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WEAK LAWS OF LARGE NUMBERS FOR DOUBLE ARRAYS OF RANDOM ELEMENTS IN BANACH SPACES

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Abstract. In this paper, we establish the weak laws of large numbers with or without random indices for double arrays of random elements in Banach spaces. Our results are more general and stronger than some well-known ones.

1. Introduction

Consider a double array $\{V_{mn}; m \geq 1, n \geq 1\}$ of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$. Let $\{u_n; n \geq 1\}$ and $\{v_n; n \geq 1\}$ be sequences of positive integers, let $\{T_n; n \geq 1\}$ and $\{\tau_n; n \geq 1\}$ be sequences of positive integer-valued random variables and let $\{a_{mn}; m \geq 1, n \geq 1\}$ and $\{b_{mn}; m \geq 1, n \geq 1\}$ be arrays of positive numbers with $a_{mn} \uparrow \infty$ and $b_{mn} \uparrow \infty$ as $\max\{m, n\} \to \infty$. In the current work, weak laws of large numbers will be established for double arrays $\max\limits_{1\leq k\leq u_m, 1\leq l\leq v_n} a_{mn}^{-1}\|\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l}(V_{ij} - c_{mnij})\|$ and for double arrays with random indices $a_{mn}^{-1}\sum\limits_{i=1}^{T_m}\sum\limits_{j=1}^{\tau_n}(V_{ij} - c_{mnij})$, where c_{mnij} is a conditional expectation.

Limit theorems for weighted sums (with or without random indices) for random variables (real-valued or Banach space-valued) are studied by many authors. The reader may refer to Wei and Taylor [15], Ordóñez Cabrera [7, 8], Adler et al. [1]. Recently, S. H. Sung et al. [14] obtained the weak law of large numbers with random indices for array of random elements, N. V. Quang and L. H. Son [11] established the weak laws of large numbers for sequences of Banach space valued random elements, N. V. Quang and N. N. Huy [12] established the weak laws of large numbers for adapted double arrays of random variables. In this paper, we establish weak laws of large numbers for double arrays of random elements in martingale type p Banach spaces.

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2. Preliminaries

For $a, b \in \mathbb{R}$, $\min\{a, b\}$ and $\max\{a, b\}$ will be denoted, respectively, by $a \wedge b$, $a \vee b$. Throughout this paper, the symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

Technical definitions relevant to the current work will be discussed in this section. Scalora [13] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element V and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(V|\mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space $\mathcal{X}$ is said to be *martingale type* p $(1 \le p \le 2)$ if there exists a finite positive constant C such that for all martingales $\{S_n; n \ge 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\ge 1} E\|S_n\|^p \le C \sum_{n=1}^{\infty} E\|S_n - S_{n-1}\|^p.$$

It can be shown using classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then for all $1 \le r < \infty$ there exists a finite constant C such that

$$E \sup_{n\ge 1} \|S_n\|^r \le CE \left(\sum_{n=1}^{\infty} \|S_n - S_{n-1}\|^p \right)^{\frac{r}{p}}.$$

Clearly every real separable Banach space is of martingale type 1 and the real line (the same as any Hilbert space) is of martingale type 2.

It follows from the Hoffmann-Jϕrgensen and Pisier [4] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier [9, 10].

The following lemma is needed to prove Lemma 2.2.

Lemma 2.1. *If* $\{X_{kl}, \mathcal{F}_l; l \ge 1\}$, $k = 1, 2, \ldots, m$ *are nonnegative submartingales, then* $\{\max_{1\le k\le m} X_{kl}, \mathcal{F}_l; l \ge 1\}$ *is a nonnegative submartingale.*

Proof. For $L > l \ge 1$,

$$E(\max_{1\le k\le m} X_{kL}|\mathcal{F}_l) \ge \max_{1\le k\le m} E(X_{kL}|\mathcal{F}_l) \ge \max_{1\le k\le m} X_{kl}.$$

□

Let $\mathcal{F}_{kl}$ be the σ-field generated by the family of random elements $\{V_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$. We have the following lemma.

