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A NOTE ON WEAKLY KOSZUL MODULES

YUAN PAN

Abstract. Let M be a weakly Koszul module. Then Martínez-Villa and Zacharia proved that M admitted a filtration of submodules $0 = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_p = M$, such that all U_{i+1}/U_i are Koszul modules (see [11]). Now further, let $\mathcal{P}^i_* \to U_i/U_{i-1} \to 0$ and $\mathcal{P}_* \to M \to 0$ be the corresponding minimal graded projective resolutions. We prove that, for all $n \geq 0$, $\mathcal{P}_n \cong \bigoplus_i \mathcal{P}^i_n$. Moreover, we also give a new characterization for a module M to be weakly Koszul in terms of the filtration of the complex $\mathcal{P}_*$, where $\mathcal{P}_* \to M \to 0$ is a minimal graded projective resolution.

1. Introduction

Throughout, $\Bbbk$ denotes an arbitrary field, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers, respectively. Each graded $\Bbbk$-algebra $A = \bigoplus_{i\geq 0} A_i$ is assumed with the following properties: (1) $A_0 = \Bbbk \times \cdots \times \Bbbk$, a finite product of $\Bbbk$; (2) $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$ and (3) each A_i is of finite dimension as a $\Bbbk$-space. The graded Jacobson radical of such a graded algebra A is obvious $\bigoplus_{i\geq 1} A_i$. Let $Gr(A)$ and $gr(A)$ denote the categories of graded A-modules and finitely generated graded A-modules, respectively.

Weakly Koszul modules, a natural generalization of Koszul modules, were first introduced by Martínez-Villa and Zacharia in [11] and one of the main results of [11] was to prove that weakly Koszul modules can be approximated by modules with linear resolutions:

- Let A be a Koszul algebra, M an arbitrary finitely generated graded A-module, and $\{S_{d_1}, S_{d_2}, \cdots, S_{d_p}\}$ a set of minimal homogeneous generating spaces of M. Suppose that $S_{d_i} \subseteq M_{d_i}$, $d_i \in \mathbb{N}$ for $1 \leq i \leq p$, and $d_1 < d_2 < \cdots < d_p$. Consider the filtration

$$\mathcal{F}M: \quad 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M$$

of M, where $U_1 = \langle S_{d_1}\rangle$, $U_2 = \langle S_{d_1}, S_{d_2}\rangle$, $\cdots$, $U_p = \langle S_{d_1}, S_{d_1}, \cdots, S_{d_p}\rangle$. Then M is a weakly Koszul module if and only if, for all $1 \leq i \leq p$, U_i/U_{i-1} are Koszul modules.

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Recently, this notion has been generalized to d-Koszul modules and piecewise-Koszul modules and so on. It turned out that such class of modules also admits a lot of good homological properties similar to Koszul modules and we refer to [4]-[8] for the further details.

Now let us recall some definitions, most of the materials of this section can be found in [11].

Definition 1.1. Let A be a graded algebra and $M \in gr(A)$. We call M a *Koszul module* if there exists a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M and a fixed integer s such that for each $n \geq 0$, P_n is generated in degree $n+s$. In particular, if the trivial A-module A_0 is a Koszul module, then we call A a *Koszul algebra.*

Definition 1.2. Let A be a Koszul algebra. We say that $M \in gr(A)$ is a *weakly Koszul module* if there exists a minimal graded projective resolution

$$\cdots \longrightarrow P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

of M such that for $i, k \geq 0$, $J^k \ker f_i = J^{k+1} P_i \cap \ker f_i$.

From the above definitions, we know that both Koszul modules and weakly Koszul modules can be defined in terms of their minimal graded projective resolutions. Motivated by the above result, one can ask the following questions: Do there exist some relationships between the minimal graded projective resolutions of M and these U_i/U_{i-1}'s? Can we characterize weakly Koszul modules in terms of the resolutions of M and U_i/U_{i-1}'s?

