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ON HARNACK'S INEQUALITY FOR NON-UNIFORMLY P-LAPLACIAN EQUATIONS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. We prove the Harnack inequality for non-uniformly p-Laplacian equations.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$, $p > 1$, t and s positive real numbers, λ and μ positive measurable functions on Ω, a a measurable function on Ω such that

$$\frac{1}{t} + \frac{1}{s} < \frac{p}{n}. \tag{1.1}$$

$$\lambda^{-1} \in L^t(\Omega), \mu \in L^s(\Omega) \tag{1.2}$$

$$\lambda(x) \leq a(x) \leq \mu(x) \qquad \forall\ x \in \Omega. \tag{1.3}$$

We study the Harnack inequality for the elliptic equation

$$-div\ (a(x)|\nabla u|^{p-2}\nabla u) = 0. \tag{1.4}$$

We need the following notations

$$\Lambda(x_0, R) = \left(⨍_{B_R} \left(\frac{1}{\lambda}\right)^t dx\right)^{\frac{1}{t}} \left(⨍_{B_R} \mu^s dx\right)^{\frac{1}{s}} \qquad \forall\ B_R \subset \Omega,$$

where B_R is an Euclidean ball $B(x_0, R)$,

$$\Lambda(R) = \Lambda(0, R)$$

$$H(\Omega) = \{u \in W^{1,p}(\Omega) : \int_\Omega (\mu u^p + \mu |Du|^p) dx < \infty\},$$

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$$\|u\|_H = \int_\Omega (\mu|u|^p + \mu|Du|^p)dx \qquad \forall u \in H(\Omega),$$

$$H_0(\Omega) = \{u \in W_0^{1,p}(\Omega) : \int_\Omega \mu|Du|^p dx < \infty\},$$

$$\|u\|_{H_0} = \int_\Omega \mu|Du|^p dx \quad \forall u \in H_0(\Omega).$$

We remark here that $(H(\Omega), \|.\|_H)$, $(H_0(\Omega), \|.\|_{H_0})$ are normed spaces and $H_0(\Omega) \subset H(\Omega)$.

Definition 1.1. A function $u \in H(\Omega)$ is called a weak solution (resp. subsolution, suppersolution) of (1.4) if

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla\varphi \, dx = 0 \ (resp. \le 0, \ge 0) \qquad \forall \, \varphi \in H_0(\Omega), \varphi \ge 0. \tag{1.5}$$

Our main results are the following theorems.

Theorem 1.2. *Assume that* (1.1), (1.2), *and* (1.3) *are satisfied. Let u be a non-negative weak subsolution of equation* (1.4) *in the ball $B(x_0, 16R) \subset \mathbb{R}^n$. We have the estimates*

$$\sup_{B(x_0,R)} u \le C_1(\Lambda(x_0, 4R))^{\overline{\gamma(\frac{p}{n}-\frac{1}{s}-\frac{1}{t})}^{\,1}} R^{-\frac{n}{\gamma}}\phi(\gamma, 2R, u) \tag{1.6}$$

for any $\gamma > \dfrac{(p-1)s}{s-1}$.

Theorem 1.3. *Assume that* (1.1), (1.2), *and* (1.3) *are satisfied. Let u be a non-negative weak supersolution of equation* (1.4) *in the ball $B(x_0, 16R) \subset \mathbb{R}^n$. Then for any* $0 < \gamma < \dfrac{(p-1)t^*}{p}$,

$$R^{-\frac{n}{\gamma}}\phi(\gamma, 2R, u) \le C_{48} \exp\left(C_{49}\Lambda(16R)^{\overline{t^*\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}^{\,1}+1} \right) \inf_{B(x_0,R)} u, \tag{1.7}$$

where $\dfrac{p}{t^*} = 1 + \dfrac{1}{t} - \dfrac{p}{n}$.

Theorems 1.2 and 1.3 directly yield the following theorems.

Theorem 1.4. *Let u be a non-negative weak supersolution of equation* (1.4) *in the ball $B(x_0, 16R) \subset \mathbb{R}^n$ with conditions* (1.1), (1.2), *and* (1.3). *Then*

$$\sup_{B(x_0,R)} u \le \exp\left(C\Lambda(x_0, 16R)^\alpha\right) \inf_{B(x_0,R)} u, \tag{1.8}$$

where C and α depend on p, n, t, s.

Theorem 1.5 (Harnack inequality). *Let u be a non-negative weak solution of equation* (1.4) *in $\Omega \subset \mathbb{R}^n$ with conditions* (1.1), (1.2), (1.3), *and $\Omega_0 \subset\subset \Omega$. Then*

$$\sup_{\Omega_0} u \le C \inf_{\Omega_0} u, \tag{1.9}$$

where C depends on $p, n, t, s, \mu, \lambda, \Omega, \Omega_0$.

Remark 1.6. The Harnack inequality for uniformly p-Laplace equations has been studied in [1–5, 11–13, 17]. Many problems on regularity of solutions to partial differential equations lead to study the Harnack inequality for nonuniformly elliptic equations (see [6–8, 10, 14–16]). In the paper [14], Trudinger established the Harnack inequality in the case that $p = 2$, $\lambda^{-1} \in L^t$ and $\mu^2\lambda^{-1} \in L^s$. In the present paper, we only need $\lambda^{-1} \in L^t$ and $\mu \in L^s$.

2. Proof of Theorem 1.2

We shall denote positive constants depending only on p, t, s, n, and γ by C_1, C_2, ... To prove the theorem we need the following lemmas.

Lemma 2.1. *Suppose that either u is a function in $H_0(B)$ or u is a function in $H(B)$ satisfying $\int_B u dx = 0$, where B is a ball in $\mathbb{R}^n$. Then if $1 + \frac{1}{t} > \frac{p}{n}$, u is in $L^{t^*}(B)$ and we have the estimate*

$$\|u\|^p_{L^{t^*}(B)} \le C \|\frac{1}{\lambda}\|_{L^t(B)} \int_B \lambda |\nabla u|^p dx,$$

where $\frac{p}{t^*} = 1 + \frac{1}{t} - \frac{p}{n}$ and C is a constant depending on n, p, t.

Proof. Put $r = \dfrac{p}{1+\frac{1}{t}}$. Then we have: $r < p$, $\dfrac{r}{p-r} = t$, $t^* = \dfrac{nr}{n-r}$.