Lemma 2.2. *Let* $0 < p \le 2$. *Let* $\{V_{ij}; 1 \le i \le m, 1 \le j \le n\}$ *be a collection of* mn *random elements in a real separable Banach space. When* $1 < p \le 2$

we assume further that the underlying Banach space is of martingale type p *and* $E(V_{ij}|\mathcal{F}_{ij}) = 0$ *for all* $1 \le i \le m, 1 \le j \le n$. *Then*

$$E \max_{\substack{1\le k\le m\\ 1\le l\le n}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} V_{ij} \right\|^p \le C \sum_{i=1}^{m}\sum_{j=1}^{n} E\|V_{ij}\|^p, \tag{2.1}$$

where the constant C *is independent of* m *and* n.

Proof. We easily obtain the conclusion (2.1) in the case of $0 < p \le 1$.

Now we consider the case of $1 < p \le 2$. In this case, we set $S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l} V_{ij}$, $Y_{ml} = \max_{1\le k\le m} \|S_{kl}\|$. If σ_l is a σ-field generated by $\{V_{ij}; 1 \le i \le m, 1 \le j \le l\}$ then for each $l\,(1 \le l \le n)$, $\sigma_l \subset \mathcal{F}_{i,l+1}$ for all $i \ge 1$. It follows that $E(V_{i,l+1}|\sigma_l) = E(E(V_{i,l+1}|\mathcal{F}_{i,l+1})|\sigma_l) = 0$. Thus, we have

$$E(S_{k,l+1}|\sigma_l) = E(S_{kl}|\sigma_l) + \sum_{i=1}^{k} E(V_{i,l+1}|\sigma_l) = S_{kl}.$$

It means that $\{S_{kl}, \sigma_l; 1 \le l \le n\}$ is a martingale. Hence, $\{\|S_{kl}\|, \sigma_l; 1 \le l \le n\}$ is a nonnegative submartingale for each $k = 1, 2, \ldots, m$, it follows from Lemma 2.1 that $\{Y_{ml}, \sigma_l; 1 \le l \le n\}$ is a nonnegative submartingale. Applying Doob's inequality (see, e.g., Chow and Teicher [2, p. 255]), we obtain

$$E\left(\max_{\substack{1\le k\le m\\ 1\le l\le n}} \|S_{kl}\|^p \right) = E\left(\max_{1\le l\le n} Y_{ml} \right)^p \le CE(Y_{mn})^p. \tag{2.2}$$

On the other hand, since $E(V_{ij}|\mathcal{F}_{ij}) = 0$ we see that $\{S_{kn}, \mathcal{G}_k = \mathcal{F}_{k+1,1}; 1 \le k \le m\}$ is a martingale. Thus

$$E(Y_{mn})^p = E \max_{1\le k\le m} \|S_{kn}\|^p \le C \sum_{k=1}^{m} E\| \sum_{j=1}^{n} V_{kj}\|^p. \tag{2.3}$$

We again have that for each $k\,(1 \le k \le m)$, $\{\sum_{j=1}^{l} V_{kj}, \mathcal{G}_{kl} = \mathcal{F}_{k,l+1}; 1 \le l \le n\}$ is a martingale. Hence,

$$E\| \sum_{j=1}^{n} V_{kj}\|^p \le E \max_{1\le l\le n} \| \sum_{j=1}^{l} V_{kj}\|^p \le C \sum_{l=1}^{n} E\|V_{kl}\|^p. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) yields the conclusion (2.1). □

