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A MODIFIED CLASSICAL ALGORITHM ALPT4C FOR SOLVING A CAPACITATED FOUR-INDEX TRANSPORTATION PROBLEM

AAID DJAMEL, NOUI AMEL, LE THI HOAI AN AND ZIDNA AHMED

Abstract. In this paper, we focus on the theoretical study and numerical solution of a capacitated four-index transportation problem. This model is not properly treated before, it is linked to significant practical problems, some theoretical properties have been shown. We constructed an algorithm for solving the problem, the implementation of the algorithm with its description has led to an encouraging finding for the digital outcome. On the other hand, the initialization method we used avoids unnecessary assignment and the detective method addresses adequately the problems of degeneracy by making the algorithm robust.

1. Introduction

In 1947 G. B. Dantzig has proposed the simplex method: a very effective technique for solving linear programming problems. Since that time, linear programming has raised great interest among researchers who have written hundreds of books and published thousands of articles on the subject. Although the complexity of this method is exponential, in practice it has proven very effective and very useful, especially in the areas of planning and organization. In 1979 the Soviet mathematician L. G. Khachiyan has implemented the first polynomial algorithm for linear programming: the algorithm of ellipsoids. However, the proposed algorithm has proven completely ineffective in practice. In 1984 N. Karmarkar has revolutionized the field of linear programming by implementing a polynomial algorithm based on methods of internal penalties. It has been a serious competitor to the simplex algorithm. Since then, an intensive research was initiated in this area and gave as result a large variety of algorithms of this type.

This work focuses on a linear problem which is particularly important because it corresponds to real and practical transportation problems which appear in different fields such as: economy, telecommunication, localization and assignment etc...

The transport problem has been formulated for the first time by F. Hitchcock in 1941. In 1949 Kantorovich and Savourine gave a first method to solve such

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a problem known potentials, regardless of the simplex method applied by G. B. Dantzig to solve this problem in 1951.

The transportation problem with two indices has been widely studied in the literature, then the study has been extended to more than two-index problems. Since the sixties, several studies have been published on the transport problem with three indices and more generally on the multiple indices without capacities. The problem of axial transport (sum of axial) with l indices (PTL) has not been widely studied for $l \geq 3$, in particular the capacitated four-index transportation problem. This case has not been previously addressed whether in its theoretical, algorithmic or numerical aspects. In 2003 R. Zitouni and A. Keraghel introduced for the first time, a method of solving a capacitated transport problem with four indices [16] [17]. On one hand, this algorithm is a variant of the simplex algorithm designed specifically to address this problem, it attacks directly the problem without any reformulation, but each time it calls for the initialization phase, and therefore the algorithm reiterates a considerable number of times which costs numerically. In order to remove the problem of degeneracy, the algorithm requires an exponential number of permutations of columns each time to test the independence of the selected columns. Hence this process cannot generate a base, which makes the algorithm diverge in the case of the degenerated problem of large sizes. On the other hand, the other existing methods require a reformulation and a problem preparation which increase the problem size and this accumulate the number of operations that lead to the optimal solution [1] [3].

Below we indicate the contribution of the suggested method.

(1) Once we exploit the characteristics and the theoretical particularities of the problem, our algorithm resolves directly the problem as it is without increasing its size and without any needs for a reformulation across the operations. This fact is less time-consuming as we compare it with the existing methods.

(2) Bearing in mind the degeneracy problem which is practically the most frequent, our method uses a detective method that generates a base in the worst cases and this allows to initiate an algorithm.

(3) In the initial phase, one uses the new heuristic method which differs from the existing methods such as: northern corner, minimal cost, Vogel and Russell method. This heuristic method uses assignments if it is necessary. Consequently, one has access to fewer operations in order to find out a feasible solution of the base. In practice, we obtain the optimal solution during the first phase.

(4) One can deal and derive from the dual problem the stop criterion. At the end of the algorithm, we obtain primal and dual solutions which are

practically very important.

(5) One may use techniques of resolution that fit the theoretical particularities and go beyond the various difficulties in order to obtain the mathematical tools which are necessary to improve the current solution.