Applying Sobolev embedding theorem and Hölder's inequality, we have

$$\left(\int_B u^{t^*} dx\right)^{\frac{r}{t^*}} \le C^r \int_B |\nabla u|^r dx \le C^r \left(\int_B \lambda |\nabla u|^p\right)^{\frac{r}{p}} \left(\int_B (\lambda^{-1})^{\frac{r}{p-r}} dx\right)^{\frac{p-r}{p}}$$

which implies the lemma. □

Lemma 2.2. *Assume that* (1.1), (1.2), *and* (1.3) *are satisfied. Let u be a non-negative weak subsolution of equation* (1.4) *in the ball $B(x_0, 4R) \subset \mathbb{R}^n$. We have the estimate*

$$\sup_{B(x_0,R)} u \le C_2 (\Lambda(x_0, 4R))^{\frac{1}{\gamma\left(\frac{p}{n} - \frac{1}{s} - \frac{1}{t}\right)}} R^{-\frac{n}{\gamma}} \phi(\gamma, 2R, u) \tag{2.1}$$

for any $\gamma > \dfrac{ps}{s-1}$.

Proof. We may assume, without loss of generality, that $x_0 = 0$, $R = 1$ and $u \geq \varepsilon > 0$.
For $\beta \geq 1$ and $0 < N < \infty$, we introduce the functions

$$F(u)(x) \equiv F_\beta^N(u)(x) = \begin{cases} u^\beta & u(x) \leq N, \\ \beta N^{\beta-1}u(x) - (\beta-1)N^\beta & u(x) > N. \end{cases}$$

$$G(u)(x) \equiv G_\beta^N(u)(x) = \begin{cases} \dfrac{p\beta^{\frac{1}{p}}}{\beta+p-1} u^{\frac{\beta+p-1}{p}}(x) & u(x) \leq N, \\ \dfrac{p\beta^{\frac{1}{p}} N^{\frac{\beta+p-1}{p}}}{\beta+p-1} + \beta^{\frac{1}{p}} N^{\frac{\beta-1}{p}}(u(x)-N) & u(x) > N. \end{cases}$$

Then

$$F'(u)(x) = \begin{cases} \beta u^{\beta-1}(x) & u(x) \leq N, \\ \beta N^{\beta-1} & u(x) > N. \end{cases}$$

$$G'(u)(x) = \begin{cases} \beta^{\frac{1}{p}} u^{\frac{\beta-1}{p}}(x) & u(x) \leq N, \\ \beta^{\frac{1}{p}} N^{\frac{\beta-1}{p}} & u(x) > N. \end{cases}$$

We have

$$\lim_{N\to\infty} G_\beta^N(u)(x) = \frac{p\beta^{\frac{1}{p}}}{\beta+p-1} u^{\frac{\beta+p-1}{p}} \qquad \forall\, x \in \Omega, \tag{2.2}$$

$$\lim_{N\to\infty} (G_\beta^N(u))'(x) = \beta^{\frac{1}{p}} u^{\frac{\beta-1}{p}} \qquad \forall\, x \in \Omega, \tag{2.3}$$

$$G'(u)(x) \leq \beta^{\frac{1}{p}} u^{\frac{\beta-1}{p}}(x) \qquad \forall\, x \in \Omega, \tag{2.4}$$

$$0 \leq F(u)(x) \leq u(x)F'(u)(x) \qquad \forall\, x \in \Omega, \tag{2.5}$$

$$0 \leq G(u)(x) \leq u(x)G'(u)(x) \qquad \forall\, x \in \Omega, \tag{2.6}$$

$$G'(u)(x) = \big(F'(u)(x)\big)^{\frac{1}{p}} \qquad \forall\, x \in \Omega. \tag{2.7}$$

It is easy to see that $\varphi(x) = \eta^p F(u) \in H_0(\Omega)$ and it can be used as a test function for (1.5) for any η in $C_c^1(B_4)$, $\eta \geq 0$. We have

$$\int_{B_4} a(x)|\nabla u|^p \eta^p F'(u)dx + \int_{B_4} a(x)|\nabla u|^{p-2}\nabla u(p\eta^{p-1}\nabla\eta F(u))dx =$$

$$= \int_{B_4} a(x)|\nabla u|^{p-2}\nabla u \left(\eta^p F'(u)\nabla u + p\eta^{p-1}\nabla\eta F(u)\right) dx$$

$$= \int_{B_4} a(x)|\nabla u|^{p-2}\nabla u \nabla(\eta^p F(u)) dx \leq 0.$$

This implies

$$\int_{B_4} a(x)|\nabla u|^p \eta^p F'(u) dx \leq \int_{B_4} a(x)|\nabla u|^{p-1}|\nabla\eta| p\eta^{p-1} F(u) dx \tag{2.8}$$

$$\leq \frac{1}{2}\int_{B_4} a(x)|\nabla u|^p \eta^p F'(u) dx + C_3 \int_{B_4} a(x)|\nabla\eta|^p u^p F'(u) dx.$$

Therefore,

$$\int_{B_4} \lambda|\nabla u|^p \eta^p F'(u) dx \leq C_4 \int_{B_4} \mu|\nabla\eta|^p u^p F'(u) dx. \tag{2.9}$$

We get

$$\int_{B_4} \lambda|\nabla G(u)|^p \eta^p dx = \int_{B_4} \lambda|\nabla u|^p \eta^p G'^p(u) dx \leq C_4 \int_{B_4} \mu|\nabla\eta|^p u^p G'^p(u) dx. \tag{2.10}$$

By (1.3), (2.6) we have

$$\int_{B_4} \lambda|\nabla\eta|^p G^p(u) dx \leq \int_{B_4} \mu|\nabla\eta|^p (uG'(u))^p dx. \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\int_{B_4} \lambda|\nabla(\eta G)|^p dx \leq C_5 \int_{B_4} \mu|\nabla\eta|^p (uG'(u))^p dx. \tag{2.12}$$

For any $0 < r' < r < 2r' \leq 4$, choose $\eta \in C_c^1(B_4)$ such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ for any $x \in B_{r'}$, $\eta(x) = 0$ for any $x \in \mathbb{R}^n \setminus B_r$, and $|\nabla\eta(x)| \leq \dfrac{4}{r - r'}$ for any $x \in \mathbb{R}^n$.