Random elements $\{V_{mn}; m \ge 1, n \ge 1\}$ are said to be *stochastically dominated* by a random element V if for some finite constant D

$$P\{\|V_{mn}\| > t\} \le DP\{\|DV\| > t\}, \quad t \ge 0,\ m \ge 1,\ n \ge 1.$$

3. The main results

In the following we let $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of random elements defined on a probability $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, $\mathcal{F}_{kl}$ be a σ-field generated by $\{V_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$. Let $\{u_n; n \geq 1\}$, $\{v_n; n \geq 1\}$ be sequences of positive integers such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$ and let $\{a_{mn}; m \geq 1, n \geq 1\}$, $\{b_{mn}; m \geq 1, n \geq 1\}$ be arrays of positive numbers with $a_{mn} \uparrow \infty$ and $b_{mn} \uparrow \infty$ as $m \vee n \to \infty$. For any set A, let $I(A)$ be the indicator function, i.e.

$$I(A)(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ 0 \text{ if } \omega \notin A. \end{cases}$$

Set $V'_{mnij} = V_{ij} I(\|V_{ij}\| \leq b_{mn})$.

Theorem 3.1. *Let $0 < p \leq 2$. When $1 \leq p \leq 2$ we assume further that the underlying Banach space is of martingale type p. If*

$$\sum_{i=1}^{u_m} \sum_{j=1}^{v_n} P\{\|V_{ij}\| > b_{mn}\} \to 0 \text{ as } m \vee n \to \infty \tag{3.1}$$

and

$$\frac{1}{a_{mn}^p} \sum_{i=1}^{u_m} \sum_{j=1}^{v_n} E\|V'_{mnij} - c_{mnij})\|^p \to 0 \text{ as } m \vee n \to \infty, \tag{3.2}$$

then

$$\max_{\substack{1\leq k\leq u_m \\ 1\leq l\leq v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V_{ij} - c_{mnij}) \right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty, \tag{3.3}$$

where $c_{mnij} = 0$ if $0 < p \leq 1$ and $c_{mnij} = E(V'_{mnij}|\mathcal{F}_{ij})$ if $1 < p \leq 2$.

Proof. For an arbitrary $\varepsilon > 0$,

$$\begin{aligned}
&P\left\{ \max_{\substack{1\leq k\leq u_m \\ 1\leq l\leq v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V_{ij} - c_{mnij}) \right\| > \varepsilon \right\} \\
\leq &P\left\{ \max_{\substack{1\leq k\leq u_m \\ 1\leq l\leq v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V_{ij} - V'_{mnij}) \right\| > \varepsilon/2 \right\} \\
&\quad + P\left\{ \max_{\substack{1\leq k\leq u_m \\ 1\leq l\leq v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V'_{mnij} - c_{mnij}) \right\| > \varepsilon/2 \right\} \\
= &P\left\{ \max_{\substack{1\leq k\leq u_m \\ 1\leq l\leq v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V_{ij} I(\|V_{ij}\| > b_{mn}) \right\| > \varepsilon/2 \right\}
\end{aligned}$$

$$+ P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\}$$

$$\le P\left\{\bigcup_{i=1}^{u_m}\bigcup_{j=1}^{v_n}(\|V_{ij}\| > b_{mn})\right\} + P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\}$$

$$\le \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(\|V_{ij}\| > b_{mn}) + P\left\{\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\}$$

$$\le \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(\|V_{ij}\| > b_{mn}) + \frac{2^p}{\varepsilon^p a_{mn}^p} E \max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\|^p$$

(by Markov's inequality)

$$\le \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(\|V_{ij}\| > b_{mn}) + \frac{C}{\varepsilon^p a_{mn}^p} \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E\|(V'_{mnij} - c_{mnij})\|^p$$

(by Lemma 2.2)

$\to 0$ as $m \vee n \to \infty$ (by (3.1) and (3.2)).

