$-saddle point of F on $A \times B$, let $x \in A$ and $y \in B$.

Since F is convex-concave at $\left(x^0, y^0\right)$ with respect to C, we have

$$(1-t)F\left(x^0, y^0\right) + tF\left(x, y^0\right) - F\left((1-t)x^0 + tx, y^0\right) \in C\left((1-t)x^0 + tx, y^0\right),$$

for all $t \in [0, 1]$, and

$$F\left(x^0, (1-t)y^0 + ty\right) - (1-t)F\left(x^0, y^0\right) - tF\left(x^0, y\right) \in C\left(x^0, (1-t)y^0 + ty\right),$$

for all $t \in [0, 1]$.

Since, for each $t \in [0, 1]$, the sets

$$C\left((1-t)x^0 + tx, y^0\right) \quad \text{and} \quad C\left(x^0, (1-t)y^0 + ty\right)$$

are cones, we deduce that

$$-\frac{1}{t}\left[F\left(x^0 + t\left(x - x^0\right), y^0\right) - F\left(x^0, y^0\right)\right] - F\left(x^0, y^0\right) + F\left(x, y^0\right) \in$$

$$\in C\left(x^0 + t\left(x - x^0\right), y^0\right),$$

for all $t \in]0, 1]$, and

$$-F\left(x^0, y\right) + F\left(x^0, y^0\right) + \frac{1}{t}\left[F\left(x^0, y^0 + t\left(y - y^0\right)\right) - F\left(x^0, y^0\right)\right] \in$$
$$\in C\left(x^0, y^0 + t\left(y - y^0\right)\right),$$

for $t \in]0, 1]$.

Since the multifunctions $C\left(\cdot, y^0\right)$ and $C\left(x^0, \cdot\right)$ are closed, the function F is Fréchet differentiable in each of the arguments at $\left(x^0, y^0\right)$, and for each $t \in [0, 1]$, the sets $C\left(x^0, (1-t)\, y^0 + ty\right)$ and $C\left((1-t)\, x^0 + tx, y^0\right)$ are solid convex cones, from the last two relations, by passing to limit, we obtain that

$$-\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle - F\left(x^0, y^0\right) + F\left(x, y^0\right) \in C\left(x^0, y^0\right)$$

and

$$-F\left(x^0, y\right) + F\left(x^0, y^0\right) + \left\langle \nabla_y F\left(x^0, y^0\right), y - y^0\right\rangle \in C\left(x^0, y^0\right).$$

From conditions (4.7) and (4.8), it follows that

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \mathrm{int} C\left(x^0, y^0\right), \tag{4.11}$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \mathrm{int} C\left(x^0, y^0\right), \tag{4.12}$$

because

$$C\left(x^0, y^0\right) + \mathrm{int} C\left(x^0, y^0\right) \subseteq \mathrm{int} C\left(x^0, y^0\right).$$

Consequently, $\left(x^0, y^0\right)$ is a weak $C\left(x^0, y^0\right)$-saddle point of F on $A \times B$.

Moreover, if (4.9) and (4.10) hold, then, from (4.11) and (4.12), it follows that

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \mathrm{int} C\left(x, y^0\right), \text{ for all } x \in A,$$

and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \mathrm{int} C\left(x^0, y\right), \text{ for all } y \in B,$$

hence $\left(x^0, y^0\right)$ is a solution of Problem (PVSPP).

The theorem is proved. □

Theorem 4.3. *Let X and Z be two normed spaces, Y be a topological linear space, A be a nonempty subset of X, B be a nonempty subset of Y, $\left(x^0, y^0\right) \in A \times B$, $F : A \times B \to Z$ be a function and $C : A \times B \to 2^Z$ be a multifunction such that*

i) *the set A is convex at x^0;*

ii) *the multifunction $C\left(\cdot, y^0\right) : A \to 2^Z$ is closed and has solid pointed closed convex cone values*;

iii) *the function $F\left(\cdot, y^0\right) : A \to Z$ is Fréchet differentiable at x^0;*

iv) *$C\left(x, y^0\right) \subseteq C\left(x^0, y^0\right)$, for all $x \in A$.*

If $\left(x^0, y^0\right)$ is a solution of $(PVSPP)$, then $\left(x^0, y^0\right)$ is a solution of $(PVVIP)$.