By Lemma 2.1 and Hölder inequality, we get

$$\|\eta G(u)\|^p_{L^{t^*}(B_4)} \leq C_6 \|\frac{1}{\lambda}\|_{L^t(B_4)} \int_{B_4} \lambda(|\nabla(\eta G)|)^p dx \tag{2.13}$$

$$\leq C_7 \|\frac{1}{\lambda}\|_{L^t(B_4)} \int_{B_4} \mu|\nabla\eta|^p (uG'(u))^p dx$$

$$\leq C_7 \|\frac{1}{\lambda}\|_{L^t(B_4)} \|\mu\|_{L^s(B_4)} \left(\int_{B_4} (|\nabla\eta| uG'(u))^{\frac{ps}{s-1}} dx\right)^{\frac{s-1}{s}}$$

$$\leq C_8\Lambda(4)\left(\int_{B_4}(|\nabla\eta|uG')^{\frac{ps}{s-1}}dx\right)^{\frac{s-1}{s}}.$$

Thus by (2.4)

$$\|G_\beta^N(u)\|^p_{L^{t^*}(B_{r'})} \leq C_8\Lambda(4)\frac{4^p}{(r-r')^p}\left(\int_{B_r}(uG')^{\frac{ps}{s-1}}dx\right)^{\frac{s-1}{s}} \tag{2.14}$$

$$\leq C_8\Lambda(4)\frac{4^p}{(r-r')^p}\left(\int_{B_r}(\beta^{\frac{1}{p}}u^{\frac{\beta+p-1}{p}})^{\frac{ps}{s-1}}dx\right)^{\frac{s-1}{s}}.$$

Letting N tend to infinity and applying Fatou's lemma, we obtain

$$\|\frac{p\beta^{\frac{1}{p}}}{\beta+p-1}u^{\frac{\beta+p-1}{p}}\|^p_{L^{t^*}(B_{r'})} \leq C_8\Lambda(4)\frac{4^p}{(r-r')^p}(\int_{B_r}(\beta^{\frac{1}{p}}u^{\frac{\beta+p-1}{p}})^{\frac{ps}{s-1}}dx)^{\frac{s-1}{s}},$$

or
(2.15)

$$\phi(t^*\frac{\beta+p-1}{p},r',u) \leq \left[\frac{C_9\Lambda(4)(\beta+p-1)^p}{(r-r')^p}\right]^{\frac{1}{\beta+p-1}}\phi\left(\frac{(\beta+p-1)s}{s-1},r,u\right),$$

where $\phi(q,\rho,u) = \|u\|_{L^q(B_\rho)}$.

Now put $\delta = t^*\dfrac{s-1}{ps}$, $r_m = \left(1+\dfrac{1}{2^m}\right)$, $\beta_0 = \dfrac{\gamma(s-1)}{s}-p+1$, $\dfrac{(\beta_{m+1}+p-1)s}{s-1} = t^*\dfrac{\beta_m+p-1}{p}$ for any positive integer m.

Since $\dfrac{1}{t}+\dfrac{1}{s}<\dfrac{p}{n}$ and $\beta_{m+1}+p-1 = \delta(\beta_m+p-1)$, we have $\delta>1$, $\lim_{m\to\infty}\beta_m = \infty$, $\dfrac{(\beta_0+p-1)s}{s-1} = \gamma$ and by (2.15)

$$\phi\left(\frac{(\beta_{m+1}+p-1)s}{s-1},1,u\right)$$

$$\leq \phi\left(\frac{(\beta_{m+1}+p-1)s}{s-1},r_{m+1},u\right) = \phi\left(\frac{(\beta_m+p-1)s\delta}{s-1},r_{m+1},u\right)$$

$$= \phi\left(t^*\frac{(\beta_m+p-1)}{p},r_{m+1},u\right)$$

$$\leq \left[\frac{C_9\Lambda(4)(\beta_m+p-1)^p}{(r_m-r_{m+1})^p}\right]^{\frac{1}{\beta_m+p-1}}\phi(\frac{(\beta_m+p-1)s}{s-1},r_m,u)$$

$$\le [C_9\Lambda(4)]^{\frac{1}{\delta^m(\beta_0+p-1)}} 2^{\frac{pm+p}{\delta^m(\beta_0+p-1)}} \delta^{\frac{pm}{\delta^m(\beta_0+p-1)}} (\beta_0+p-1)^{\frac{p}{\delta^m(\beta_0+p-1)}}$$
$$\times \phi(t^*\frac{\beta_{m-1}+p-1}{p}, r_m, u)$$
$$\le \left[(C_9\Lambda(4))^{\sum_{k=0}^m \frac{1}{\delta^k}} 2^{\sum_{k=0}^m \frac{pk+p}{\delta^k}} \delta^{\sum_{k=0}^m \frac{pk}{\delta^k}} (\beta_0+p-1)^{\sum_{k=0}^m \frac{p}{\delta^k}}\right]^{\frac{1}{(\beta_0+p-1)}} \phi(\frac{(\beta_0+p-1)s}{s-1}, r_0, u)$$
$$\le \left[(C_9 2^{\sum_{k=0}^m \frac{pk+p}{\delta^k}} \delta^{\sum_{k=0}^m \frac{pk}{\delta^k}} (\beta_0+p-1)^{\sum_{k=0}^m \frac{p}{\delta^k}}\right]^{\frac{1}{(\beta_0+p-1)}} \Lambda(4)^{\frac{1}{(\beta_0+p-1)}\sum_{k=0}^m \frac{1}{\delta^k}}$$
$$\times \phi(\frac{(\beta_0+p-1)s}{s-1}, r_0, u).$$