The proof is complete. □

Corollary 3.2. *Let $1 \le p \le 2$ and let $\mathcal{X}$ be a martingale type p Banach space. If*

$$\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P\{\|V_{ij}\| > b_{mn}\} \to 0 \text{ as } m \vee n \to \infty,$$

$$\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} E(V'_{mnij}|\mathcal{F}_{ij})\right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty \tag{3.4}$$

and

$$\frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E\|V'_{mnij} - E(V'_{mnij}|\mathcal{F}_{ij}))\|^p \to 0 \text{ as } m \vee n \to \infty,$$

then

$$\max_{\substack{1\le k\le u_m\\1\le l\le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k}\sum_{j=1}^{l} V_{ij}\right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty. \tag{3.5}$$

Remark 3.3. If the condition (3.4) is replaced by the condition

$$\frac{1}{a_{mn}} \left\| \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E(V'_{mnij}|\mathcal{F}_{ij})\right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty,$$

then the conclusion (3.5) will be replaced by

$$\frac{1}{a_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}V_{ij}\xrightarrow{P}0 \text{ as } m\vee n\to\infty.$$

The following result is random indices version of Theorem 3.1.

Theorem 3.4. *Let $0<p\le 2$. When $1\le p\le 2$, we assume further that the underlying Banach space is of martingale type p. Suppose that $\{T_n; n\ge 1\}$ and $\{\tau_n; n\ge 1\}$ are sequences of positive integer-valued random variables such that*

$$\lim_{n\to\infty}P\{T_n>u_n\}=\lim_{n\to\infty}P\{\tau_n>v_n\}=0. \tag{3.6}$$

If

$$\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}P\{\|V_{ij}\|>b_{mn}\}\to 0 \text{ as } m\wedge n\to\infty \tag{3.7}$$

and

$$\frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}E\|V'_{mnij}-c_{mnij}\|^p\to 0 \text{ as } m\wedge n\to\infty, \tag{3.8}$$

then

$$\frac{1}{a_{mn}}\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V_{ij}-c_{mnij})\xrightarrow{P}0 \text{ as } m\wedge n\to\infty,$$

where $c_{mnij}=0$ if $0<p\le 1$ and $c_{mnij}=E(V'_{mnij}|\mathcal{F}_{ij})$ if $1<p\le 2$.

Proof. For arbitrary $\varepsilon>0$,

$$\begin{aligned}
&P\left\{\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V_{ij}-c_{mnij})\|>\varepsilon\right\}\\
\le &P\left\{\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V_{ij}-V'_{mnij})\|>\varepsilon/2\right\}\\
&\quad+P\left\{\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V'_{mnij}-c_{mnij})\|>\varepsilon/2\right\}\\
= &P\left\{\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V_{ij}I(\|V_{ij}\|>b_{mn})\|>\varepsilon/2\right\}\\
&\quad+P\left\{\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V'_{mnij}-c_{mnij})\|>\varepsilon/2\right\}\\
= &A_{mn}+B_{mn}.
\end{aligned}$$

For A_{mn}, we have

$$\begin{aligned}
A_{mn} \le\ & P\left\{\left(\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n} V_{ij}I(\|V_{ij}\| > b_{mn})\| > \varepsilon/2\right)\bigcap(T_m \le u_m)\bigcap(\tau_n \le v_n)\right\} \\
& + P\{T_m > u_m\} + P\{\tau_n > v_n\} \\
\le\ & P\left\{\bigcup_{i=1}^{u_m}\bigcup_{j=1}^{v_n}(\|V_{ij}\| > b_{mn})\right\} + P\{T_m > u_m\} + P\{\tau_n > v_n\} \\
\le\ & \sum_{i=1}^{u_m}\sum_{j=1}^{v_n} P(\|V_{ij}\| > b_{mn}) + P\{T_m > u_m\} + P\{\tau_n > v_n\} \to 0 \text{ as } m \wedge n \to \infty,
\end{aligned}$$

by (3.6) and (3.7).
Next, for B_{mn} we have

$$\begin{aligned}
B_{mn} \le\ & P\left\{\left(\frac{1}{a_{mn}}\|\sum_{i=1}^{T_m}\sum_{j=1}^{\tau_n}(V'_{mnij} - c_{mnij})\| > \varepsilon/2\right)\bigcap(T_m \le u_m)\bigcap(\tau_n \le v_n)\right\} \\
& + P\{T_m > u_m\} + P\{\tau_n > v_n\} \\
\le\ & P\left\{\frac{1}{a_{mn}}\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\} \\
& + P\{T_m > u_m\} + P\{\tau_n > v_n\}.
\end{aligned}$$

