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ATTRACTORS FOR NON-AUTONOMOUS SEMILINEAR PARABOLIC EQUATIONS WITH DELAYS

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Abstract. We study the asymptotic behavior of solutions to a class of retarded non-autonomous semilinear parabolic equations with nonlinearities of polynomial type, general delays and time-dependent external forces. The existence of weak solutions for the equations is proved by using the Galerkin method. We then prove the existence of a pullback attractor without restriction on the growth order of polynomial type nonlinearity and on exponential growth of the external force. When the time-dependent external force is a translation bounded function, the existence of a uniform attractor is proved. Finally, we give a relationship between the pullback attractor and the uniform attractor.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics and biology. One way to treat this problem for a system having some dissipativity properties is to analysis the existence and structure of its attractor. The existence of the attractor has been derived for a large class of PDEs without delays and ODEs with delays (see e.g. [3, 12, 14, 23] and the references therein). However, to the best of our knowledge, little seems to be known about the existence of the attractor for PDEs with delays in the non-autonomous case.

PDEs with delays are often considered in the model such as maturation time for population dynamics in mathematical biology and other fields. Such equations are naturally more difficult since they are infinite dimensional both in time and space variables. We refer to the monograph [24] for a theory of PDEs with delays. Recently, the long-time behavior of PDEs with delays, including the stability of solutions and the existence of attractors, has attracted the attention of many researchers (see e.g. [1, 2, 4-10, 13, 15-16, 19-22]).

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In this paper we study the long-time behavior for the following non-autonomous equation:

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) + Au(t,x) + f(u(t,x)) &= F(u_t)(x) + g(t,x),\ x \in \Omega,\ t > \tau, \\ u(\tau,x) &= u^0(x), \quad x \in \Omega, \\ u(\tau+\theta,x) &= \varphi(\theta,x),\ \theta \in (-r,0),\ x \in \Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, and other symbols satisfy the following conditions, c.f. [15, 22]:

($H1$) The initial data $u^0 \in L^2(\Omega)$ and $\varphi \in L^2(-r,0;L^2(\Omega))$ are given;

($H2$) A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent (for example, $-\Delta$ with the homogeneous Dirichlet condition);

($H3$) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that

$$C_1|u|^p - C_0 \leq f(u)u \leq C_2|u|^p + C_0,\ p > 2, \tag{1.2}$$

$$f'(u) \geq -C_3 \text{ for all } u \in \mathbb{R}, \tag{1.3}$$

where C_0, C_1, C_2 and C_3 are positive constants;

($H4$) $F : L^2(-r,0;L^2(\Omega)) \to L^2(\Omega)$ is locally Lipschitz continuous for the initial data, i.e., for any $M > 0$, there exists $L_{F,M} > 0$ such that for $u, v \in L^2(-r,0;L^2(\Omega))$ satisfying $(u(0),u),(v(0),v) \in B(0,M)$, the closed ball in $L^2(\Omega) \times L^2(-r,0;L^2(\Omega))$ centered at 0 with radius M, one has

$$\|F(u) - F(v)\| \leq L_{F,M}\left(\|u(0)-v(0)\|^2 + \|u-v\|^2_{L^2(-r,0;L^2(\Omega))}\right)^{1/2}, \tag{1.4}$$

and there exist $k_1, k_2, k_3 \geq 0$, such that for all $\xi \in L^2(-r,0;L^2(\Omega))$, $\eta \in L^2(\Omega)$, one has

$$|\langle F(\xi),\eta\rangle| \leq k_1\|\eta\|^2 + k_2\int_{-r}^{0}\|\xi(\theta)\|^2 d\theta + k_3; \tag{1.5}$$

hereafter we denote by $\langle\cdot,\cdot\rangle$ and $\|.\|$ the inner product and norm in $L^2(\Omega)$;

($H5$) The external force $g \in L^2_{loc}(\mathbb{R};L^2(\Omega))$ satisfies

$$\int_{-\infty}^{0} e^{cs}\|g(s)\|^2 ds < +\infty \text{ and } \int_{-\infty}^{0}\int_{-\infty}^{s} e^{cy}\|g(y)\|^2 dy ds < +\infty,$$

where c is a fixed positive constant.