Proof. Assume that $\left(x^0, y^0\right) \in A \times B$ is a parameterized weak C-saddle point of the function F on $A \times B$. Then

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \operatorname{int}C\left(x, y^0\right), \tag{4.13}$$

for all $x \in A$, and

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \operatorname{int}C\left(x^0, y\right), \tag{4.14}$$

for all $y \in B$.

From (4.14) it follows that

$$\begin{aligned} y^0 \in T\left(x^0\right) &= \\ &= \left\{y \in B : \ F\left(x^0, v\right) - F\left(x^0, y\right) \notin \operatorname{int}C\left(x^0, y\right), \text{ for all } v \in B\right\} \end{aligned} \tag{4.15}$$

We will show that $\left(x^0, y^0\right)$ is a solution of $(PVVIP)$, i.e. $x^0 \in A$ and $y^0 \in T\left(x^0\right)$ satisfy

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle \notin -\operatorname{int}C\left(x, y^0\right), \text{ for all } x \in A. \tag{4.16}$$

For this, let $x \in A$. Since A is convex at x^0, we have

$$(1-t)\,x^0 + tx \in A, \text{ for all } t \in [0,1].$$

Then (4.13) implies

$$F\left(x^0, y^0\right) - F\left((1-t)\,x^0 + tx, y^0\right) \notin \operatorname{int}C\left((1-t)\,x^0 + tx, y^0\right), \tag{4.17}$$

for all $t \in [0,1]$.

Since $C\left(\cdot, y^0\right)$ has convex cone values, from (4.17) it follows that

$$\frac{1}{t}\left[F\left(x^0 + t\left(x - x^0\right), y^0\right) - F\left(x^0, y^0\right)\right] \notin -\operatorname{int}C\left(x^0 + t\left(x - x^0\right), y^0\right), \tag{4.18}$$

for all $t \in]0,1]$.

Since for each $t \in [0,1]$ the set $Z \setminus \left(-\operatorname{int}C\left(x^0 + t\left(x - x^0\right), y^0\right)\right)$ is closed, the multifunction $C\left(\cdot, y^0\right) : A \to 2^Z$ is closed, and the function $F\left(\cdot, y^0\right) : X \to Z$ is Fréchet differentiable at x^0, from (4.18), by passing to limit, it follows that

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle \notin -\operatorname{int}C\left(x^0, y^0\right),$$

and hence

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle \notin -\operatorname{int}C\left(x, y^0\right),$$

Consequently, $\left(x^0, y^0\right)$ is a solution of $(PVVIP)$. The theorem is proved. □

Theorem 4.4. *Let X and Z be two normed spaces, Y be a nonempty set, A be a nonempty subset of X, B be a nonempty subset of Y, $\left(x^0, y^0\right) \in A \times B$, $F : A \times B \to Z$ be a function, and $C : A \times B \to 2^Z$ be a multifunction such that:*

i) *the set A is convex at x^0;*

ii) *the multifunction $C\left(\cdot, y^0\right) : A \to 2^Z$ is closed and has solid pointed closed convex cone values;*

iii) *the function* $F\left(\cdot, y^0\right) : A \to Z$ *is convex at* x^0 *with respect to* $C\left(\cdot, y^0\right)$, *and Fréchet differentiable at* x^0;

iv) $C\left(x^0, y^0\right) \subseteq C\left(x, y^0\right)$, *for all* $x \in A$.

If $\left(x^0, y^0\right) \in A \times B$ *is a solution of* $(PVVIP)$, *then* $\left(x^0, y^0\right)$ *is a solution of* $(PVSPP)$.

