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SOME FRIEDRICHS TYPE INEQUALITIES IN THE FULL EUCLIDEAN SPACE

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper we prove the inequality

$$\int\limits_{\mathbb{R}^n} \mu_R(|x|)|u(x)|^p dx \leqslant M\Bigg[\int\limits_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\,dx + \Bigg|\int\limits_{|x|=R} u(x)\,ds\Bigg|^p\Bigg],$$

where $w(|x|) > 0$ and $\mu_R(|x|) > 0$ are the weight functions, $R > 0$ is an arbitrary number. In doing so, we first show some "two-sides" Hardy type inequalities.

1. Problem A

Let $w(r) > 0$ be a given function on $\mathbb{R}_+ = (0, \infty)$, $R > 0$ and $p > 1$. It is necessary to find a function $\mu_R(r) > 0$, such that

$$\int\limits_0^\infty \mu_R(r)\Bigg|\int\limits_R^r f(t)\,dt\Bigg|^p dr \leqslant M\int\limits_0^\infty |f(r)|^p w(r)\,dr, \tag{1}$$

where $M > 0$ is independent of R. We have proved the following result.

Theorem A. *Let $w^{-s} \in L_1^{loc}(\mathbb{R}_+)$, where $s = 1/(p-1)$. Then*

(1) *The inequality* (1) *holds if*

$$\mu_R(r) = (p-1)^p\Bigg|\int\limits_R^r w^{-s}(t)\,dt\Bigg|^{-p} w^{-s}(r).$$

Moreover, $M = p^p$.

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(2) *If w^{-s} is summable at the point $r=0$, then as $R\to 0$ we obtain the direct Hardy type inequality*

$$(p-1)^p\int_0^\infty\Big(\int_0^r w^{-s}(t)\,dt\Big)^{-p} w^{-s}(r)\Big|\int_0^r f(t)\,dt\Big|^p dr \leqslant p^p\int_0^\infty |f(r)|^p w(r)\,dr.$$

(3) *If w^{-s} is summable at the point $r=+\infty$, then as $R\to+\infty$ we obtain the inverse Hardy type inequality*

$$(p-1)^p\int_0^\infty\Big(\int_r^\infty w^{-s}(t)\,dt\Big)^{-p} w^{-s}(r)\Big|\int_r^\infty f(t)\,dt\Big|^p dr \leqslant p^p\int_0^\infty |f(r)|^p w(r)\,dr.$$

The proof of this theorem can be found in [2]–[4] (in [5] the final result is given).

Example 1. Let $w(r)=r^{p-1}$. Then $\mu_R(r)=(p-1)^p r^{-1}|\ln\frac{r}{R}|^{-p}$ and

$$(p-1)^p\int_0^\infty \frac{1}{r|\ln\frac{r}{R}|^p}\Big|\int_R^r f(t)\,dt\Big|^p dr \leqslant p^p\int_0^\infty |f(r)|^p r^{p-1} dr.$$

2. Problem B

Let $\mu(r)>0$ be a given function. It is necessary to find a function $w_R(r)>0$, such that

$$\int_0^\infty \mu(r)\Big|\int_R^r f(t)\,dt\Big|^p dr \leqslant M\int_0^\infty |f(r)|^p w_R(r)\,dr, \tag{2}$$

where as before $R>0$ and $M>0$ is independent of R.

Theorem B. *Let $\mu(r)>0$, and $\mu(r)$ be locally summable on the real half-line $[0,+\infty]$, excluding (may be) the point $r=R$. Then*

$$\int_0^\infty \mu(r)\Big|\int_R^r f(t)\,dt\Big|^p dr \leqslant p^p\int_0^\infty |f(r)|^p w_R(r)\,dr,$$

where

$$w_R(r)=\begin{cases}\Big(\int_0^r \mu(t)\,dt\Big)^p \mu^{-(p-1)}(r), & \text{for } r\in(0,R);\\ \Big(\int_r^\infty \mu(t)\,dt\Big)^p \mu^{-(p-1)}(r), & \text{for } r\in(R,\infty),\end{cases}$$

and $f\in L_{p,w_R}(0,\infty)$ is an arbitrary function.