Since $\frac{(\beta_0+p-1)s}{s-1} = \gamma$ and $r_0 = 2$ we have $\phi(\frac{(\beta_0+p-1)s}{s-1}, r_0, u) = \phi(\gamma, 2, u)$ and

$$\left[\frac{s}{s-1}\sum_{k=0}^m \frac{1}{\delta^k}\right]^{-1} = \left[\frac{s}{s-1}\frac{1}{1-\frac{1}{\delta}}\right]^{-1} = \left[\frac{s}{s-1}\frac{1}{1-\frac{ps}{t^*(s-1)}}\right]^{-1} = \left[\frac{s}{s-1}\frac{t^*(s-1)}{t^*(s-1)-ps}\right]^{-1}$$
$$= \left[\frac{st^*}{t^*(s-1)-ps}\right]^{-1} = \frac{t^*(s-1)-ps}{st^*} = 1-\frac{1}{s}-\frac{p}{t^*} = 1-\frac{1}{s}-\frac{p(n-r)}{nr} = 1-\frac{1}{s}-\frac{p}{r}+\frac{p}{n}$$
$$= 1-\frac{1}{s}-\frac{p}{\frac{p}{1+\frac{1}{t}}}+\frac{p}{n} = 1-\frac{1}{s}-1-\frac{1}{t}+\frac{p}{n} = \frac{p}{n}-\frac{1}{s}-\frac{1}{t}.$$

This implies

$$\frac{1}{\beta_0+p-1}\sum_{k=0}^m \frac{1}{\delta^k} = \frac{s}{\gamma(s-1)}\sum_{k=0}^m \frac{1}{\delta^k} = \frac{1}{\gamma(\frac{p}{n}-\frac{1}{s}-\frac{1}{t})}$$

and

$$\sup_{x\in B_1} u = \lim_{m\to\infty} \phi(\frac{(\beta_{m+1}+p-1)s}{s-1}, 1, u)$$
$$\le C_{10}\Lambda(4)^{\frac{1}{\gamma(\frac{p}{n}-\frac{1}{s}-\frac{1}{t})}} \phi(\gamma, 2, u). \tag{2.16}$$
□

Proof of Theorem 1.2. We may assume, without loss of generality, that $x_0 = 0$, $R = 1$ and $u \ge \varepsilon > 0$. By Lemma 2.2, u is bounded in B_4. Let β be a positive real number and use $\varphi \equiv \eta^p u^\beta$ as a test function in (1.5). As in (2.8) we have

$$\int_{B_4} a(x)|\nabla u|^p \eta^p u^{\beta-1}\beta dx \le \int_{B_4} a(x)|\nabla u|^{p-1}|\nabla\eta| p\eta^{p-1}u^\beta dx$$
$$\le \frac{1}{2}\int_{B_4} a(x)|\nabla u|^p\eta^p u^{\beta-1}\beta dx + C_{11}\int_{B_4} a(x)|\nabla\eta|^p u^{\beta+p-1}\frac{1}{\beta^{p-1}}dx.$$

This implies that

$$\begin{aligned}
&\int_{B_4} \lambda \left|\nabla \left(\eta u^{\frac{\beta+p-1}{p}}\right)\right|^p dx \\
&\leq 2^{p-1} \int_{B_4} \lambda |\nabla \eta|^p u^{\beta+p-1} dx + \frac{(\beta+p-1)^p 2^{p-1}}{p^p} \int_{B_4} \lambda |\nabla u|^p \eta^p u^{\beta-1} dx \\
&\leq 2^{p-1} \int_{B_4} \mu |\nabla \eta|^p u^{\beta+p-1} dx + \frac{(\beta+p-1)^p 2^{p-1}}{p^p} \int_{B_4} \lambda |\nabla u|^p \eta^p u^{\beta-1} dx \\
&\leq 2^{p-1} \int_{B_4} \mu |\nabla \eta|^p u^{\beta+p-1} dx + \frac{C_{11}(\beta+p-1)^p 2^{p-1}}{\beta^p p^p} \int_{B_4} \mu |\nabla \eta|^p u^{\beta+p-1} dx \\
&\leq C_{12} \left(\frac{(\beta+p-1)^p}{\beta^p} + 1\right) \int_{B_4} \mu |\nabla \eta|^p u^{\beta+p-1} dx.
\end{aligned}$$

Arguing as in (2.14) and (2.15), we get

$$\begin{aligned}
&\left(\int_{B_4} \left(\eta u^{\frac{\beta+p-1}{p}}\right)^{t^*} dx\right)^{\frac{p}{t^*}} \\
&\leq C_{13}\Lambda(4) \left(\frac{(\beta+p-1)^p}{\beta^p} + 1\right) \left(\int_{B_4} \left(|\nabla \eta| u^{\frac{\beta+p-1}{p}}\right)^{\frac{ps}{s-1}} dx\right)^{\frac{s-1}{s}}
\end{aligned}$$

and

$$\begin{aligned}
(2.17) \qquad &\phi\left(t^* \frac{\beta+p-1}{p}, r', u\right) \\
&\leq \left[\left(\frac{(\beta+p-1)^p}{\beta^p} + 1\right) \frac{C_{14}\Lambda(4)}{(r-r')^p}\right]^{\frac{1}{\beta+p-1}} \phi\left(\frac{(\beta+p-1)s}{s-1}, r, u\right),
\end{aligned}$$

for any $0 < r' < r < 2r' \leq 4$.

Let δ and r_m be as in the proof of Lemma 2.2, and $\beta_0 + p - 1 = \frac{\gamma(s-1)}{s}$, $\frac{(\beta_{m+1}+p-1)s}{s-1} = t^* \frac{\beta_m + p - 1}{p}$ for any positive integer m. We have $\lim_{m\to\infty} \beta_m = \infty$ and

$$\begin{aligned}
&\phi\left(\frac{(\beta_{m+1}+p-1)s}{s-1}, 1, u\right) \\
&\leq \phi\left(\frac{(\beta_{m+1}+p-1)s}{s-1}, r_{m+1}, u\right) = \phi\left(t^* \frac{(\beta_m+p-1)}{p}, r_{m+1}, u\right) \\
&\leq \left(C_{14}\Lambda(4) \frac{(\beta_m+p-1)^p}{\beta_m^p} \frac{1}{(r_m - r_{m+1})^p}\right)^{\frac{1}{\beta_m+p-1}} \phi\left(\frac{(\beta_m+p-1)s}{s-1}, r_m, u\right)
\end{aligned}$$

$$\le [C_{14}\Lambda(4)]^{\frac{1}{\delta^m(\beta_0+p-1)}} 2^{\frac{pm+p}{\delta^m(\beta_0+p-1)}} (\frac{\beta_0+p-1}{\beta_0})^{\frac{p}{\delta^m(\beta_0+p-1)}} \delta^{\frac{mp}{\delta^m(\beta_0+p-1)}}$$
$$\times \phi(\frac{(\beta_m+p-1)s}{s-1}, r_m, u)$$
$$\le \left[2\delta C_{14}\frac{\beta_0+p-1}{\beta_0}\right]^{\sum_{k=0}^{m}\frac{pk+p}{\delta^k(\beta_0+p-1)}} \Lambda(4)^{\sum_{k=0}^{m}\frac{1}{\delta^k(\beta_0+p-1)}} \phi(\frac{(\beta_0+p-1)s}{s-1}, r_0, u).$$
$$\le \left[2\delta C_{14}\frac{\beta_0+p-1}{\beta_0}\right]^{\sum_{k=0}^{\infty}\frac{2k+4}{\delta^k(\beta_0+p-1)}} \Lambda(4)^{\frac{\delta}{(\delta-1)(\beta_0+p-1)}} \phi(\gamma, 2, u).$$