Let us give some comments about the conditions of g in the paper. The assumption ($H5$) is used to prove the existence of a weak solution to problem (1.1) and of a pullback attractor for the process associated to Problem (1.1). When proving the existence of a uniform attractor, we need a stronger condition ($H5bis$) (see Sect. 4) of g, that is, g is translation bounded in $L^2_{loc}(\mathbb{R};L^2(\Omega))$. This assumption ensures that the symbol space $\Sigma = \mathcal{H}_w(g)$, the closure of the set $\{g(s+\cdot)|s \in \mathbb{R}\}$ in $L^{2,w}_{loc}(\mathbb{R};L^2(\Omega))$ with the weak topology, is weakly compact, and this enables us to use the abstract theorem of Lu et al. in [18] to prove the existence and structure of the uniform attractor.

Given $T > \tau$ and $u : (\tau - r, T) \to L^2(\Omega)$, as in [14], for each $t \in [\tau, T]$ we denote by u_t the function defined on $(-r, 0)$ by the relation $u_t(\theta) = u(t+\theta)$, for all $\theta \in (-r, 0)$. In this paper, we first construct the process associated to (1.1) in the space $L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$, so the pair $(u(t), u_t) \in L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$ presents the state of the system. Then we investigate the long-time behavior of the process by showing the existence of a pullback/uniform attractor. It is noticed that the obtained results improve and extend some existing ones in [1, 15, 22].

Since $A : D(A) \to L^2(\Omega)$ is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, A has a discrete spectrum that only contains positive eigenvalues $\{\lambda_k\}_{k=1}^\infty$ satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots, \quad \lambda_k \to \infty \quad \text{as } k \to \infty,$$

and the corresponding eigenfunctions $\{e_k\}_{k=1}^\infty$ compose an orthonormal basis of the Hilbert space $L^2(\Omega)$ such that

$$(e_j, e_k) = \delta_{jk} \text{ and } Ae_k = \lambda_k e_k, \quad k = 1, 2, \dots$$

Hence we can define the fractional power spaces and operators as

$$X^\alpha = D(A^\alpha) = \{u = \sum_{k=1}^\infty c_k e_k \in H : \sum_{k=1}^\infty c_k^2 \lambda_k^{2\alpha} < \infty\},$$

$$A^\alpha u = \sum_{k=1}^\infty c_k \lambda_k^\alpha e_k, \text{ where } u = \sum_{k=1}^\infty c_k e_k.$$

It is known (see e.g. [12]) that if $\alpha > \beta$ then the space $D(A^\alpha)$ is compactly embedded into $D(A^\beta)$. In particular,

$$D(A^{\frac{1}{2}}) \hookrightarrow L^2(\Omega) \hookrightarrow D(A^{-\frac{1}{2}}),$$

where the injections are dense and compact.

Note that by the Riesz Representation Theorem, we have

$$\|F(\xi)\| = \|F(\xi)\|_{op} = \sup_{\|\eta\|=1} |\langle F(\xi), \eta\rangle| \le k_1 + k_2 \int_{-r}^0 \|\xi(\theta)\|^2 d\theta + k_3, \tag{1.6}$$

which implies that F is a bounded map from $L^2(-r, 0; L^2(\Omega))$ to $L^2(\Omega)$.

Now we introduce some notations which will be used in this paper:

- $H = L^2(\Omega)$,
- $V = D(A^{\frac{1}{2}})$ with the associated product $\langle u, v\rangle_V = \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v\rangle_H$,
- $V' = D(A^{-\frac{1}{2}})$ is the dual space of V,
- $L^2_H = L^2(-r, 0; H)$, $L^2_V = L^2(-r, 0; V)$ are Hilbert spaces with the norms

$$\|u\|^2_{L^2_X} = \int_{-r}^0 \|u(s)\|^2_X ds,$$

- $M^2_H = H \times L^2_H$, $M^2_V = V \times L^2_V$ are Hilbert spaces with the norms

$$\|(u, \varphi)\|^2_{M^2_X} = \|u\|^2_X + \|\varphi\|^2_{L^2_X},$$

- $W = L^2(\tau, T; V) \cap L^p(\tau, T; L^p(\Omega)), W^* = L^2(\tau, T; V') + L^{p'}(\tau, T; L^{p'}(\Omega))$, where p' is the conjugate of p.

The paper is organized as follows. In Section 2 we recall some results about pullback attractors and uniform attractors which will be used in the paper. In Section 3, we prove the existence of a pullback attractor in M_H^2 for the process associated to Problem (1.1) when the external force has an exponential growth. The existence of a uniform attractor in M_H^2 for the family of processes associated to Problem (1.1) is discussed in Section 4 when the external force is a translation bounded function. In the last section, we give a relationship between the pullback attractor and the uniform attractor.