The proof of this theorem is similar to that of Theorem A.

Example 2. Let $\mu(r)=(p-1)^p r^{-1}|\ln\frac{r}{R}|^{-p}$. Then $w_R(r)=r^{p-1}$ and we obtain the same inequality as in Example 1.

3. Connections between $(w,\, \mu_R)$ and $(\mu,\, w_R)$

Theorem $\mathbf{A_1}$. *Let $w(r)$ and $\mu_R(r)$ be the functions in the inequality (1) such that*

$$\int_0^R w^{-s}(r)\,dr = \infty, \qquad \int_R^\infty w^{-s}(r)\,dr = \infty.$$

Then

$$w(r) = \Big(\int_0^r \mu_R(t)\,dt\Big)^p \mu_R^{-p+1}(r), \quad r \in (0,R);$$

$$w(r) = \Big(\int_r^\infty \mu_R(t)\,dt\Big)^p \mu_R^{-p+1}(r), \quad r \in (R,\infty).$$

Theorem $\mathbf{B_1}$. *Let $\mu(r)$ and $w_R(r)$ be the functions in the inequality (2) such that*

$$\int_0^R \mu(r)\,dr = \infty, \qquad \int_R^\infty \mu(r)\,dr = \infty.$$

Then

$$\mu(r) = (p-1)^p\Big(\int_r^R w_R^{-s}(t)\,dt\Big)^{-p} w_R^{-s}(r), \quad r \in (0,R);$$

$$\mu(r) = (p-1)^p\Big(\int_R^r w_R^{-s}(t)\,dt\Big)^{-p} w_R^{-s}(r), \quad r \in (R,\infty).$$

The proof of these theorems is based on the following lemmas.

Lemma 1. *For any $0 < r_1 < r_2 < R$*

$$(p-1)^{-p}\int_{r_1}^{r_2} \mu_R(r)\,dr = \frac{1}{p-1}\Big(\int_{r_2}^R w^{-s}(r)\,dr\Big)^{-p+1} - \frac{1}{p-1}\Big(\int_{r_1}^R w^{-s}(r)\,dr\Big)^{-p+1}.$$

Lemma 2. *For any $0 < r_1 < r_2 < R$*

$$\int_{r_1}^{r_2} w_R^{-s}(r)\,dr = \frac{1}{p'-1}\Big(\int_0^{r_1} \mu(r)\,dr\Big)^{-p'+1} - \frac{1}{p'-1}\Big(\int_0^{r_2} \mu(r)\,dr\Big)^{-p'+1}.$$

Analogous inequalities take place for the interval (R,∞).

Remark. It is obvious that for functions $u(r)$ with $u(R) = 0$ the inequalities (1) and (2) can be written in the form

$$\int_0^\infty \mu_R(r)|u(r)|^p dr \leqslant p^p \int_0^\infty |u'(r)|^p w(r)\, dr, \tag{3}$$

$$\int_0^\infty \mu(r)|u(r)|^p dr \leqslant p^p \int_0^\infty |u'(r)|^p w_R(r)\, dr. \tag{4}$$

4. Friedrichs type inequality

Let $u \in L_1^{loc}(\mathbb{R}^n)$ $(n > 1,\ x = (x_1, \dots, x_n))$ be such that $\nabla u \in L_{p,w}(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} |\nabla u(x)| w(|x|)\, dx < \infty.$$

Introduce the weight function $w_n(r) = w(r)r^{n-1}$, where $r = |x|$. We suppose that $w^{-s} \in L_1^{loc}(0, \infty)$. Further, let

$$\mu_{R,n}(r) = \left| \int_R^r w_n^{-s}(t)\, dt \right|^{-p} w_n^{-s}(r)$$

be the "canonical" weight function defined in Theorem A.