Proof. Assume that $\left(x^0, y^0\right)$ is a solution of $(PVVIP)$; then $y^0 \in T\left(x^0\right)$ and

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle \notin -\text{int}C\left(x, y^0\right), \text{ for all } x \in A. \tag{4.19}$$

In order to prove that $\left(x^0, y^0\right)$ is a solution of $(PVSPP)$, let $(x, y) \in A \times B$. From

$$y^0 \in T\left(x^0\right) = \{y \in B : \ F\left(x^0, v\right) - F\left(x^0, y\right) \notin \text{int}C\left(x^0, y\right), \text{ for all } v \in B\},$$

it results that

$$F\left(x^0, y\right) - F\left(x^0, y^0\right) \notin \text{int}C\left(x^0, y\right). \tag{4.20}$$

Since $F\left(\cdot, y^0\right)$ is convex at x^0 with respect to $C\left(\cdot, y^0\right)$, we have

$$(1-t) F\left(x^0, y^0\right) + tF\left(x, y^0\right) - F\left((1-t) x^0 + tx, y^0\right) \in C\left((1-t)x^0 + tx, y^0\right),$$

for all $t \in [0, 1]$. Since, for each $t \in [0, 1]$, the set $C\left((1-t) x^0 + tx, y^0\right)$ is a cone, we have

$$-\frac{1}{t}\left[F\left(x^0 + t\left(x - x^0\right), y^0\right) - F\left(x^0, y^0\right)\right] - F\left(x^0, y^0\right) + F\left(x, y^0\right) \in$$
$$\in C\left((1-t)x^0 + tx, y^0\right), \text{ for all } t \in]0, 1].$$

Since $F\left(\cdot, y^0\right) : X \to Z$ is Fréchet differentiable at x^0, and $C\left(\cdot, y^0\right)$ is closed, from the last relation, by passing to limit, we obtain that

$$-\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0\right\rangle + F\left(x, y^0\right) - F\left(x^0, y^0\right) \in C\left(x^0, y^0\right).$$

From condition (4.19), and hypothesis iv), it follows that

$$F\left(x^0, y^0\right) - F\left(x, y^0\right) \notin \text{int}C\left(x, y^0\right) \tag{4.21}$$

because

$$C\left(x, y^0\right) + \text{int}C\left(x, y^0\right) \subseteq \text{int}C\left(x, y^0\right).$$

Conditions (4.20) and (4.21) prove that $\left(x^0, y^0\right)$ is a solution of $(PVSPP)$. The theorem is proved. □

In what follows, we use the following statement.

Theorem 4.5. (Fan – KKM Theorem) *Let* U *be a subset of the topological linear space* X *and* $S : U \to 2^X$ *be a multifunction such that*

(i) *for each* $u \in U$ *the set* $S(u)$ *is closed;*

(ii) *there exists* $\widehat{u} \in U$ *such that* $S(\widehat{u})$ *is compact.*

If, for each finite subset $\{u^1, \cdots, u^m\}$ *of* U, *we have*

$$\text{conv}\{u^1, \cdots, u^m\} \subseteq \cup\{S\left(u^i\right) : i \in \{1, \ldots, m\}\},$$

then

$$\cap\{S(u) : u \in U\} \neq \emptyset,$$

where conv (R) *denotes the convex hull of the set* R.

The following statement is an existence theorem for Problem (VSPP) to have a solution.

Theorem 4.6. *Let* X *and* Z *be two normed spaces, and* Y *be a topological linear space. Let* A *be a nonempty closed subset of* X, B *be a nonempty compact subset of* Y, $C : A \times B \to 2^Z$ *be a multifunction, and* $F : A \times B \to Z$ *be a function such that:*

i) *the set* A *is convex and closed and the set* B *is compact;*

ii) *the mulifunction* $C : A \times B \to 2^Z$ *is closed and has solid pointed closed convex cone values*;

iii) *for each* $y \in B$ *the function* $F(\cdot, y) : X \to Z$ *is convex on* A *with respect to* $C(\cdot, y)$, *and Fréchet diferentiable on* A;

iv) *the function* $\nabla_x F$ *is continuous in both* x *and* y *on* $A \times B$.