It is noticed that the restriction $p > 2$ in (1.2) is made for the coherence of the presentation only; some comments about results in the case $p = 2$ are given in Remarks 3.1 and 4.1.

2. Preliminaries

For the convenience of readers, in this section we recall some results about pullback attractors and uniform attractors which will be used in the paper.

2.1. **Pullback attractors.** Let X be a complete metric space and $B_X(a, r)$ be the ball in X centered at a with radius r. A process on X is a two parameters process $U(t, \tau) : X \to X$ satisfying the following properties:

$$U(t, r)U(r, \tau) = U(t, \tau) \text{ for all } t \geq r \geq \tau,$$
$$U(\tau, \tau) = \text{Id for all } \tau \in \mathbb{R}.$$

We usually use the Hausdorff semi-distance $\text{dist}_X(.,.)$ defined by

$$\text{dist}_X(A, B) := \sup_{a\in A} \inf_{b\in B} d(a, b) \text{ for } A, B \subset X.$$

Definition 2.1. [9, Definition 2]. Let $U(t, \tau)$ be a process in the complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a pullback attractor in X for $U(t, \tau)$ if, for every $\tau \in \mathbb{R}$, it satisfies

(1) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$ (invariance), and
(2) $\lim_{s\to+\infty} \text{dist}_X(U(t, t-s)D, \mathcal{A}(t)) = 0$ for all bounded subsets D of X.

The pullback attracting property (2) considers the state of the system at time t when the initial time $t - s$ goes to $-\infty$.

Definition 2.2. [9, Definition 4]. A family of sets $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ is said to be pullback absorbing in X with respect to the process $U(t, \tau)$ if for any bounded subset B of X and any $t \in \mathbb{R}$, there exists $\tau(t, B) \leq t$ such that $U(t, \tau)B \subset \mathcal{B}(t)$ for all $\tau \leq \tau(t, B)$.

The following theorem shows the sufficient conditions for the existence of a pullback attractor in X.

Theorem 2.3. [9, Theorem 5]. *Let $U(t,\tau)$ be a continuous two-parameter process on X. If there exists a family of compact pullback absorbing sets $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ in X with respect to the process $U(t,\tau)$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in X, and $\mathcal{A}(t)\subset\mathcal{B}(t)$ for all $t\in\mathbb{R}$. Furthermore,*

$$\mathcal{A}(t)=\overline{\bigcup_{\substack{D\subset X\\ bounded}}\Lambda_D(t)},\ \text{where } \Lambda_D(t)=\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{s\geq n}U(t,t-s)D}.$$

2.2. **Uniform attractors.** Consider a family of processes $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}$ on a Banach space E depending on a parameter $\sigma\in\Sigma$. The parameter σ, chosen as the collection of all time-dependent coefficients of the equation, is said to be the symbol of the process $\{U_\sigma(t,\tau)\}$ and the set Σ is said to be the symbol space. By $\mathcal{B}(E)$ we denote the collection of the bounded sets of E.

Definition 2.4. [11, Chapter 4, Definition 3.3]. A set $\mathcal{B}_0\subset E$ is said to be uniformly (w.r.t. $\sigma\in\Sigma$) absorbing for the family of processes $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}$, if for any $\tau\in\mathbb{R}$ and any $B\in\mathcal{B}(E)$ there exists $t_0=t_0(\tau,B)\geq\tau$ such that

$$\bigcup_{\sigma\in\Sigma}U_\sigma(t,\tau)B\subset\mathcal{B}_0,$$

for all $t\geq t_0$. A family of processes possessing a compact uniformly absorbing set is called uniformly compact.

Definition 2.5. [11, Chapter 4, Definition 3.5]. A closed set $\mathcal{A}_\Sigma\subset E$ is said to be a uniform (w.r.t. $\sigma\in\Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}$, if it is uniformly (w.r.t. $\sigma\in\Sigma$) attracting (attracting property) and it is contained in any closed uniformly (w.r.t. $\sigma\in\Sigma$) attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}:\mathcal{A}_\Sigma\subset\mathcal{A}'$ (minimality property).