In this paper, we mainly answer the above questions and the following are the main results:

Theorem 1.3. *Let M be a weakly Koszul module and $\mathcal{F}M : 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M$ its submodule filtration. Let $\mathcal{P}_* \to M \to 0$ and $\mathcal{P}_*^i \to U_i/U_{i-1} \to 0$ be the minimal graded projective resolutions. Then for $n \geq 0$, we have*

$$\mathcal{P}_n \cong \bigoplus_{i=1}^{p} \mathcal{P}_n^i.$$

Theorem 1.4. *Let $M \in gr(A)$ and*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

a minimal projective resolution of M. Set

$$\mathcal{P}_* := \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to 0.$$

Then M is a weakly Koszul module if and only if the complex $\mathcal{P}_*$ has a filtration

$$\mathcal{F}\mathcal{P}_* : 0 = \mathcal{P}_*^0 \subset \mathcal{P}_*^1 \subset \cdots \subset \mathcal{P}_*^{p-1} \subset \mathcal{P}_*^p = \mathcal{P}_*,$$

such that for all $1 \leq j \leq p$,

$$\cdots \to P_i^j/P_i^{j-1} \to P_{i-1}^j/P_{i-1}^{j-1} \to \cdots \to P_1^j/P_1^{j-1} \to P_0^j/P_0^{j-1} \to 0$$

has only one non-zero homology K_j *at* P_0^j/P_0^{j-1}*, which is a Koszul module. In fact,* $P_i = \bigoplus_{j=1}^p P_i^j/P_i^{j-1}$*. Moreover,* M *has a filtration,* $\mathcal{F}M : 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M$ *such that* $U_j/U_{j-1} \cong K_j$ *and all* K_j *are Koszul modules.*

2. Proofs of the main results

Lemma 2.1. ([3]) *Let* A *be a Koszul algebra and* $M \in gr(A)$ *generated in a single degree. Let*

$$\mathcal{P} = \cdots \longrightarrow P_n \xrightarrow{f_n} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal graded projective resolution of M*. Then the following two statements are equivalent:*

(a) M *is a Koszul module.*

(b) $\mathcal{P}$ *satisfies that* $J\ker f_n = J^2P_n \cap \ker f_n$ *for all* $n \geq 0$*.*

Lemma 2.2. *Let* A *be a graded algebra and*

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an exact sequence in $Gr(A)$*. Then the following are equivalent for some* $k \geq 0$*:*

(a) $J^kK = K \cap J^kM$*;*

(b) $A/J^k \otimes_A K \to A/J^k \otimes_A M$ *is a monomorphism;*

(c) $0 \to J^kK \to J^kM \to J^kN \to 0$ *is exact;*

(d) $0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$ *is exact;*

(e) $0 \to J^kK/J^mK \to J^kM/J^mM \to J^kN/J^mN \to 0$ *is exact for all* $m > k$*.*

Proof. The equivalence of (a) and (b) has been proved in [11]. We only prove (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). For all $k \geq 0$, $J^kK = K \cap J^kM$ is equivalent to that the sequence

$$0 \longrightarrow J^kK \longrightarrow J^kM \longrightarrow J^kN \longrightarrow 0$$

is exact, which is equivalent to that the following diagram with exact rows and columns is commutative

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^m K & \longrightarrow & J^m M & \longrightarrow & J^m N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^k K & \longrightarrow & J^k M & \longrightarrow & J^k N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^k K/J^m K & \longrightarrow & J^k M/J^m M & \longrightarrow & J^k N/J^m N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $m > k$ and $m \geq 0$. Now by "3×3" Lemma, we are done. □

Lemma 2.3. *Let A be a graded algebra and* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *an exact sequence in $gr(A)$ with $JK = K \cap JM$. Then we have the following commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_0, Q_0 and L_0 are graded projective covers, $\Omega^1(K)$, $\Omega^1(M)$ and $\Omega^1(N)$ are the first syzygies of K, M and N respectively.

Proof. We can obtain the exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0$$

since $JK = K \cap JM$ by Lemma 2.2. Note that for a finitely generated graded module over a positively graded algebra, M/JM is the minimal generating space of M and we denote $M/JM := S_M = S_M^{d_1} \oplus S_M^{d_2} \oplus \cdots \oplus S_M^{d_p}$, where $S_M^{d_i}$ is the set of homogeneous elements of M of degree d_i and the "$\oplus$" is with respect to A_0-modules. Therefore, we get an exact sequence of A_0-modules

$$0 \longrightarrow S_K \longrightarrow S_M \longrightarrow S_N \longrightarrow 0.$$

It should be noted that

$$A \otimes_{A_0} S_K \xrightarrow{f} K \longrightarrow 0\ , \quad f(\sum a_i \otimes s_K^i) = \sum a_i \cdot s_K^i,$$

$$A \otimes_{A_0} S_M \xrightarrow{g} M \longrightarrow 0\ , \quad g(\sum a_i \otimes s_M^i) = \sum a_i \cdot s_M^i$$

and

$$A \otimes_{A_0} S_N \xrightarrow{h} N \longrightarrow 0\ , \quad h(\sum a_i \otimes s_N^i) = \sum a_i \cdot s_N^i$$

are graded projective covers since it is clear that $A \otimes_{A_0} S_K$, $A \otimes_{A_0} S_M$ and $A \otimes_{A_0} S_N$ are graded projective A-modules.