Let $T : A \to 2^Y$ *the multifunction defined by*

$$T(x) = \{y \in B : \ F(x, v) - F(x, y) \notin \operatorname{int}C(x, v), \ \textit{for all } v \in B\}, \ \textit{for all } x \in A.$$

If there exist a nonempty compact subset U *of* X *and* $\widehat{u} \in A \cap U$ *such that for each* $u \in A \setminus (A \cap U)$ *and* $y \in T(u)$ *we have*

$$\langle \nabla_x F(u, y), \widehat{u} - u \rangle \in -\operatorname{int}C(u, y),$$

then there exists a point $(x^0, y^0) \in A \times B$ *such that* $y^0 \in T(x^0)$ *and*

$$\langle \nabla_x F(x^0, y^0), x - x^0 \rangle \notin -\operatorname{int}C(x^0, y^0), \ \textit{for all } x \in A.$$

If, in addition, the multifunction $C : A \times B \to 2^Z$ *is constant with respect to the first argument* x, *i.e. there exists a multifunction* $\widehat{C} : B \to 2^Z$ *such that*

$$\widehat{C}(y) = C(x, y), \ \textit{for all } (x, y) \in A \times B,$$

then Problem $(PVSPP)$ *has a solution.*

Proof. Let $S : A \to 2^X$ be the multifunction defined, for all $u \in A$, by

$$S(u) = \{x \in A : \text{ there exists } t \in T(x) \text{ such that}$$
$$\langle \nabla_x F(x, t), u - x \rangle \notin -\operatorname{int}C(x, t)\}.$$

Let us show that

$$\cap\{S(u) : u \in A\} \neq \emptyset. \tag{4.22}$$

In order to prove (4.22), we use Fan-KKM Theorem (Theorem 4.5).

First, we prove that for each finite subset $\{u^1, u^2, \ldots, u^m\}$ of A, the convex hull of the set $\{u^1, u^2, ..., u^m\}$ is contained in the set

$$\cup\{S(u^i) : \ i \in \{1, 2, \ldots, m\}\},$$

that is,

$$\operatorname{conv}\left\{u^1, u^2, \ldots, u^m\right\} \subseteq \cup\left\{S\left(u^i\right):\ i \in \{1, 2, \ldots, m\}\right\}. \tag{4.23}$$

For this, we assume the contrary that there exist $u^1, u^2, \ldots, u^m \in A$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ with

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1,$$

such that

$$u := \sum_{i=1}^{m} \alpha_i u^i \notin \cup\left\{S\left(u^i\right):\ i \in \{1, 2, \ldots, m\}\right\}. \tag{4.24}$$

Since A is convex, we obtain that $u \in A$.

On the other hand, from (4.24), we deduce that $u \notin S\left(u^i\right)$, for all $i \in \{1, 2, \ldots, m\}$. It follows that, for each $i \in \{1, \ldots, m\}$, and each $t \in T\left(u^i\right)$ we have

$$\left\langle \nabla_x F(u, t), u^i - u \right\rangle \in -\operatorname{int} C(u, t).$$

Let now, $t \in T\left(u^i\right)$, $i \in \{1, ..., m\}$. Since $\operatorname{int} C(u, t)$ is convex

$$\sum_{i=1}^{m} \alpha_i \left\langle \nabla_x F(u, t), u^i - u \right\rangle \in -\operatorname{int} C(u, t).$$

But the operator $\nabla_x F(u, t)$ is linear, and then

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_i \left\langle \nabla_x F(u, t), u^i - u \right\rangle \\ &= \left\langle \nabla_x F(u, t), \sum_{i=1}^{m} \alpha_i u^i - \sum_{i=1}^{m} \alpha_i u \right\rangle \\ &= \langle \nabla_x F(u, t), u - u \rangle = 0. \end{aligned}$$

It follows that $0 \in -\operatorname{int} C(u, t)$ which is a contradiction. Thus (4.23) is true.

Now, we show that the multifunction T is closed. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of elements from A and $u \in A$ such that $u^n \to u$. Let $(y^n)_{n\in\mathbb{N}}$ be a sequence of elements from B such that $y^n \in T(u^n)$, for all $n \in \mathbb{N}$. From $y^n \in T(u^n)$, $(n \in \mathbb{N})$ we deduce that

$$F(u^n, v) - F(u^n, y^n) \notin \operatorname{int} C(u^n, v), \text{ for all } v \in B \text{ and } n \in \mathbb{N}. \tag{4.25}$$

Since $\{y^n : n \in \mathbb{N}\}$ is a subset of B and B is compact, the sequence (y^n) contains a convergent subsequence. Without loss of generality, we can assume that the sequence itself is convergent. Hence there exists $y \in B$ such that $y^n \to y$.