The kernel $\mathcal{K}_\sigma$ of a process $\{U_\sigma(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U_\sigma(t,\tau)\}$:

$$\mathcal{K}_\sigma=\{u(.)\mid U_\sigma(t,\tau)u(\tau)=u(t),\operatorname{dist}(u(t),u(0))\leq C_u,\forall t\geq\tau,\tau\in\mathbb{R}\}.$$

The set $\mathcal{K}_\sigma(s)=\{u(s):u(.)\in\mathcal{K}_\sigma\}$ is said to be the kernel section at time $t=s,s\in\mathbb{R}$.

The following result, a direct consequence of Theorem 2.5 in [18], gives a sufficient conditions on the existence and structure of the uniform attractor for a family of (weakly continuous) processes.

Theorem 2.6. *Let Σ be a weakly compact set and the family of processes $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}$ is $(E\times\Sigma,E)$-weakly continuous. If $\{U_\sigma(t,\tau)\mid\sigma\in\Sigma\}$ has a uniformly (w.r.t. $\sigma\in\Sigma$) compact absorbing set $\mathcal{B}_0$, then it possesses a uniform compact attractor $\mathcal{A}_\Sigma$ in E. Moreover,*

$$\mathcal{A}_\Sigma=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s)\ \ \forall s\in\mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the kernel section at $t=s$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma\in\Sigma$.

A set Y is said to be uniformly (w.r.t. $\tau \in \mathbb{R}$) attracting for a process $\{U(t,\tau)\}$ if

$$\sup_{\tau \in \mathbb{R}} \operatorname{dist}_X(U(t+\tau,\tau)B, Y) \to 0 \text{ as } t \to +\infty$$

for any bounded set B. In particular, a closed set $\mathcal{A}_0$ is said to be a uniform (w.r.t. $\tau \in \mathbb{R}$) attractor for $\{U(t,\tau)\}$, if it is contained in any closed uniformly attracting set. Given a symbol σ_0, let $\Sigma_0 = \{\sigma_0(\cdot + h) | h \in \mathbb{R}\}$ be a subset of some Banach space. If the process $\{U_{\sigma_0}(t,\tau)\}$ satisfies the following translation identity:

$$U_{\sigma_0}(t+h, \tau+h) = U_{T(h)\sigma_0}(t,\tau), \quad \forall t \geq \tau, \tau \in \mathbb{R}, h \geq 0, \tag{2.1}$$

then obviously, the uniformly (w.r.t. $\tau \in \mathbb{R}$) attracting property of $\{U_{\sigma_0}(t,\tau)\}$ is equivalent to the uniformly (w.r.t. $\sigma \in \Sigma_0$) attracting property of $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma_0$. It is easy to see that the uniform (w.r.t. $\tau \in \mathbb{R}$) attractor $\mathcal{A}_0$ of $\{U_{\sigma_0}(t,\tau)\}$ coincides with the uniform (w.r.t. $\sigma \in \Sigma_0$) attractor $\mathcal{A}_{\Sigma_0}$ of the family of processes $\{U_\sigma(t,\tau) \mid \sigma \in \Sigma_0\}$.

3. Existence of a pullback attractor

Definition 3.1. A function u is called a weak solution of Problem (1.1) on the interval (τ, T) if $u \in L^2(\tau - r, T; H) \cap W$, $\dfrac{\partial u}{\partial t} \in W^*$, $u(\tau) = u^0$, $u(\tau+\theta) = \varphi(\theta)$ for $\theta \in (-r, 0)$, and

$$\int_\tau^T \Big(\langle \frac{\partial u}{\partial t}, \varphi \rangle + \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \varphi \rangle + \langle f(u), \varphi \rangle \Big) dt = \int_\tau^T \Big(\langle F(u_t), \varphi \rangle + \langle g, \varphi \rangle \Big) dt,$$

for all test functions $\varphi \in W$.

Repeating the arguments used in the autonomous case [1], we get the following.

Theorem 3.2. *Under conditions $(H1) - (H5)$, for any $\tau \in \mathbb{R}$, $T > \tau$ given, Problem (1.1) has a unique weak solution u on (τ, T) which satisfies*

$$u(t) \in C([\tau, T]; H).$$

Moreover, the solution is defined over the interval $[\tau, \infty)$.