2. The problem position

Given m origins $A_1, \ldots, A_m$ of availabilities $\alpha_1, \ldots, \alpha_m$, n destinations $B_1, \ldots, B_n$ of demands $\beta_1, \ldots, \beta_n$, p means of transportation chosen suitably $S_1, \ldots, S_p$ of reserved charges $\gamma_1, \ldots, \gamma_p$ and q qualities of the goods taken in even units $H_1, \ldots, H_q$ of quantities $\delta_1, \ldots, \delta_q$. We denote by d_{ijkl} the capacities of the roads of transport and by c_{ijkl} the cost unit of transport of a quantity x_{ijkl} of goods H_l transported from the origin A_i towards the destination B_j through the means of transportation S_k.

2.1. **The problem formulation.**

The capacitated four-index transportation problem that we denote by $(C4TP)$ is formulated as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} c_{ijkl}x_{ijkl} \tag{2.1}$$

subject to the constraints:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \text{ for all } i = 1, ..., m \tag{2.2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \text{ for all } j = 1, ..., n \tag{2.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \text{ for all } k = 1, ..., p \tag{2.4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \text{ for all } l = 1, ..., q \tag{2.5}$$

$$0 \leq x_{ijkl} \leq d_{ijkl} \text{ for all } (i, j, k, l). \tag{2.6}$$

In this problem, α_i, β_j, γ_k, δ_l, d_{ijkl} and c_{ijkl} are given and are such that for all i,j,k,l, we have $\alpha_i > 0$, $\beta_j > 0$, $\gamma_k > 0$, $\delta_l > 0$, $d_{ijkl} > 0$ and $c_{ijkl} \geq 0$. This formulation is equivalent to the following linear program:

$$\min[\ c^t x : Ax = b, \quad 0 \leq x \leq d\ ],$$

where $x = (x_{ijkl})^t \in R^N$, $c = (c_{ijkl})^t \in R^N$, $d = (d_{ijkl})^t \in R^N$, $b = (\alpha_i, \beta_j, \gamma_k, \delta_l) \in R^M$, and A is a $M \times N$ matrix with $M = m + n + p + q$ and $N = mnpq$.

2.2. **Conditions of feasibility** [15]

(1) For the problem $C4TP$ to have a feasible solution, the following necessary conditions should be verified :

$$\sum_{i=1}^{m}\alpha_i = \sum_{j=1}^{n}\beta_j = \sum_{k=1}^{p}\gamma_k = \sum_{l=1}^{q}\delta_l = H \tag{2.7}$$

$$\begin{aligned} \alpha_i &\leq \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} d_{ijkl} \text{ for all } i = 1, ..., m, \\ \beta_j &\leq \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} d_{ijkl} \text{ for all } \ j = 1, ..., n, \\ \gamma_k &\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} d_{ijkl} \text{ for all } \ k = 1, ..., p, \\ \delta_l &\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} d_{ijkl} \text{ for all } \ l = 1, ..., q, \end{aligned} \tag{2.8}$$

(2) For the problem $C4TP$ to have a feasible solution, the sufficient condition (2.7), and the following condition should be verified:

$$\frac{\alpha_i\beta_j\gamma_k\delta_l}{d_{ijkl}} \leq H^3 \qquad \text{for all } (i, j, k, l). \tag{2.9}$$

2.3. **Conditions of optimality** [15]**.** Suppose the problem $C4TP$ is feasible, then a feasible solution x of this problem is optimal if and only if there exists a vector

$$(u_1 \ , \ldots , u_m, v_1, \ldots , v_n, \ldots , w_1, \ldots , w_p, \ldots , t_1, \ldots , t_q)^t$$

in $\mathbb{R}^N$ such that

$$\begin{aligned} u_i + v_j + w_k + t_l &\leq c_{ijkl} && \text{if } x_{ijkl} = 0 \\ u_i + v_j + w_k + t_l &= c_{ijkl} && \text{if } \ 0 < x_{ijkl} < d_{ijkl} \\ u_i + v_j + w_k + t_l &\geq c_{ijkl} && \text{if } x_{ijkl} = d_{ijkl}. \end{aligned} \tag{2.10}$$

2.4. **Constraints matrix**

Lemma 2.1. *The constraints matrix is of* rank $M - 3$.