Therefore, we have

$$\sup_{x\in B_1} u \le C_{15}(\Lambda(4))^{\frac{1}{\gamma\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}} \phi(\gamma, 2, u). \qquad \square$$

3. Proof of Theorem 1.3

We prove the theorem by two following lemmas. In these lemmas we assume all conditions in the theorem are satisfied, and $x_0 = 0$, $R = 1$ and $u \ge \varepsilon > 0$.

Lemma 3.1. *Under the hypothesis of Theorem* 1.3 *we have*

$$\exp(-C_{23}\Lambda(8R)^{\frac{1}{t^*(\frac{p}{n}-\frac{1}{s}-\frac{1}{t})}+1}) \exp(|B_4|^{-1}\int_{B_4} \log u dx) \le \inf_{B_2} u. \tag{3.1}$$

Proof. For any positive real number s we put

$W(B_s) = \{v \in W_0^{1,p}(B_s) \cap L^\infty(B_s) : v \ge 0, Dv \in L^\infty(B_s), \mathrm{supp}(v) \subset\subset B_s\}$.

Let η be in $W(B_8)$. Using $\varphi = \eta u^{-(p-1)}$ as test functions in (1.5), we obtain

$$\int_{B_8} a(x)|\nabla(\log\frac{k}{u})|^{p-2}\nabla(\log\frac{k}{u})\nabla\eta dx = -\int_{B_8} a(x)\left|\frac{\nabla u}{u}\right|^{p-2}\frac{\nabla u\nabla\eta}{u}dx$$
$$\le -\int_{B_8} a(x)|\nabla u|^{p-2}\nabla u(\frac{\nabla\eta}{u^{p-1}} - (p-1)\eta\frac{\nabla u}{u^p})dx$$
$$= -\int_{B_8} a(x)|\nabla u|^{p-2}\nabla u\nabla(\frac{\eta}{u^{p-1}})dx \le 0 \qquad \forall\ k > 0. \tag{3.2}$$

Since $W(B_8)$ is dense in $H_0(B_8)$, we get

$$\int_{B_8} a(x)|\nabla(\log\frac{k}{u})|^{p-2}\nabla(\log\frac{k}{u})\nabla\eta dx \le 0 \qquad \forall\ k > 0,\ \forall\ \eta \in H_0(B_8). \tag{3.3}$$

Hence $\log\dfrac{k}{u}$ is a subsolution of (1.4) in B_8. Thus $(\log\dfrac{k}{u})^+$ also is a subsolution of (1.4) in B_8. By Lemma 2.2, we have

$$\sup_{x\in B_2}\log\frac{k}{u}\leq\sup_{x\in B_2}(\log\frac{k}{u})^+\leq C_{16}(\Lambda(8))^{\overline{\frac{1}{t^*\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}}}\phi(t^*,4,(\log\frac{k}{u})^+). \tag{3.4}$$

Let η be in $W(B_8)$. Using $\varphi\equiv\dfrac{\eta^p}{u^{p-1}}$ as test functions in (1.5) we get

$$\int_{B_8}a(x)|\nabla u|^p(p-1)\eta^pu^{-p}dx\leq\int_{B_8}a(x)|\nabla u|^{p-2}\nabla u\nabla\eta\; p\left(\frac{\eta}{u}\right)^{p-1}dx$$
$$\leq\frac{1}{2}\int_{B_8}a(x)|\nabla u|^p(p-1)\eta^pu^{-p}dx+C_{17}\int_{B_8}a(x)|\nabla\eta|^pdx.$$

Therefore,

$$\begin{aligned}\int_{B_8}\lambda|\nabla(\log(\frac{k}{u}))|^p\eta^pdx&=\int_{B_8}\lambda|\nabla u|^p\eta^pu^{-p}dx\\ &\leq\int_{B_8}a(x)|\nabla u|^p\eta^pu^{-p}dx\leq C_{18}\int_{B_8}a(x)|\nabla\eta|^pdx\\ &\leq C_{18}\int_{B_8}\mu|\nabla\eta|^pdx.\end{aligned}$$

For any $0<r'<r<2r'\leq 8$, choosing $\eta\in C_c^1(B_8)$ with the following properties: $0\leq\eta\leq 1$, $\eta(x)=1$ for all x in B_4, $\eta(x)=0$ for all x in $\mathbb{R}^n\setminus B_6$, and $|\nabla\eta|\leq 2$, we have

$$\int_{B_4}\lambda|\nabla(\log(\frac{k}{u}))|^pdx\leq C_{19}\int_{B_6}\mu dx. \tag{3.5}$$

By choosing $k=\exp(\dfrac{1}{|B_4|}\displaystyle\int_{B_4}\log u dx)$, we get

$$\left(\int_{B_4}|\log(\frac{k}{u})|^{t^*}\right)^{\frac{p}{t^*}}\leq C_{20}\|\frac{1}{\lambda}\|_{L^t(B_4)}\int_{B_4}\lambda|\nabla(\log(\frac{k}{u}))|^pdx. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we have

$$\begin{aligned}\sup_{x\in B_2}\log\frac{k}{u}&\leq C_{21}(\Lambda(8))^{\overline{\frac{1}{t^*\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}}}\|\frac{1}{\lambda}\|_{L^t(B_4)}\int_{B_6}\mu dx\\ &\leq C_{22}(\Lambda(8))^{\overline{\frac{1}{t^*\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}}+1}.\end{aligned}$$