Due to the result of Theorem 3.2, we can define the process $U(t,\tau) : M_H^2 \to M_H^2$ associated to Problem (1.1) as follows.

$$U(t,\tau)(u^0, \varphi) = (u(t; \tau, (u^0, \varphi)), u_t(.; \tau, (u^0, \varphi))) \text{ for } (u^0, \varphi) \in M_H^2, \ t \geq \tau,$$

where $u(t) = u(t; \tau, u^0, \varphi)$ is the unique weak solution of Problem (1.1) with initial datum $(u^0, \varphi) \in M_H^2$.

Lemma 3.3. *Under assumptions $(H1)-(H5)$, the operator $U(.,.)$ is a continuous process on M_H^2.*

Proof. The composition properties for the process $U(.,.)$ follows from the uniqueness of solutions to Problem (1.1).

To prove the continuity of $U(t,\tau)$, let us consider two initial data (u^0,φ), $(v^0,\psi) \in M_H^2$ and their corresponding solutions $u(.)$, $v(.)$. Then $w = u - v$ satisfies

$$\frac{\partial w}{\partial t} + Aw + f(u) - f(v) = F(u_t) - F(v_t) \text{ in } W^* \text{ for a.e. } t \in [\tau,\infty).$$

Multiplying this equation by w and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\|w(t)\|_V^2+\int_\Omega [f(u(t))-f(v(t))][u(t)-v(t)]dx = \langle F(u_t)-F(v_t), w(t)\rangle.$$

Using condition (1.3), we have

$$\int_\Omega [f(u(t)) - f(v(t))][u(t) - v(t)]dx \geq -C_3 \int_\Omega [u(t) - v(t)]^2 dx = -C_3\|w(t)\|^2.$$

By the Cauchy inequality, we get

$$\langle F(u_t)-F(v_t), w(t)\rangle \leq \|F(u_t)-F(v_t)\|.\|w(t)\| \leq \frac{1}{4\lambda_1}\|F(u_t)-F(v_t)\|^2+\lambda_1\|w(t)\|^2.$$

Using condition (1.4) and noting that $\lambda_1\|w(t)\|^2 \leq \|w(t)\|_V^2$, we get

$$\begin{aligned}\langle F(u_t) - F(v_t), w(t)\rangle &\leq \frac{L_{F,M}^2}{4\lambda_1}\left(\|u_t(0) - v_t(0)\|^2 + \|u_t - v_t\|_{L_H^2}^2\right) + \|w(t)\|_V^2 \\ &= \frac{L_{F,M}^2}{4\lambda_1}\left(\|w(t)\|^2 + \|w_t\|_{L_H^2}^2\right) + \|w(t)\|_V^2,\end{aligned}$$

where $u_t(0) - v_t(0) = u(t) - v(t) = w(t)$. Hence

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2 &\leq \frac{L_{F,M}^2}{2\lambda_1}\left(\|w(t)\|^2 + \|w_t\|_{L_H^2}^2\right) + 2C_3\|w(t)\|^2 \\ &= C_4\|w(t)\|^2 + C_5\int_{-r}^0 \|w_t(s)\|^2 ds.\end{aligned}$$

Integrating this inequality from τ to t, we obtain

$$\begin{aligned}\|w(t)\|^2 - \|w(\tau)\|^2 &\leq C_4\int_\tau^t \|w(s)\|^2 ds + C_5\int_\tau^t\int_{-r}^0 \|w(s+\theta)\|^2 d\theta ds \\ &\leq C_4\int_\tau^t \|w(s)\|^2 ds + C_5\int_{-r}^0\int_{\tau-r}^t \|w(s)\|^2 ds d\theta \\ &\leq (C_4 + C_5 r)\int_\tau^t \|w(s)\|^2 ds + C_5 r\int_{\tau-r}^\tau \|w(s)\|^2 ds.\end{aligned}$$

The Gronwall lemma implies that

$$\|w(t)\|^2 \leq \left(\|w(\tau)\|^2 + C_5 r\int_{\tau-r}^\tau \|w(s)\|^2 ds\right) e^{(C_4+C_5 r)(t-\tau)}, \qquad t \in [\tau, T].$$

We rewrite the last inequality as

$$\|u(t) - v(t)\|^2 \leq \left(\|u^0 - v^0\|^2 + C_5 r\|\varphi - \psi\|^2_{L^2_H}\right) e^{(C_4+C_5r)(t-\tau)}. \tag{3.1}$$