Now set $P_0 := A \otimes_{A_0} S_K$, $Q_0 := A \otimes_{A_0} S_M$ and $L_0 := A \otimes_{A_0} S_N$. We have the following exact sequence since A_0 is semisimple

$$0 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow L_0 \longrightarrow 0\ .$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the columns, the middle and the bottom rows are exact. Now by "3×3" Lemma, we get the following exact sequence

$$0 \longrightarrow \Omega^1(K) \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(N) \longrightarrow 0.$$

Therefore, we get the desired diagram. □

Lemma 2.4. ([11]) *Let $M = \bigoplus_{i\geq 0} M_i$ be a weakly Koszul module with $M_0 \neq 0$. Set $K = \langle M_0 \rangle$. Then*

(a) *K is a Koszul module;*

(b) *$K \cap J^k M = J^k K$ for each $k \geq 0$;*

(c) *M/K is a weakly Koszul module.*

Lemma 2.5. *Let $M = \bigoplus_{i\geq 0} M_i$ be a weakly Koszul module with $M_0 \neq 0$, $K := \langle M_0 \rangle$ and $N := M/K$. Then we have the following commutative diagram with* *exact rows and columns*

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_2 & \longrightarrow & P_2 \oplus Q_2 & \longrightarrow & Q_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the columns are minimal graded projective resolutions of K, M and N, respectively.

Proof. By Lemma 2.4 (b), we get $JK = K \cap JM$. By Lemma 2.3, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where P_0, Q_0 and L_0 are graded projective covers, $\Omega^1(K)$, $\Omega^1(M)$ and $\Omega^1(N)$ are the first syzygies of K, M and N, respectively. Of course, $L_0 = P_0 \oplus Q_0$ since the sequence $0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow Q_0 \longrightarrow 0$ is exact and Q_0 is a graded projective module.

Clearly, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JP_0 & \longrightarrow & JL_0 & \longrightarrow & JQ_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Applying the functor $A/J\otimes_A$ to the above diagram, note that M and N are weakly Koszul modules, by Lemmas 2.4 and 2.2, we get the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & A/J\otimes_A \Omega^1(K) & \xrightarrow{\beta} & A/J\otimes_A \Omega^1(M) & \longrightarrow & A/J\otimes_A \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/J\otimes_A JP_0 & \longrightarrow & A/J\otimes_A JL_0 & \longrightarrow & A/J\otimes_A JQ_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A/J\otimes_A JK & \longrightarrow & A/J\otimes_A JM & \longrightarrow & A/J\otimes_A JN & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By Lemma 2.4, K is a Koszul module. By Lemma 2.1, $J\Omega^1(K) = \Omega^1(K)\cap J^2P_0$. By Lemma 2.2, α is a monomorphism, which implies that β is also a monomorphism. By Lemma 2.2 again, we have $J\Omega^1(K) = \Omega^1(K)\cap J\Omega^1(M)$. Now repeating the above argument and by an easy induction, we finish the proof. □

Now we are ready to prove Theorem 1.3.

Proof. Consider the following exact sequence

$$0 \longrightarrow U_1 \longrightarrow M \longrightarrow M/U_1 \longrightarrow 0.$$

By Lemma 2.5, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^1 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & M & \longrightarrow & M/U_1 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}_*^1$, $\mathcal{P}_*$ and $\mathcal{L}_*^1$ are the minimal graded projective resolutions of U_1, M and M/U_1, respectively. Clearly, $\mathcal{P}_* = \mathcal{P}_*^1 \oplus \mathcal{L}_*^1$. Set $W = M/U_1$. Then $\langle W_{d_2} \rangle = U_2/U_1$. Consider the following exact sequence

$$0 \longrightarrow U_2/U_1 \longrightarrow W \longrightarrow W/(U_2/U_1) \longrightarrow 0.$$

By Lemma 2.5 again, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^2 & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & \mathcal{L}_*^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_2/U_1 & \longrightarrow & W & \longrightarrow & W/(U_2/U_1) & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}_*^2$, $\mathcal{L}_*^1$ and $\mathcal{L}_*^2$ are the minimal graded projective resolutions of U_2/U_1, W and $W/(U_2/U_1)$, respectively. Clearly, $\mathcal{L}_*^1 = \mathcal{P}_*^2 \oplus \mathcal{L}_*^2$. Repeating the above argument and by induction, we are done. □