Hence

$$(\inf_{B_2} u)^{-1} \leq \exp(C_{23}(\Lambda(8))^{\frac{1}{t^*\left(\frac{p}{n}-\frac{1}{s}-\frac{1}{t}\right)}+1} - |B_4|^{-1}\int_{B_4} \log u dx), \tag{3.7}$$

which implies the lemma. □

Lemma 3.2. *Under the hypothesis of Theorem* 1.3 *we have*

$$\phi(\gamma, 1, u) \leq C_{24}\exp(C_{25}\Lambda(8)^{\frac{1}{t^*(\frac{p}{n}-\frac{1}{s}-\frac{1}{t})}+1})\exp(|B_4|^{-1}\int_{B_4} \log u dx). \tag{3.8}$$

Proof. Let β be a real number > 1 and η be a nonnegative function in $C_c^1(B_8)$. Put

$$w = (\log \frac{u}{k})^+, \tag{3.9}$$

$$\varphi(x) = \eta^p u^{-(p-1)}(w^\beta(x) + (\frac{2}{p-1}\beta)^\beta). \tag{3.10}$$

It is easy to see that $\varphi \in H_0(B_8)$, and

$$\nabla\varphi = p\eta^{p-1}\frac{\nabla\eta}{u^{p-1}}(w^\beta + (\frac{2\beta}{p-1})^\beta) + \eta^p u^{-p}[\beta w^{\beta-1} - (p-1)(w^\beta + (\frac{2\beta}{p-1})^\beta)]\nabla u.$$

Taking φ as a test function for (1.5) and using the inequality $\beta w^{\beta-1} \leq \frac{p-1}{2}(w^\beta + (\frac{2}{p-1}\beta)^\beta)$, we have

$$\begin{aligned}&\int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}\left((p-1)(w^\beta + (\frac{2}{p-1}\beta)^\beta) - \beta w^{\beta-1}\right)dx\\ &\leq \int_{B_8} a(x)|\nabla u|^{p-1}|\nabla\eta|\ p\eta^{p-1}u^{-(p-1)}\left(w^\beta + (\frac{2}{p-1}\beta)^\beta\right)dx.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}\frac{p-1}{2}(w^\beta + (\frac{2}{p-1}\beta)^\beta)dx\\ &\leq \int_{B_8} a(x)|\nabla u|^{p-1}|\nabla\eta|\ p\eta^{p-1}u^{-(p-1)}(w^\beta + (\frac{2}{p-1}\beta)^\beta)dx.\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_{B_8} a(x)|\nabla u|^{p-1}|\nabla\eta|\ p\eta^{p-1}u^{-(p-1)}(w^\beta + (\frac{2}{p-1}\beta)^\beta)dx\\ &= \int_{B_8} a(x)|\nabla u|^{p-1}|\nabla\eta|\ p\eta^{p-1}u^{-(p-1)}w^\beta dx\end{aligned}$$

$$+\int_{B_8} a(x)|\nabla u|^{p-1}|\nabla\eta|\ p\eta^{p-1}u^{-(p-1)}(\frac{2}{p-1}\beta)^{\beta}dx$$
$$\leq \frac{(p-1)^2}{4(p+1)}\int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}w^{\beta-1}dx + C_{26}\int_{B_8} a(x)|\nabla\eta|^p(w)^{\beta+p-1}dx$$
$$+\frac{(p-1)^2}{4(p+1)}\int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}(\frac{2}{p-1}\beta)^{\beta}dx + C_{27}\int_{B_8} a(x)|\nabla\eta|^p(\frac{2}{p-1}\beta)^{\beta}dx.$$

These inequalities imply

$$\int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}\left[\frac{p-1}{2}w^{\beta}+\frac{p-1}{2}\left(\frac{2\beta}{p-1}\right)^{\beta}-\frac{(p-1)^2w^{\beta-1}}{4(p+1)}-\frac{(p-1)^2}{4(p+1)}\left(\frac{2\beta}{p-1}\right)^{\beta}\right]dx$$
$$\leq C_{26}\int_{B_8} a(x)|\nabla\eta|^p(w)^{\beta+p-1}dx + C_{27}\int_{B_8} a(x)|\nabla\eta|^p\left(\frac{2}{p-1}\beta\right)^{\beta}dx.$$

Therefore,

$$\begin{aligned}\int_{B_8}\lambda|\nabla u|^p\eta^p u^{-p}\beta w^{\beta-1}dx &\leq \int_{B_8} a(x)|\nabla u|^p\eta^p u^{-p}\beta w^{\beta-1}dx\\ &\leq C_{28}\int_{B_8} a(x)|\nabla\eta|^p\left(w^{\beta+p-1}+\left(\frac{2}{p-1}\beta\right)^{\beta}\right)dx\\ &\leq C_{28}\int_{B_8}\mu|\nabla\eta|^p\left(w^{\beta+p-1}+\left(\frac{2}{p-1}\beta\right)^{\beta}\right)dx.\end{aligned}$$

Put $q=\dfrac{\beta+p-1}{p}$. We have $pq-p+1=\beta>1$, $|\nabla u|^p u^{-p}\beta w^{\beta-1}=|\nabla w^q|^p\dfrac{pq+1-p}{q^p}$ and

$$\int_{B_8}\lambda|\nabla(\eta w^q)|^p dx \leq 2^{p-1}\left[\int_{B_8}\lambda|\nabla w^q|^p\eta^p dx+\int_{B_8}\lambda|\nabla\eta|^p w^{pq}dx\right]$$
$$\leq C_{28}\frac{q^p}{pq+1-p}\int_{B_8}\mu|\nabla\eta|^p\left(w^{pq}+\left(\frac{2pq+2-2p}{p-1}\right)^{pq-p+1}\right)\eta^p dx$$
$$+2^{p-1}\int_{B_8}\mu|\nabla\eta|^p w^{pq}dx$$
$$\leq C_{29}\ q^p\int_{B_8}\mu|\nabla\eta|^p\left(w^{pq}+\left(\frac{2pq+2-2p}{p-1}\right)^{pq-p+1}\right)dx.$$