Note that if $t \geq \tau + r$, then we obtain from (3.1) that

$$\begin{aligned}
\|u_t - v_t\|^2_{L^2_H} &= \int_{-r}^0 \|u(t+\theta) - v(t+\theta)\|^2 d\theta \\
&\leq \int_{-r}^0 \sup_{s\in[-r,0]} \|u(t+s) - v(t+s)\|^2 d\theta \\
&\leq r\left(\|u^0 - v^0\|^2 + C_5 r\|\varphi - \psi\|^2_{L^2_H}\right) e^{(C_4+C_5r)(t-\tau)}.
\end{aligned}$$

Next, if $\tau \leq t < \tau + r$, we deduce immediately that

$$\begin{aligned}
\|u_t - v_t\|^2_{L^2_H} &= \int_{-r}^0 \|u(t+\theta) - v(t+\theta)\|^2 d\theta \\
&\leq \int_{\tau-r}^{\tau} \|u(s) - v(s)\|^2 ds + \int_{\tau}^{\tau+r} \|u(s) - v(s)\|^2 ds \\
&\leq \left(r\|u^0 - v^0\|^2 + (C_5 r^2 + 1)\|\varphi - \psi\|^2_{L^2_H}\right) e^{(C_4+C_5r)(t-\tau)}.
\end{aligned}$$

Thus, we have for all $t \geq \tau$,

$$\|u_t - v_t\|^2_{L^2_H} \leq \left(r\|u^0 - v^0\|^2 + (C_5 r^2 + 1)\|\varphi - \psi\|^2_{L^2_H}\right) e^{(C_4+C_5r)(t-\tau)},$$

which joints with (3.1) imply the continuity of $U(t,\tau)$. □

Lemma 3.4. *Let $u \in L^p(\Omega)$, $p > 2$. Then for any $\xi > 0$, there exists a positive constant $C(\xi, p) > 0$ such that*

$$\|u\|^p_{L^p(\Omega)} \geq \xi\|u\|^2_{L^2(\Omega)} - C(\xi, p). \tag{3.2}$$

Proof. Using Young's inequality, we have

$$\begin{aligned}
\xi\|u\|^2 = \int_\Omega \xi|u|^2 dx &\leq \int_\Omega \left(\frac{2}{p}(|u|^2)^{\frac{p}{2}} + \frac{p-2}{p}\xi^{\frac{p}{p-2}}\right) dx \\
&= \frac{2}{p}\|u\|^p_{L^p(\Omega)} + \frac{p-2}{p}\xi^{\frac{p}{p-2}}|\Omega|.
\end{aligned}$$

Since $p > 2$, $\frac{2}{p} < 1$. Putting $C(\xi, p) = \frac{p-2}{p}\xi^{\frac{p}{p-2}}|\Omega|$, we get (3.2). □

Lemma 3.5. *Under assumptions $(H1) - (H5)$, the solution u of (1.1) satisfies*

$$\|u(t)\|^2 \leq e^{-c(t-\tau)}\|u^0\|^2 + 2k_2 r e^{-c(t-\tau-r)}\|\varphi\|^2_{L^2_H} + M_2 + e^{-ct}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds, \tag{3.3}$$

where M_2 is a positive constant independent of t, τ.

Proof. From (1.1), in view of (1.5), (1.2) and the Cauchy inequality, we have

$$\frac{d}{dt}\|u(t)\|^2 + 2\lambda_1\|u(t)\|^2 + 2C_1\|u(t)\|^p_{L^p(\Omega)} \le 2C_0|\Omega| + 2k_3 + (2k_1+1)\|u(t)\|^2 + 2k_2\int_{-r}^0 \|u_t(\theta)\|^2 d\theta + \|g(t)\|^2. \tag{3.4}$$

Now for any $\xi > 0$, by Lemma 3.4, there exists a number $C(\xi, p) > 0$ such that

$$2C_1\|u(t)\|^p_{L^p(\Omega)} \ge \xi\|u(t)\|^2 - C(\xi, p).$$

Therefore,

$$\frac{d}{dt}\|u(t)\|^2 \le (2k_1 + 1 - 2\lambda_1 - \xi)\|u(t)\|^2 + 2k_2\int_{-r}^0 \|u_t(\theta)\|^2 d\theta + \|g(t)\|^2 + M_1,$$

where $M_1 = C(p,\xi) + 2C_0|\Omega| + 2k_3$. We have

$$\begin{aligned}\frac{d}{dt}\left(e^{ct}\|u(t)\|^2\right) &= ce^{ct}\|u(t)\|^2 + e^{ct}\frac{d}{dt}\|u(t)\|^2 \\ &\le (2k_1 + 1 + c - 2\lambda_1 - \xi)e^{ct}\|u(t)\|^2 + M_1e^{ct} \\ &\quad + 2k_2\int_{-r}^0 e^{ct}\|u_t(\theta)\|^2 d\theta + e^{ct}\|g(t)\|^2.\end{aligned}$$