Then, by the continuity of F and the hypothesis ii), from (4.25), by passing to limit, we deduce that

$$F(u, v) - F(u, y) \notin \operatorname{int} C(u, v), \text{ for all } v \in B,$$

which shows that $y \in T(u)$. Consequently, the multifunction T is closed.

Now, we show that, for each $u \in A$, the set $S(u)$ is closed. For this let $u \in A$. Let now $s \in X$ and $(s^n)_{n\in\mathbb{N}}$ be a sequence of elements from $S(u)$ such that $s^n \to s$. The set A being closed, the element s belongs to A. Since, for each $n \in \mathbb{N}$, we have $s^n \in S(u)$, it follows that, for each $n \in \mathbb{N}$, there exists $t^n \in T(s^n)$ such that

$$\langle \nabla_x F(s^n, t^n), u - s^n \rangle \notin -\mathrm{int}C(s^n, t^n). \tag{4.26}$$

The set B is compact and $\{t^n : n \in \mathbb{N}\}$ is a subset of B, then the sequence (t^n) contains a convergent subsequence. Without loss of generality, we can assume that the sequence itself is convergent. Let $t \in B$ be the limit of the sequence (t^n). Since T is closed, it follows that $t \in T(u)$.

On the other hand, the function $\nabla_x F$ is continuous, and for each $n \in \mathbb{N}$, the set $Z \setminus (-\mathrm{int}C(s^n, t^n))$ is closed, then, from (4.26), by passing to limit, it follows that

$$\langle \nabla_x F(s, t), u - s \rangle \notin -\mathrm{int}C(s, t),$$

which shows that the set $S(u)$ is closed. Hence, for each $u \in A$, the set $S(u)$ is closed.

Now we show that $S(\widehat{u})$ is compact. Since $S(\widehat{u})$ is closed and U is compact, it is sufficient to show that $S(\widehat{u}) \subseteq U$. Assume the contrary, i.e. there exists $\widetilde{u} \in S(\widehat{u})$ such that $\widetilde{u} \notin U$. From $\widetilde{u} \in S(\widehat{u})$, it follows that $\widetilde{u} \in A$ and there exists $\widetilde{y} \in T(\widetilde{u})$ such that

$$\langle \nabla_x F(\widetilde{u}, \widetilde{y}), \widehat{u} - \widetilde{u} \rangle \notin -\mathrm{int}C(\widetilde{u}, \widetilde{y}). \tag{4.27}$$

On the other hand, $\widetilde{u} \in A \setminus (A \cap U)$ because $\widetilde{u} \in A$ and $\widetilde{u} \notin U$. Since $\widetilde{y} \in T(\widetilde{u})$, by the hypotheses of the theorem, we have

$$\langle \nabla_x F(\widetilde{u}, \widetilde{y}), \widehat{u} - \widetilde{u} \rangle \in -\mathrm{int}C(\widetilde{u}, \widetilde{y}),$$

which contradicts (4.27).

Consequently, (4.22) holds. Then there exists $x^0 \in A$ such that $x^0 \in S(u)$, for all $u \in A$, that is there exist $x^0 \in A$ and $y^0 \in T(x^0)$ such that

$$\left\langle \nabla_x F\left(x^0, y^0\right), x - x^0 \right\rangle \notin -\mathrm{int}C\left(x^0, y^0\right), \text{ for all } x \in A.$$

If, in addition, the multifunction C is constant with respect to the first argument x, then we can apply Theorem 4.4. It follows that $\left(x^0, y^0\right)$ is a solution of Problem (PVSPP).

The theorem is proved. □

References

[1] Q. H. Ansari and J. C. Yao, An existence result for the generalized vector equilibrium problem, *Appl. Math. Lett.* **12** (1999), 53–56.

[2] C. Berge, Sur une convexité régulière et ses applications a la théorie des jeux, *Bull. Soc. Math. France* **82** (1954), 301–319.

[3] G. Y. Chen, Existence of solutions for a vector variational inequality; An extension of the Hartman-Stampacchia Theorem, *J. Optim. Theory Appl.* **74** (1992), 445–456.

[4] D. I. Duca and L. Lupşa: Bi-(φ,ψ) convex sets, *Mathematica Pannonica* **14** (2) (2003), 193–203.