Proof. Let us prove by recurrence that the rank of the matrix A is equal to $M-3$.
1) Suppose that we have a problem of transport $C4TP$ with $m = n = p = q = 1$, then $A = [1111]^t$ and rank $(A) = 4 - 3 = 1 = M - 3$.
2) Let us suppose that this lemma is true for the problem $C4TP$ with $M = \eta \geq 4$, and let us prove that it is true also for the problem with $M = \eta + 1$, bearing in mind that the last constituent of the last row is non-zero, we denote $\rho \neq 0$ (see the table of transport [18]). According to the hypothesis of recurrence we can extract a $(\eta - 3) \times (\eta - 3)$ sub-matrix A_η of size such that $\det(A_\eta) \neq 0$.

Afterward we permute the last row (column) by the $(\eta-2)^{nd}$ row (column), we obtain the sub-matrix $A_{(\eta+1)}$ which is the sub-matrix A_η, increased by the new row and the new column, then we have $\det(A_{(\eta+1)}) = \rho \times \det(A_\eta) \neq 0$. So we can extract a $(\eta-2)\times(\eta-2)$ sub-matrix of size whose determinant is non-zero, then rank $(A) \geq \eta - 2 = M - 3$. On the other hand, from the condition of feasibility (2.7), we have rank $(A) \leq M-3$. We conclude that rank $(A) = M-3$. □

3. The algorithm description for solving

To handle the problem of degeneracy, one proposes the following method:

3.1. **The detective method** (DM). Let N_b be the number of vectors P_{ijkl} such that $0 < x_{ijkl} < d_{ijkl}$. If $N_b < M-3$, then the problem $C4TP$ is degenerated, we consider the matrix D that consists of M rows and N columns constructed of N_b first vectors P_{ijkl} such that $0 < x_{ijkl} < d_{ijkl}$ and the remaining vectors P_{ijkl} such that $x_{ijkl} = 0$ or $x_{ijkl} = d_{ijkl}$.

One considers the sub-matrix D_B which contains $M-3$ first columns of D. If rank $(D_B) = M-3$, then D_B is a base, otherwise, we proceed as follows in order to detect the vectors of the base: let $N_d = M - 3 - \text{rank}\ (D_B)$.

Consider the sub-matrix D_L consisting of N_b first columns of the matrix D, from the $(N_{b+1})^{nd}$ vector of the matrix D. Let $P_{ijkl} = P_d$ (from the left to the right) with $d = 1, \ldots, N - N_b$, consider the matrix increased $[D_L, P_1, \ldots, P_d]$; if rank $[D_L, P_1, \ldots, P_d] >$ rank $[D_L, P_1, \ldots, P_{d-1}]$, then P_d is a basic vector, incrementing d until the N_d basic vectors are found and consequently a base will be determined : $I_b = \{(i,j,k,l)$ such that P_{ijkl} are independent$\}$. We divide the boxes which are not interesting in two disjoint sets $H_0 = \{(i,j,k,l)$ such that $x_{ijkl} = 0\}$, $H_d = \{(i,j,k,l)$ such that $x_{ijkl} = d_{ijkl}\}$. To determine the boxes that form a cycle, one uses the following resolution technique.

3.2. **The cycle resolution technique** (CRT). The vector $P_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}}$ which enters the base is taken as coefficient

$$\alpha_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}} = \begin{cases} 1 \text{ if } (\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) \in H_0 \\ -1 \text{ if } (\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) \in H_d. \end{cases}$$

Taking into account of the equation $c_{ijkl} = u_i + v_j + w_k + t_l\ \forall (i,j,k,l) \in I_b$, we construct the following linear system:

$BX = b$ where B is a $M \times (M-3)$ matrix consisting of the vectors P_{ijkl} with $(i,j,k,l) \in I_b$, i.e. B is a base, $X = (\alpha_{ijkl})^t \in R^{M-3}$ and $b \in R^M$;

$$b = \begin{cases} P_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}} \ \text{ if } (\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) \in H_0 \\ -P_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}} \text{ if } (\bar{\imath},\bar{\jmath},\bar{k},\bar{l}) \in H_d\ . \end{cases}$$

As the matrix B possesses M rows and $M-3$ columns and the vector of the unknown X possesses $M-3$ components, then the system is not compatible, one proceeds as follows in order to remove this difficulty. Considering the increased matrix $[B, b]$, by using Gauss elimination with only permutation of rows, we get a staggered matrix $[\tilde{B}, \tilde{b}]$ of $M-3$ rows and $M-2$ columns. Now we can solve the

linear system $\tilde{B}X = \tilde{b}$. Solutions $\alpha_{ijkl} \neq 0$ of the previous system determine the boxes forming the cycle. Let $F = \{(i, j, k, l)$ such that $\alpha_{ijk} = 1\}$, $B = \{(i, j, k, l)$ such that $\alpha_{ijk} = -1\}$, the set F contains receivable boxes and the set B contains sending boxes.
In order to calculate the dual variables we use the following resolution technique:

3.3. **The resolution technique of the dual variables** $(RTDV)$. We consider the following linear system:

$$c_{ijkl} = u_i + v_j + w_k + t_l \; \forall (i, j, k, l) \in I_b.$$

As the system contains more unknowns (M) than equations $(M - 3)$, it admits an infinity of solutions. We construct the system: $AX = b$ such that for all $(\bar{\imath}, \bar{\jmath}, \bar{k}, \bar{l}) \in I_b$ we have

- if $I \leq M - 3$

$$a_{iJ} = \begin{cases} 1 & \text{if } i = \bar{\imath} & \text{for all} & J \leq m, \\ 1 & \text{if } j = \bar{\jmath} & \text{for all} & m < J \leq m + n, \\ 1 & \text{if } k = \bar{k} & \text{for all} & m + n < J \leq m + n + p, \\ 1 & \text{if } l = \bar{l} & \text{for all} & m + n + p < J \leq m + n + p + q, \\ 0 & \text{elsewhere.} \end{cases}$$

- if $I > M - 3$
  $a_{iJ} = 0$ except $a_{1,M-2} = 1$, $a_{m+1,M-1} = 1$, $a_{m+n+1,M} = 1$.

3.4. **The stop criterion**

Lemma 3.1. *Let define the following difference for all* $(i, j, k, l) \notin I_b$

$$D_{ijkl} = \begin{cases} u_i + v_j + w_k + t_l - c_{ijkl} & \textit{for all } (i, j, k, l) \in H_0 \\ c_{ijkl} - (u_i + v_j + w_k + t_l) & \textit{for all } (i, j, k, l) \in H_d. \end{cases}$$

and let the set $\Pi = \{D_{ijkl} \;\; \textit{as } D_{ijkl} > 0, \; \textit{for all } (i, j, k, l) \in (H_0 \cup H_d)\}$. *If* $\Pi \neq \emptyset$, *then the current solution* $x = (x_{ijkl})$ *is not an optimal solution.*

Proof. Suppose $\Pi = \emptyset$, that is to say, for all $(i, j, k, l) \in (H_0 \cup H_d)$, we have $D_{ijkl} \leq 0$, which indicates that

$$\begin{cases} u_i + v_j + w_k + t_l - c_{ijkl} \leq 0 & \text{for all } (i, j, k, l) \in H_0 \\ \text{and} \\ c_{ijkl} - (u_i + v_j + w_k + t_l) \leq 0 & \text{for all } (i, j, k, l) \in H_d. \end{cases}$$

That is to say

$$\begin{array}{cc} u_i + v_j + w_k + t_l - c_{ijkl} \leq 0 & \text{if } x_{ijkl} = 0 \\ \text{and} & \\ c_{ijkl} - (u_i + v_j + w_k + t_l) \leq 0 & \text{if } x_{ijkl} = d_{ijkl}. \end{array} \tag{3.1}$$

According to the theorem of optimality, the solution x_{ijkl} verifying (2.10) is optimal. Consequently, if $\Pi \neq \emptyset$, then the solution x_{ijkl} is not optimal and hence it can be improved. $\square$

3.5. **Calculation of a basic feasible solution.** By giving a feasible problem $C4TP$, initially all variables are zero, this means no goods has passed yet. To find a feasible solution, we order the costs in an ascending order; we assign a value to the variable x_{ijkl} corresponding to the first box, indicating that this box is not taken into account in the next step, using an indicator a_{ijkl} that takes the value 0 if the x_{ijkl} is concerned by the assignment and the value 1 otherwise. On the other hand, we update our settings, so if $\alpha_i \neq 0$, $\beta_j \neq 0$, $\gamma_k \neq 0$ and $\delta_l \neq 0$, so it is necessary to assign a value to that variable x_{ijkl}. Otherwise, we pass to the following box that verifies the necessary condition of assignment. We repeat this process until all the concerned boxes are affected. Consequently we obtain a basic feasible solution to our problem. We summarize the previous techniques in an algorithm called algorithm of assignment if it is necessary for a capacitated four-index problem of transportation that we denote by $(AAIN_{C4PT})$.