By (3.6), in order to prove that $B_{mn} \to 0$ as $m \wedge n \to \infty$, we need to show that

$$P\left\{\frac{1}{a_{mn}}\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\} \to 0 \text{ as } m \wedge n \to \infty.$$

Note that in the case of $1 < p \le 2$, we have $E(V'_{mnij} - E(V'_{mnij}|\mathcal{F}_{ij})|\mathcal{F}_{ij}) = 0$ for all $m \ge 1, n \ge 1, 1 \le i \le u_m$, $1 \le j \le v_n$. Applying Markov's inequality and Lemma 2.2 we obtain

$$\begin{aligned}
& P\left\{\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}}\frac{1}{a_{mn}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\| > \varepsilon/2\right\} \\
\le & \frac{2^p}{\varepsilon^p a_{mn}^p} E\left(\max_{\substack{1\le k\le u_m\\ 1\le l\le v_n}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}(V'_{mnij} - c_{mnij})\right\|^p\right) \\
\le & \frac{C}{\varepsilon^p a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n} E\|V'_{mnij} - c_{mnij})\|^p \to 0 \text{ as } m \wedge n \to \infty \quad \text{(by (3.8))},
\end{aligned}$$

which completes the proof. □

Theorem 3.5. *Let $0 < p \le 2$. When $1 \le p \le 2$ we assume further that the underlying Banach space is of martingale type p. Let $\{k_{mn}; m \ge 1, n \ge 1\}$ be an array of positive integers such that $k_{mn} \to \infty$ as $m \vee n \to \infty$ and*

$$\frac{k_{mn}}{a^p_{mn}} \to 0 \text{ as } m \vee n \to \infty. \tag{3.9}$$

Suppose that there exists a positive nondecreasing function g on $[0, \infty)$ satisfying

$$\lim_{a \to 0} g(a) = 0, \quad \sum_{j=1}^{\infty} g^p(1/j) < \infty \tag{3.10}$$

and

$$\sup_{m \ge 1, n \ge 1} \frac{k_{mn}}{a^p_{mn}} \sum_{j=1}^{k_{mn}-1} \frac{g^p(j+1) - g^p(j)}{j} < \infty. \tag{3.11}$$

If

$$\sup_{a>0} \sup_{m \ge 1, n \ge 1} \frac{1}{k_{mn}} \sum_{i=1}^{u_m} \sum_{j=1}^{v_n} aP\{\|V_{ij}\| > g(a)\} < \infty, \tag{3.12}$$

and

$$\lim_{a \to \infty} \sup_{m \ge 1, n \ge 1} \frac{1}{k_{mn}} \sum_{i=1}^{u_m} \sum_{j=1}^{v_n} aP\{\|V_{ij}\| > g(a)\} = 0, \tag{3.13}$$

then

$$\max_{\substack{1 \le k \le u_m \\ 1 \le l \le v_n}} \frac{1}{a_{mn}} \left\| \sum_{i=1}^{k} \sum_{j=1}^{l} (V_{ij} - c_{mnij}) \right\| \xrightarrow{P} 0 \text{ as } m \vee n \to \infty. \tag{3.14}$$

Moreover, let $\{T_n; n \ge 1\}$ and $\{\tau_n; n \ge 1\}$ be sequences of positive integer-valued random variables satisfying (3.6)*, then*

$$\frac{1}{a_{mn}} \sum_{i=1}^{T_m} \sum_{j=1}^{\tau_n} (V_{ij} - c_{mnij}) \xrightarrow{P} 0 \text{ as } m \wedge n \to \infty, \tag{3.15}$$

where $c_{mnij} = 0$ if $0 < p \le 1$; $c_{mnij} = E(V_{ij} I(\|V_{ij}\| \le g(k_{mn}) | \mathcal{F}_{ij})$ if $1 < p \le 2$.