[5] D. I. Duca and L. Lupşa, Bi-(φ,ψ) convex-concave functions, *Scientific Bulletin of the "Politehnica" University of Timişoara* **48** (62) (2) (2003), 67–72.

[6] D. I. Duca and L. Lupşa, On the E-Epigraph of an E-convex function, *J. Optim. Theory Appl.* **129** (2) (2006), 343–348.

[7] D. I. Duca and L. Lupşa, Saddle points for vector valued functions, (*to appear*).

[8] D. I. Duca and L. Lupşa, Saddle points with respect to a set, (*to appear*).

[9] D. I. Duca and L. Lupşa, Saddle points for vector valued functions: Existence, necesary and sufficient theorems, (*to appear*).

[10] K. Fan, Minimax Theorems, *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 42–47.

[11] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305–310.

[12] F. Ferro, A minimax theorem for vector-valued functions, *J. Optim. Theory Appl.* **60** (1989), 19–31.

[13] F. Giannessi, Theorems of the alternative, quadratic programs, and complementarity problems, pp. 151-186 in "*Variational Inequalities and Complemntarity Problems*", Edited by R.W. Cottle, F. Giannessi, and J.L. Lions, John Wiley, New York NY, 1980.

[14] F. Giannessi, On Minty variational principle, in "*New Trends in Mathematical Programming*", Edited by F. Giannessi, T. Rapcsak and S. Komlosi, Kluwer Academic Publishers, Dordrecht, Netherland, 1997.

[15] G. H Greco and C. D. Horvath, A topological minimax theorem, *J. Optim. Theory Appl.* **113** (2002), 513–536.

[16] J. Hartung, An extension of Sion's Min-Max theorem with an application to a method for constrained games, *Pacific J. Math.* **103** (1982), 401–408.

[17] K. R. Kazmi and S. Khan, Existence of solutions for a vector saddle point problem, *Bull. Austral. Math. Soc.* **61** (2000), 201–206.

[18] K. Kimura and T. Tanaka, Existence theorems of saddle points for vector valued functions, pp. 169-178 in "*Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis*", edited by W. Takahashi and T. Tanaka, Yokohama Publishers, 2003.

[19] H. Kneser, Sur un théorème fondamental de la théorie des jeux, *C.R. Acad. Sci.*, Paris, **234** (25) (1952), 2418–2420.

[20] J. von Neumann, Zur Theorie der Gesellschafsspiele, *Math. Ann.* **100** (1928), 295–320.

[21] J. W. Nieuwenhuis, Some minimax theorems in vector-valued functions, *J. Optim. Theory Appl.*, **40** (1983), 463–475.

[22] H. Nikaido, On von Neumann's minmax theorem, *Pacific J. Math.* **4** (1954), 65–72.

[23] U. Passy and E.Z. Prisman, A duality approach to minimax results for quasisaddle functions in finite dimensions, *Math. Programming* **55** (1992), 81–98.

[24] M. Sion, On general minmax theorems, *Pacific J. Math.* **8** (1958), 171–176.

[25] M. Schiffman, On the equality minmax-maxmin, and the theory of games, *RAND, Report* RM -243, 1949.

[26] S. Simons, A flexible minimax theorem, *Acta Mathematica Hungarica* **63** (1994), 119–132.

[27] T. Tanaka, Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions, *J. Optim. Theory Appl.* **81** (1994), 355–377.

[28] T. Tanaka, Generalized semicontinuity and existence theorems for cone saddle points, *Appl. Math. Optim.* **36** (1997), 313–322.

[29] H. Tuy, On a general minimax theorem, *Soviet Math. Dokl.* **15** (1974), 1689–1693.

[30] H. Tuy, On the general minimax theorem, *Colloquium Mathematicum* **33** (1975), 145–158.

[31] H. Tuy, Minimax theorems revisited, *Acta Math. Vietnam.* **29** (2004), 217–229.

[32] C. Yen, A minimax inequality and its applications to variational inequalities, *Pacific. J. Math.* **97** (1981), 477–481.

[33] J. Yu and X. Z. Yuan, New saddle point theorem beyond topological vector spaces, *J. Optim. Theory Appl.* **96**(1998), 211–219.

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