4. The algorithm $AAIN_{C4PT}$

Phase 1

(1) **Initialization:**
- $x_{ijkl} = 0$, $a_{ijkl} = 0$ $\forall (i,j,k,l,)$
- $N_b = 0$
- order the costs c_{ijkl} in an ascending way

(2) **Iteration:**

While $(\exists (i,j,k,l)$ such that $(\alpha_i \beta_j \gamma_k \delta_l > 0 \quad \text{and} \quad a_{ijkl} = 0))$ **do**
- Take $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) =$ first box
- Take $x_{\overline{ijkl}} = \min(\alpha_{\bar{i}}, \beta_{\bar{j}}, \gamma_{\bar{k}}, \delta_{\bar{l}}, d_{\overline{ijkl}})$, $a_{\overline{ijkl}} = 1$
  if $x_{\overline{ijkl}} < d_{\overline{ijkl}}$ then $N_b = N_b + 1$
- Update $\alpha_{\bar{i}}, \beta_{\bar{j}}, \gamma_{\bar{k}}, \delta_{\bar{l}}$ as follows:
  $\alpha_{\bar{i}} = \alpha_{\bar{i}} - x_{\overline{ijkl}}$, and the same for $\beta_{\bar{j}}, \gamma_{\bar{k}}, \delta_{\bar{l}}$

  End while.

(3) If $N_b < M - 3$ apply DM in order to determine a base I_b

Phase 2

(1) **Initialization:**
- Calculate the u_i, v_j, w_k and t_l by using $RTDV$
- Determine the set Π by using the lemma 2 (Stop Criterion)

(2) **Iteration:**

While $(\Pi \neq \emptyset)$ **do**
- Find (i_0, j_0, k_0, l_0) such that $D_{i_0 j_0 k_0 l_0} = \max\limits_{(i,j,k,l)} \Pi$
- Use the CRT in order to determine the coefficients of the cycle.
- Calculate $\theta^* = \min \left\{ \min\limits_{(i,j,k,l) \in B} (x_{ijkl}), \min\limits_{(i,j,k,l) \in F} (d_{ijkl} - x_{ijkl}) \right\} = \theta_{\overline{ijkl}}$
- Update the current solution and the basic boxes.

$$- \; x_{ijkl} = \begin{cases} x_{ijkl} + \alpha_{ijkl}\, \theta^* & \text{if } (i,j,k,l) \in (F \cup B) \\ x_{ijkl} & \text{otherwise} \end{cases}$$

– $I_B = \{I_B \backslash \{(\bar{i}, \bar{j}, \bar{k}, \bar{l})\} \cup \{(i_0, j_0, k_0, l_0)\}\}$

• Calculate again u_i, v_j, w_k , t_l and determine the set Π.

End while.

End algorithm.

Lemma 4.1. *Let* $x^{(r)}$ *and* $x^{(r+1)}$ *be two consecutive non degenerated feasible solutions. Then their corresponding objective functions* $Z^{(r)}$ *and* $Z^{(r+1)}$ *satisfy*

(1) $Z^{(r+1)} = Z^{(r)} - \theta^{(r)} D^{(r+1)}_{i_0 j_0 k_0 l_0}$,

(2) $Z^{(r+1)} < Z^{(r)}$.

Proof. There are two cases to be considered : the first case with $(i_0, j_0, k_0, l_0) \in H_0$ and the second case with $(i_0, j_0, k_0, l_0) \in H_d$. We put $\sigma^{(r)} = \{(i, j, k, l)$ boxes forming the cycle$\}$. Let consider the first case. We have

$$Z^{(r+1)} = \sum_{(i,j,k,l)\in\sigma^{(r)}} c_{ijkl}\; x^{(r+1)}_{ijkl} + \sum_{(i,j,k,l)\notin\sigma^{(r)}} c_{ijkl}\; x^{(r)}_{ijkl}.$$