Theorem $\mathbf{A_n}$. *Let*

$$\int_{|x|=R} u(s)\, ds = 0. \tag{5}$$

Then the following inequality

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x)|^p dx \leqslant M \int_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\, dx$$

holds. Here

$$\nu_{R,n}(r) = r^{-n+1} \min\{w_n(r)r^{-p}, \mu_{R,n}(r)\} = \min\{w(r)r^{-p}, \mu_{R,n}(r)r^{-n+1}\}.$$

Proof. Let us consider the integral

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x)|^p dx = \int_0^\infty \nu_{R,n}(r) r^{n-1} \int_{|x|=1} |u(r,s)|^p ds\, dr.$$

Using Poincaré's inequality on the unit sphere S_1^n, we obtain that

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x)|^p dx \leqslant M\Bigg[\int_0^\infty \nu_{R,n}(r)r^{n-1}\Bigg|\int_{|x|=1} u(r,s)ds\Bigg|^p dr + \int_0^\infty \nu_{R,n}(r)r^{n-1}\int_{|x|=1} |\nabla_s u(r,s)|^p ds\, dr\Bigg] := M(I_1+I_2),$$

where $\nabla_s u(r,s)$ is the tangent gradient.

Firstly, we estimate the integral

$$I_1 = \int_0^\infty \nu_{R,n}(r)r^{n-1}|f(r)|^p dr,$$

where $f(r) = \int_{|x|=1} u(r,s)ds$. Let us note that $f(R) = 0$ and

$$\nu_{R,n}(r)r^{n-1} \leqslant \mu_{R,n}(r).$$

Then due to inequality (3)

$$I_1 \leqslant \int_0^\infty \mu_{R,n}(r)|f(r)|^p dr \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty |f'(r)|^p w(r)r^{n-1}dr =$$

$$= \left(\frac{p}{p-1}\right)^p \int_0^\infty w(r)r^{n-1}\Bigg|\int_{|x|=1} \frac{\partial u(r,s)}{\partial r}ds\Bigg|^p dr \leqslant$$

$$\leqslant M\int_0^\infty w(r)r^{n-1}\int_{|x|=1} |\nabla u(r,s)|^p ds\, dr = M\int_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\, dx.$$

We turn to the integral

$$I_2 = \int_0^\infty \nu_{R,n}(r)r^{n-1}\int_{|x|=1} |\nabla_s u(r,s)|^p ds\, dr.$$

Since the coordinates $x_1,\dots,x_n$ are linear with respect to r, we have $|\nabla_s u| \leqslant Mr|\nabla u|$. Bearing in mind the inequality $\nu_{R,n}(r) \leqslant w(r)r^{-p}$ we find that

$$I_2 \leqslant M\int_0^\infty \nu_{R,n}(r)r^{n-1+p}\int_{|x|=1} |\nabla u(r,s)|^p ds\, dr \leqslant$$

$$\leqslant M\int_0^\infty w(r)r^{n-1}\int_{|x|=1} |\nabla u(r,s)|^p ds\, dr = M\int_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\, dx.$$

Summing these calculations we obtain the initial inequality of the theorem. □

Corollary. (Friedrichs type inequality)

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x) - C|^p dx \leqslant M \int_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\, dx,$$

where

$$C = \frac{1}{\text{mes } S_R^n} \int_{S_R^n} u(s)\, ds$$

is the mean value of $u(x)$ on $S_R^n = \{x \in \mathbb{R}^n : |x| = R\}$, *or, that is the same,*

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x)|^p dx \leqslant M \Bigg[\int_{\mathbb{R}^n} |\nabla u(x)|^p w(|x|)\, dx + \Bigg| \int_{S_R^n} u(x)\, ds \Bigg|^p \Bigg].$$

Example 3. Let $n \geqslant 2$, $w(|x|) = |x|^{p-n}$. Then for any function $u(x)$ with the condition (5)

$$\int_{\mathbb{R}^n} \nu_{R,n}(|x|)|u(x)|^p dx \leqslant M \int_{\mathbb{R}^n} |\nabla u(x)|^p |x|^{p-n} dx,$$

where

$$\nu_{R,n}(r) = \min\{r^{-n} \big|\ln \tfrac{r}{R}\big|^{-p}, r^{-n}\}.$$

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