Arguing as in (2.13), we get

$$(\int_{B_8}(\eta w^q)^{t^*}dx)^{p/t^*} \le C_{30}\, q^p\Lambda(B_8)[\int_{B_8}(|\nabla\eta|^p(w^{pq}+(\frac{2pq+2-2p}{p-1})^{pq-p+1}))^{\frac{s}{s-1}}dx]^{\frac{s-1}{s}}$$

$$\le C_{31}\Lambda(8)q^p[(\int_{B_8}(|\nabla\eta|^p w^{pq})^{\frac{s}{s-1}}dx)^{\frac{s-1}{s}}+(\int_{B_8}(|\nabla\eta|^p(\frac{2pq+2-2p}{p-1})^{pq-p+1})^{\frac{s}{s-1}}dx)^{\frac{s-1}{s}}].$$

For any $0 < r' < r < 2r' \le 8$, choose $\eta \in C_c^1(B_8)$ such that $0 \le \eta \le 1$, $\eta(x) = 1$ for any x in $B_{r'}$, $\eta(x) = 0$ for any x in $\mathbb{R}^n \setminus B_r$, and $|\nabla\eta| \le \frac{4}{r-r'}$. Since $q > 1$, we have

$$(\int_{B_{r'}} w^{qt^*}dx)^{\frac{1}{qt^*}} = [(\int_{B_{r'}} w^{qt^*}dx)^{p/t^*}]^{\frac{1}{pq}}$$

$$\le [\frac{C_{32}\Lambda(8)q^p}{(r-r')^p}]^{\frac{1}{pq}}[(\int_{B_r} w^{\frac{pqs}{s-1}}dx)^{\frac{s-1}{s}}+(\int_{B_r}\left(\frac{2pq+2-2p}{p-1}\right)^{\frac{(pq-p+1)s}{s-1}}dx)^{\frac{s-1}{s}}]^{\frac{1}{pq}}$$

$$\le [\frac{C_{32}\Lambda(8)q^p}{(r-r')^p}]^{\frac{1}{pq}}[(\int_{B_r} w^{\frac{pqs}{s-1}}dx)^{\frac{s-1}{s}}+\left(\frac{2pq+2-2p}{p-1}\right)^{pq-p+1}|B_r|^{\frac{s-1}{s}}]^{\frac{1}{pq}}$$

$$\le [\frac{C_{32}\Lambda(8)q^p}{(r-r')^p}]^{\frac{1}{pq}}\left[(\int_{B_r} w^{\frac{pqs}{s-1}}dx)^{\frac{s-1}{pqs}}+\frac{2pq}{p-1}\right].$$

Therefore,

$$\phi(qt^*, r', w) \le \left(\frac{C_{32}\Lambda(8)q^p}{(r-r')^p}\right)^{\frac{1}{pq}}\left(\phi(\frac{pqs}{s-1}, r, w)+\frac{2pq}{p-1}\right). \tag{3.11}$$

Put $\delta = t^*\frac{s-1}{ps}$,$q_0 = 1$, $q_{m+1} = \frac{t^*(s-1)}{ps}q_m = \delta^{m+1}$, $r_m = 2(1+\frac{1}{2^m})$ for any positive integer m. Since $\frac{1}{t}+\frac{1}{s}<\frac{p}{n}$, we have $\delta > 1$, $\beta_m > 0$ and

$$\phi(q_{m+1}t^*, r_{m+1}, w) \le (\frac{C_{32}\Lambda(8)q_{m+1}^p}{(r_m-r_{m+1})^p})^{\frac{1}{pq_{m+1}}}(\phi(\frac{pq_{m+1}s}{s-1}, r_m, w)+\frac{2pq_{m+1}}{p-1})$$

$$= (\frac{C_{32}\Lambda(8)q_{m+1}^p}{(r_m-r_{m+1})^p})^{\frac{1}{pq_{m+1}}}(\phi(q_m t^*, r_m, w)+\frac{2pq_{m+1}}{p-1})$$

$$\le \prod_{k=0}^{m}(C_{32}\Lambda(8)\delta^{p(k+1)}2^{pk})^{\frac{1}{p\delta^{k+1}}}(\phi(t^*, 4, w)+\frac{2p}{p-1}\sum_{k=0}^{m}q_{k+1})$$

or

$$\phi(q_{m+1}t^*, r_{m+1}, w) \le C_{33}(\Lambda(8))^{\frac{1}{p(\delta-1)}}(\phi(t^*, 4, w)+\frac{2p}{p-1}\delta\frac{\delta^{m+1}-1}{\delta-1}). \tag{3.12}$$

Let z be in $[t^*, \infty)$, there exists an integer $m \geq 0$ such that $q_m t^* \leq z < q_{m+1} t^*$. We have

$$\phi(z,2,w) \leq |B_2|^{\frac{\delta-1}{t^*}} \phi(q_{m+1}t^*, 2, w) \leq |B_2|^{\frac{\delta-1}{t^*}} \phi(q_{m+1}t^*, r_{m+1}, w)$$

$$\leq C_{34} \left(\Lambda(8)\right)^{\frac{1}{p(\delta-1)}} \left(\phi(t^*,4,w) + z\right). \tag{3.13}$$

Applying Fatou's lemma, we obtain

$$\int_{B_2} e^{zw} dx \leq \liminf_{k\to\infty} \int_{B_2} \sum_{i=0}^{k} \frac{(zw)^i}{i!} dx = \liminf_{k\to\infty} \sum_{i=0}^{k} \frac{(z)^i}{i!} \int_{B_2} w^i dx$$

$$\leq \liminf_{k\to\infty} \sum_{i=0}^{k} \frac{(z)^i}{i!} (C_{34}(\Lambda(8))^{\frac{1}{p(\delta-1)}} (\phi(t^*,4,w) + i))^i$$

$$\leq \liminf_{k\to\infty} \sum_{i=0}^{k} \frac{(z)^i}{i!} (C_{34}(\Lambda(8))^{\frac{1}{p(\delta-1)}})^i 2^i (\phi(t^*,4,w)^i + i^i).$$