Then

$$Z^{(r+1)} = K + \sum_{(i,j,k,l)\in\sigma^{(r)}} c_{ijkl}\; (x^{(r+1)}_{ijkl} + \alpha^{*}_{ijkl}),$$

where

$$K = \sum_{(i,j,k,l)\notin\sigma^{(r)}} c_{ijkl}\; x^{(r)}_{ijkl}.$$

So if $\alpha_{i_0 j_0 k_0 l_0} = 1$ and $\widehat{\sigma}^{(r)} = \sigma^{(r)} - \{(i_0, j_0, k_0, l_0)\}$ then

$$Z^{(r+1)} = Z^{(r)} - \theta^{(r)}(c_{i_0 j_0 k_0 l_0} + \sum_{(i,j,k,l)\in\widehat{\sigma}^{(r)}} \alpha_{ijkl}\; (u^{(r)}_i + v^{(r)}_j + w^{(r)}_k + t^{(r)}_l). \tag{4.1}$$

Let

$$\begin{array}{rcl} i(\widehat{\sigma}^{(r)}) & = & \{(j,k,l) \text{ such that } (i,j,k,l) \in \widehat{\sigma}^{(r)}\} \\ j(\widehat{\sigma}^{(r)}) & = & \{(i,k,l) \text{ such that } (i,j,k,l) \in \widehat{\sigma}^{(r)}\} \\ k(\widehat{\sigma}^{(r)}) & = & \{(i,j,l) \text{ such that } (i,j,k,l) \in \widehat{\sigma}^{(r)}\} \\ l(\widehat{\sigma}^{(r)}) & = & \{(i,j,k) \text{ such that } (i,j,k,l) \in \widehat{\sigma}^{(r)}\}. \end{array}$$

It is easy to verify that for all $i \neq i_0$, we have

$$\sum_{(i,j,k,l)\in i(\widehat{\sigma}^{(r)})} \alpha_{ijkl} = 0, \tag{4.2}$$

and for $i = i_0$,

$$u^{(r)}_{i_0} \left(\sum_{(i_0,j,k,l)\in\widehat{\sigma}^{(r)}} \alpha_{i_0 jkl} + 1 \right) = 0.$$

Consequently,

$$\sum_{(i_0,j,k,l)\in\widehat{\sigma}^{(r)}} \alpha_{i_0 jkl}\; u^{(r)}_{i_0} = -u^{(r)}_{i_0}.$$

We obtain results similar to (4.1) and (4.2) for j, k, l. Therefore

$$\sum_{(i,j,k,l)\in i(\widehat{\sigma}^{(r)})} \alpha_{ijkl}(u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + t_l^{(r)}) = -(u_0^{(r)} + v_0^{(r)} + w_0^{(r)} + t_0^{(r)}).$$

By substituting this value in (4.1), we obtain

$$Z^{(r+1)} = Z^{(r)} - \theta^{(r)} D^{(r+1)}_{i_0j_0k_0l_0}.$$

This shows that $Z^{(r+1)} < Z^{(r)}$ since $\theta^{(r)} > 0$ and $D^{(r+1)}_{i_0j_0k_0l_0} > 0$. □

One can prove in a similar way the second case $(i_0, j_0, k_0, l_0) \in H_d$.

Theorem 4.2. *Suppose that the problem is not degenerated, then the algorithm $AAIN_{C4PT}$ converges in a finite number of iterations.*

Proof. The previous lemma shows that the algorithm $AAIN_{C4PT}$ guarantees that the same base can never appear in two distinct iterations and as the number of visited vertices is necessarily finite, the algorithm converges and its convergence is over. □

5. Numerical tests

(i,j,k,l)	1111	1121	1211	1122	1221	2111	2121	2221
d_{ijkl}	15	10	8	9	7	11	11	9
c_{ijkl}	5	4	6	7	2	1	5	4

Table 1. Quantities c_{ijkl} and d_{ijkl} of the problem

Problem	**Size** $M \times N$	**Number of iterations**		**Optimal value** z^*		**Execution time in seconds**	
		$AAIN_{C4TP}$	AL_{PTAC}	$AAIN_{C4TP}$	AL_{PTAC}	$AAIN_{C4TP}$	AL_{PTAC}
1	8×16	3	38	9	9	0	0
2	9×24	3	45	11	11	0	0.015
3	10×36	5	55	140	140	0	0.015
4	11×54	7	66	180	1.031	0	0.015
5	12×81	7	76	186	186	0	0.047
6	13×108	8	87	193	193	0	0.078
7	14×144	8	98	2950	2950	0	0.141
8	18×360	12	152	253	253	0	0.687
9	19×600	13	185	1100	1100	0	6.125
10	20×648	14	184	269	269	0	1.406
11	21×720	14	202	235.93	235.93	0	8.578
12	21×750	14	203	237.75	237.75	0	2.734
13	22×900	14	220	242.18	242.18	0	4.312