Lemma 2.6. *Let* $M = \bigoplus_{i \geq 0} M_i$ *be a weakly Koszul module with its natural filtration:*

$$\mathcal{F}M: \quad 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M.$$

Then for each exact sequence, $0 \longrightarrow U_j \longrightarrow U_{j+1} \longrightarrow U_{j+1}/U_j \longrightarrow 0$, *we also have the similar conclusion and commutative diagram stated in Lemma 2.5.*

Proof. By Lemma 2.5, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^1 & \longrightarrow & \mathcal{P}_*^2 & \longrightarrow & \mathcal{Q}_*^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_2/U_1 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ are the minimal graded projective resolutions of U_1, U_2 and U_2/U_1, respectively. Clearly, for each i, the terms P^1_i, P^2_i and Q^2_i in the complexes $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ respectively satisfy $P^2_i = P^1_i \oplus Q^2_i$.

Similarly, we also have the following two commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{Q}^2_* & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{Q}^3_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_2/U_1 & \longrightarrow & U_3/U_1 & \longrightarrow & U_3/U_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{Q}^2_*$, $\mathcal{Q}$ and $\mathcal{Q}^3_*$ are the minimal graded projective resolutions of U_2/U_1, U_3/U_1 and U_3/U_2, respectively,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}^1_* & \longrightarrow & \mathcal{P}^3_* & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & U_3 & \longrightarrow & U_3/U_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}^1_*$, $\mathcal{P}^3_*$ and $\mathcal{Q}$ are the minimal graded projective resolutions of U_2/U_1, U_3/U_1 and U_3/U_2, respectively. If we further denote the terms in complexes $\mathcal{P}^3_*$ and $\mathcal{Q}^3_*$ by P^3_i and Q^3_i, then it is clear that $P^3_i = P^1_i \oplus Q^2_i \oplus Q^3_i$.

For the exact sequence $0 \longrightarrow U_2 \longrightarrow U_3 \longrightarrow U_3/U_2 \longrightarrow 0$, by 'Horseshoe Lemma', we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}^2_* & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{Q}^3_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & U_3/U_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where $\mathcal{P}^2_*$ and $\mathcal{Q}^3_*$ are the minimal graded projective resolutions of U_2 and U_3/U_2, respectively. For each term P_i in $\mathcal{P}_*$, it is clear that $P_i = P^2_i \oplus Q^3_i = P^1_i \oplus Q^2_i \oplus Q^3_i$, which shows that $\mathcal{P}_*$ is the minimal graded projective resolution of U_3. Then we can get the desired result by induction. □

Now we can prove Theorem 1.4.

Proof. ($\Rightarrow$) Suppose $M = \bigoplus_{i\geq 0} M_i$ is a weakly Koszul module with its natural filtration:

$$\mathcal{F}M: \quad 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M.$$

By Lemma 2.6, we can get the following commutative diagram with exact columns,

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{P}_*^1 & \xrightarrow{\subset} & \mathcal{P}_*^2 & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & \mathcal{P}_*^{p-1} & \xrightarrow{\subset} & \mathcal{P} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \xrightarrow{\subset} & U_2 & \xrightarrow{\subset} & \cdots & \xrightarrow{\subset} & U_{p-1} & \xrightarrow{\subset} & M & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

which induces minimal graded projective resolutions of $U_j/U_{j-1} = K_j$ for each $1 \leq j \leq p$,

$$\cdots \to P_i^j/P_i^{j-1} \to P_{i-1}^j/P_{i-1}^{j-1} \to \cdots \to P_1^j/P_1^{j-1} \to P_0^j/P_0^{j-1} \to K_j \to 0.$$

Thus, each K_j is a Koszul module.

($\Leftarrow$) If $M \in gr(A)$ has the minimal projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

and the complex

$$\mathcal{P}_* : \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to 0$$

has a filtration as stated in the theorem, then it is not hard to check that for each $1 \leq j \leq p$,

$$\mathcal{P}_j : \cdots \to P_i^j \to P_{i-1}^j \to \cdots \to P_1^j \to P_0^j \to 0$$

has only one non-zero homology, say U_j, at P_0^j. Therefore, the filtration of the complex $\mathcal{P}_*$ induces a filtration of the module M:

$$\mathcal{F}M: \quad 0 = U_0 \subset U_1 \subset \cdots \subset U_{p-1} \subset U_p = M.$$

Moreover, for each $1 \leq j \leq p$, we have $K_j = U_j/U_{j-1}$ is a Koszul module. Then of course, M is a weakly Koszul module, as desired. □

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Yiwu Industrial and Commercial College
Yiwu, Zhejiang, 322000, P.R. China

E-mail address: mathpanyuan@126.com