Proof. By (3.13), take $a = k_{mn}$ and $b_{mn} = g(k_{mn})$ we immediately have (3.1).

Now we need to verify the condition (3.2) with $b_{mn} = g(k_{mn})$. Since g is a nondecreasing function, it follows that

$$\begin{aligned}
&\frac{1}{a^p_{mn}} \sum_{i=1}^{u_m} \sum_{j=1}^{v_n} E\|V'_{mnij} - c_{mnij}\|^p \\
\le &C \frac{1}{a^p_{mn}} \sum_{i=1}^{u_m} \sum_{j=1}^{v_n} E\|V'_{mnij}\|^p \text{ (by } c_r\text{- inequality)}
\end{aligned}$$

$$=C\frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}E\|V_{ij}I(\|V_{ij}\|$$
$$\leq g(1))\|^p + C\frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\sum_{l=2}^{k_{mn}}E\|V_{ij}I(g(l-1) < \|V_{ij}\| \leq g(l))\|^p$$
$$=C.M_{mn} + C.N_{mn}.$$

By (3.9), (3.10) and (3.12), we have

$$\begin{aligned}
M_{mn} &= \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\sum_{l=1}^{\infty}E\|V_{ij}I(g(1/(l+1)) < \|V_{ij}\| \leq g(1/l))\|^p \\
&\leq \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\sum_{l=1}^{\infty}g^p(1/l)P\{g(1/(l+1)) < \|V_{ij}\| \leq g(1/l)\} \\
&\leq \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\left(\sum_{l=2}^{\infty}[(g^p(1/(l-1)) - g^p(1/l))P\{\|V_{ij}\| > g(1/l)\}]\right. \\
&\qquad \left. - g^p(1)P\{\|V_{ij}\| > g(1)\}\right) \\
&\leq \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\left(\sum_{l=2}^{\infty}[(g^p(1/(l-1)) - g^p(1/l))P\{\|V_{ij}\| > g(1/l)\}]\right) \\
&\leq \frac{k_{mn}}{a_{mn}^p}\sum_{l=2}^{\infty}l(g^p(1/(l-1)) - g^p(1/l))\times \\
&\qquad \times \sup_{m\geq 1, n\geq 1}\left\{\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\frac{1}{l}P\{\|V_{ij}\| > g(1/l)\}\right\} \\
&\leq \frac{k_{mn}}{a_{mn}^p}\left(g^p(1) + \sum_{l=1}^{\infty}g^p(1/l)\right)\sup_{a>0}\sup_{m\geq 1, n\geq 1}\left\{\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}aP\{\|V_{ij}\| > g(a)\}\right\} \\
&\to 0 \text{ as } m\vee n\to\infty.
\end{aligned}$$

For N_{mn}, we have

$$\begin{aligned}
N_{mn} &\leq \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\sum_{l=2}^{k_{mn}}g^p(l)P\{g(l-1) < \|V_{ij}\| \leq g(l)\} \\
&\leq g^p(1)\frac{1}{a_{mn}^p}\left(\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}P\{\|V_{ij}\| > g(1)\}\right) \\
&\quad + \frac{1}{a_{mn}^p}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}\left(\sum_{l=1}^{k_{mn}-1}(g^p(l+1) - g^p(l))P\{\|V_{ij}\| > g(l)\}\right)
\end{aligned}$$

$$= g^p(1)\frac{k_{mn}}{a^p_{mn}}\left(\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}P\{\|V_{ij}\| > g(1)\}\right)$$
$$+ \frac{1}{a^p_{mn}}\sum_{l=1}^{k_{mn}-1}\frac{(g^p(l+1)-g^p(l))}{l}\left(\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}lP\{\|V_{ij}\| > g(l)\}\right)$$
$$\le g^p(1)\frac{k_{mn}}{a^p_{mn}}\left(\sup_{a>0}\sup_{m\ge1,n\ge1}\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}aP\{\|V_{ij}\| > g(a)\}\right)$$
$$+ \frac{k_{mn}}{a^p_{mn}}\sum_{l=1}^{k_{mn}-1}\frac{(g^p(l+1)-g^p(l))}{l}\left(\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}lP\{\|V_{ij}\| > g(l)\}\right).$$