Table 2. Comparison of the two methods $AAIN_{C4PT}$ and AL_{PT4C} with sample problems generated randomly

Firstly, let consider a transportation problem with C4TP of the form: $m = n = p = 2$, $q = 1$, $\alpha_1 = 5$, $\alpha_2 = 10$, $\beta_1 = 9$, $\beta_2 = 6$, $\gamma_1 = 12$, $\gamma_2 = 3$, $\delta_1 = 15$. The quantities c_{ijkl} and d_{ijkl} are given in Table 1 above. In this illustrative example,

Example Number	**Size of Problem** $M \times N$	**Number of Iterations**	**Optimal value** z^*	**Execution Time in seconds**
14	23×1080	6	189.063	0.062
15	24×1296	6	212.625	0.110
16	25×1512	7	7150	0.125
17	26×1764	7	3777.5	0.187
18	27×2058	7	8425	0.265
19	28×2401	7	2744	0.407
20	29×2744	8	3096	0.422
21	30×3136	8	492.5	0.610
22	31×3584	8	401	0.843
23	32×4096	8	714	1.187
24	33×4608	9	113.25	1.250
25	34×5184	9	691.5	1.719
26	35×5832	9	351.75	2.297
27	36×6561	9	477	3.078
28	37×7290	10	130.75	3.250
29	38×8100	10	266	4.312
30	39×9000	10	405.75	5.610
31	40×10000	10	550	7.282
32	41×11000	11	298.5	7.625
33	42×12100	11	303.5	9.828
34	43×13310	11	115.688	12.469
35	44×14641	11	117.536	15.782
36	45×15972	12	42.1875	16.469
37	46×17424	12	42.875	20.781
38	47×19008	12	30.688	25.871
39	48×20736	12	44.25	32.016
40	49×22446	13	47.3125	33.281

Table 3. Numerical result for the proposed method $AAIN_{C4PT}$ with sample problem generated randomly

we only present the optimal solution by their non-zero components. Solving this problem by $AAIN_{C4PT}$ leads to the results: The non-zero components of the optimal solution $x^* = (x_{ijkl})$ are: $x_{1211} = 5$, $x_{2111} = 6$, $x_{2121} = 3$, and $x_{2211} = 1$. The number of iterations $= 2$. The time of execution $= 0,000$ sec. The optimal value is $z^* = 50$.

Secondly, we compare our method to the method (AL_{PT4C}) described in [18], on 40 sample problems with different sizes as shown in Tables 2 and 3 above. All these problems are generated randomly. We note that the degenerated problems of large sizes affected highly the method AL_{PT4C}. Beyond the problem 14 of size (23×108), the algorithm AL_{PT4C} could not reach an optimal solution. This means that it could not go beyond the degeneracy phenomenon. That is why, in Table 2, we summarize the numerical results corresponding to the problems 1 to

13, which are concerned with the two methods. In Table 3, we only expose the results of our method for the problems 14 to 40. Our tests are performed on an Intel (R) Pentium (R) Dual Windows and the programs are written in C.

Through numerical tests that we made, we realize the stability and robustness of our algorithm against the existing method [1] [3]. However, for the degenerated problems (most common in practice), the algorithm uses the detective method $AAIN_{C4PT}$, previously introduced, to overcome this phenomenon.

6. Conclusion

A new algorithm for solving a problem of transport capacity with four indices has been proposed in this paper. For problems of variant size, the optimal solution is often reached during the initialization phase, as it avoids unnecessary assignments and uses a detective method to handle the problems of degeneracy, making the algorithm very robust even in the worst cases. We were able to innovate successfully the solving techniques which play a crucial role in improving an initial basic solution for the optimum.

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Department of Mathematics, University of Constantine, Algeria
E-mail address: djamelaaid@gmail.com

Department of Mathematics, University of Sétif,, Algeria
E-mail address: amelnoui@gmail.com

Université Paul Verlaine - Metz, Ile du Saulcy, 57045 Metz, France
E-mail address: lethi@univ-metz.fr

Université Paul Verlaine - Metz, Ile du Saulcy, 57045 Metz, France
E-mail address: zidna@univ-metz.fr