Choose $z = (4e)^{-1}(C_{34}(\Lambda(8))^{\frac{1}{p(\delta-1)}})^{-1}$. Applying Stirling's formula (see [9], p.52), (3.5) and (3.6), we have

$$[\phi(z,2,\frac{u}{k})]^z = \int_{B_2} e^{zw} dx \leq \liminf_{k\to\infty} \sum_{i=0}^{k} (\frac{1}{2e})^i \frac{1}{i!} \left(\phi(t^*,4,w)^i + i^i\right) \tag{3.14}$$

$$= e^{C_{35}\phi(t^*,4,w)} + C_{36} \leq e^{C_{37}\Lambda(8)} + C_{36}.$$

Let $\beta \in (1-p, 0)$ and $\eta \in C_c^1(B_4)$. Using $\varphi \equiv \eta^p u^\beta$ as a test function in (1.5), and arguing as in (2.17) we have

$$\phi(t^* \frac{\beta+p-1}{p}, r', u) \leq [(\frac{(\beta+p-1)^p}{|\beta|^p} + 1)\frac{C_{37}\Lambda(4)}{(r-r')^p}]^{\frac{1}{\beta+p-1}} \phi(\frac{(\beta+p-1)s}{s-1}, r, u).$$

Denote $\delta^* = \dfrac{ps}{t^*(s-1)}$, $\beta_0 = \dfrac{p\gamma}{t^*} - p + 1$, $\dfrac{s}{s-1}(\beta_m + p - 1) = \dfrac{t^*}{p}(\beta_{m+1} + p - 1)$ and $r_m = (2 - \dfrac{1}{2^m})$. Since $\dfrac{1}{t} + \dfrac{1}{s} < \dfrac{p}{n}$ and $(\dfrac{(\beta_m+p-1)^p}{|\beta_m|^p}) \leq (\dfrac{(\beta_0+p-1)^p}{|\beta_0|^p})$, we have $\delta^* < 1$ and $0 > \beta_m > 1-p$ and

$$\phi(\frac{(\beta_m+p-1)t^*}{p}, r_m, u) \leq$$

$$\leq [\frac{C_{38}\Lambda(4)}{(r_{m+1}-r_m)^p}(\frac{(\beta_m+p-1)^p}{|\beta_m|^p} + 1)]^{\frac{1}{\beta_m+p-1}} \phi(\frac{(\beta_m+p-1)s}{s-1}, r_{m+1}, u)$$

$$\leq [\frac{C_{38}\Lambda(4)}{(r_{m+1}-r_m)^p}(\frac{(\beta_m+p-1)^p}{|\beta_m|^p} + 1)]^{\frac{1}{\beta_m+p-1}} \phi(\frac{(\beta_{m+1}+p-1)t^*}{p}, r_{m+1}, u).$$

$$\leq [C_{39}\Lambda(4)2^{pm}]^{\frac{1}{\beta_m+p-1}} \phi(\frac{(\beta_{m+1}+p-1)t^*}{p}, r_{m+1}, u).$$

This implies

$$\phi(\frac{(\beta_0+p-1)t^*}{p}, r_0, u) \le \prod_{k=0}^{m} [C_{39}\Lambda(4)2^{pk}]^{\frac{1}{\beta_k+p-1}} \phi(\frac{(\beta_{m+1}+p-1)t^*}{p}, r_{m+1}, u)$$

$$\le \prod_{k=0}^{m} [C_{39}\ \Lambda(4)\ 2^{pm}]^{\frac{1}{\beta_k+p-1}} \phi(\frac{(\beta_{m+1}+p-1)t^*}{p}, r_{m+1}, u).$$

Let $z \in [t^*, \infty)$, there exists a nonnegative integer N such that $\delta^{*N+1}\gamma < z \le \delta^{*N}\gamma$. Since $N < \frac{\log z - \log\gamma}{\log \delta^*}$ and

$$\sum_{k=0}^{N} \frac{1}{\beta_k+p-1} = \sum_{k=0}^{N} \frac{1}{\delta^{*k}(\beta_0+p-1)} = \frac{1}{(\beta_0+p-1)} \frac{1}{\delta^{*N}} \frac{1-\delta^{*N+1}}{1-\delta^*}$$
$$\le \frac{1}{(\beta_0+p-1)} \frac{\gamma}{z(1-\delta^*)},$$

we get

$$\phi(\gamma,1,u) \le [C_{39}\Lambda(4) 2^{p\frac{\log z - \log\gamma}{\log\delta^*}}]^{\frac{t^*}{pz(1-\delta^*)}} \phi(\frac{(\beta_{N+1}+p-1)t^*}{p}, r_{N+1}, u)$$
$$\le [C_{39}\ \Lambda(4)\ 2^{p\frac{\log z - \log\gamma}{\log\delta^*}}]^{\frac{t^*}{pz(1-\delta^*)}} |B_2|^{\frac{(1-\delta^*)}{z\delta^*}} \phi(z,2,u)$$
$$\le \left[C_{40}\ \Lambda(4)\ 2^{p\frac{\log z - \log\gamma}{\log\delta^*}}\right]^{\frac{t^*}{pz(1-\delta^*)}} \phi(z,2,u). \tag{3.15}$$

Combining (3.14) and (3.15), we get

$$\phi(\gamma,1,\frac{u}{k}) \le [C_{40}\Lambda(4)\ 2^{p\frac{\log z - \log\gamma}{\log\delta^*}}]^{\frac{t^*}{pz(1-\delta^*)}} [e^{C_{37}\Lambda(8)} + C_{36}]^{\frac{1}{z}}.$$

Since $\frac{\log x}{x} \le \frac{1}{e}$ and $\frac{x^\alpha}{e^x} \le C(\alpha)$ for any positive real numbers x and α, we obtain

$$\phi(\gamma,1,\frac{u}{k}) \le C_{41} \left[C_{42} e^{C_{43}\Lambda(8)}\right]^{\frac{1}{z}}$$
$$\le C_{41} \left[\frac{1}{\Lambda(8)} C_{42}\Lambda(8)\ e^{C_{43}\Lambda(8)}\right]^{C_{44}(\Lambda(8))^{\frac{1}{2(\delta-1)}}}$$
$$\le C_{41} \left[\frac{1}{\Lambda(8)}\right]^{C_{44}(\Lambda(8))^{\frac{1}{2(\delta-1)}}} \left[e^{C_{45}\Lambda(8)}\right]^{C_{44}(\Lambda(8))^{\frac{1}{2(\delta-1)}}}$$
$$\le C_{46}[e^{C_{45}\Lambda(8)}]^{C_{44}(\Lambda(8))^{\frac{1}{2(\delta-1)}}} \le C_{46}\exp[C_{47}\Lambda(8)^{1+\frac{1}{2(\delta-1)}}],$$

which automatically implies (3.8). □

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