Because of (3.9) and (3.12) we have

$$\frac{k_{mn}}{a^p_{mn}}\left(\sup_{a>0}\sup_{m\ge1,n\ge1}\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}aP\{\|V_{ij}\| > g(a)\}\right) \to 0 \text{ as } m\vee n\to\infty.$$

On the other hand, it follows by (3.11), (3.13) and the Toeplitz lemma (see, e.g., Loève, 1977, p. 250) that

$$\frac{k_{mn}}{a^p_{mn}}\sum_{l=1}^{k_{mn}-1}\frac{(g^p(l+1)-g^p(l))}{l}\left(\frac{1}{k_{mn}}\sum_{i=1}^{u_m}\sum_{j=1}^{v_n}lP\{\|V_{ij}\| > g(l)\}\right) \to 0 \text{ as } m\vee n\to\infty.$$

Thus, the expression $N_{mn} \to 0$ as $m \vee n \to \infty$.

Applying Theorems 3.1 and 3.4 we obtain the conclusions (3.14) and (3.15) respectively. □

Remark 3.6. Note that condition (3.11) can be difficult to check. It is analogous to Proposition 1 of D. H. Hong et al. [5], a sufficient condition for (3.11) is given as follows:

$$\frac{g(k_{mn})}{a_{mn}} = O(1) \text{ and } \sum_{l=1}^{k_{mn}}\frac{g^p(l)}{l^2} = O\left(\frac{a^p_{mn}}{k_{mn}}\right). \tag{3.16}$$

Corollary 3.7. *Let $0 < r < p \le 2$. When $1 \le p \le 2$ we assume further that the underlying Banach space is of martingale type p. Suppose that $\{V_{mn}; m \ge 1, n \ge 1\}$ is stochastically dominated by a random element V. If*

$$\lim_{a\to\infty} aP\{\|V\|^r > a\} = 0,$$

then

$$\max_{\substack{1\le k\le m\\1\le l\le n}}\frac{1}{(mn)^{1/r}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}(V_{ij}-c_{mnij})\right\| \xrightarrow{P} 0 \text{ as } m\vee n\to\infty,$$

where $c_{mnij} = 0$ if $0 < p \le 1$ and $c_{mnij} = E(V_{ij}I(\|V_{ij}\|^r \le mn)|\mathcal{F}_{ij})$ if $1 < p \le 2$.

Proof. Let $g(t) = t^{1/r}$, $u_n = v_n = n$, $k_{mn} = mn$ and $a_{mn} = (mn)^{1/r}$. Then conditions (3.9), (3.10), (3.12) and (3.13) are clearly satisfied. On the other hand, by the inequality $\sum_{l=1}^{k_{mn}} l^{\frac{p}{r}-2} \le C.k_{mn}^{\frac{p}{r}-1}$, it follows that (3.16) holds. Thus, condition (3.11) holds. □

The following corollary is stronger than the sufficient condition of Feller's weak law of large numbers (see, e.g., [3], Section VII.7.).

Corollary 3.8. *Let $0 < r < 2$. Suppose that $\{X_n; n \ge 1\}$ is a sequence of random variables which is stochastically dominated by a random variable X. If*

$$\lim_{a\to\infty} aP\{|X|^r > a\} = 0,$$

then

$$\frac{1}{n^{1/r}} \max_{1\le l\le n} \left| \sum_{j=1}^{l} (X_j - E(X_j I(|X_j|^r \le n)|\mathcal{F}_j)) \right| \xrightarrow{P} 0 \text{ as } n \to \infty,$$

where $\mathcal{F}_j = \sigma(X_i; 1 \le i < j)$.

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