Integrating from τ to t ($t \ge \tau$), we have

$$\begin{aligned}e^{ct}\|u(t)\|^2 - e^{c\tau}\|u(\tau)\|^2 &\le (2k_1 + 1 + c - 2\lambda_1 - \xi)\int_\tau^t e^{cs}\|u(s)\|^2 ds + \frac{M_1}{c}e^{ct} \\ &\quad + 2k_2\int_\tau^t\int_{-r}^0 e^{cs}\|u_s(\theta)\|^2 d\theta ds + \int_\tau^t e^{cs}\|g(s)\|^2 ds.\end{aligned}$$

Notice that

$$\begin{aligned}\int_\tau^t\int_{-r}^0 \left(e^{cs}\|u_s(\theta)\|^2 d\theta\right) ds &\le e^{cr}\int_\tau^t\int_{-r}^0 e^{c(s+\theta)}\|u(s+\theta)\|^2 d\theta ds \\ &\le e^{cr}\int_{-r}^0\int_{\tau-r}^t e^{cs}\|u(s)\|^2 ds d\theta \\ &\le re^{c(r+\tau)}\|\varphi\|^2_{L^2_H} + re^{cr}\int_\tau^t e^{cs}\|u(s)\|^2 ds,\end{aligned} \tag{3.5}$$

thus,

$$\begin{aligned}e^{ct}\|u(t)\|^2 \le & e^{c\tau}\|u^0\|^2 + 2k_2re^{c(\tau+r)}\|\varphi\|^2_{L^2_H} \\ &+ (2k_1 + 1 + c + 2k_2re^{cr} - 2\lambda_1 - \xi)\int_\tau^t e^{cs}\|u(s)\|^2 ds \\ &+ \frac{M_1}{c}e^{ct} + \int_{-\infty}^t e^{cs}\|g(s)\|^2 ds.\end{aligned}$$

Now we choose ξ large enough such that $2k_1 + 1 + c + 2k_2re^{cr} - 2\lambda_1 - \xi < 0$ to obtain (3.3). □

As a consequence of Lemma 3.5, we obtain the following result.

Lemma 3.6. *Assume that* $(H1)-(H5)$ *hold. Then there exists a family* $\{\mathcal{B}_H(t)\}_{t\in\mathbb{R}}$ *of bounded pullback absorbing sets in* M_H^2 *for the process* $U(t,\tau)$ *associated to Problem* (1.1).

Proof. One can see that, there exists $\hat{\tau} = \hat{\tau}(t, u^0, \varphi)$ such that, for all $\tau \le \hat{\tau}$, the following inequality holds:

$$e^{-c(t-\tau)}\|u^0\|^2 + 2k_2 r e^{-c(t-\tau-r)}\|\varphi\|^2_{L^2_H} \le e^{-ct}\int_{-\infty}^{t} e^{cs}\|g(s)\|^2 ds.$$

Therefore, by Lemma 3.5, we have

$$\|u(t)\|^2 \le 2e^{-ct}\int_{-\infty}^{t} e^{cs}\|g(s)\|^2 ds + M_2 < +\infty,$$

for all $t \ge \hat{\tau}$. Now, taking $t \ge \hat{\tau} + r$, we have for $\theta \in (-r, 0)$,

$$\begin{aligned}\|u(t+\theta)\|^2 &\le 2e^{-c(t+\theta)}\int_{-\infty}^{t+\theta} e^{cs}\|g(s)\|^2 ds + M_2\\ &\le 2e^{cr}e^{-ct}\int_{-\infty}^{t} e^{cs}\|g(s)\|^2 ds + M_2.\end{aligned}$$

It follows that

$$\int_{-r}^{0}\|u_t(\theta)\|^2 d\theta \le 2re^{cr}e^{-ct}\int_{-\infty}^{t} e^{cs}\|g(s)\|^2 ds + M_2 r.$$

Hence, it is obvious that

$$\begin{aligned}\|U(t,\tau)(u^0,\varphi)\|^2_{M_H^2} &= \|u(t)\|^2 + \int_{-r}^{0}\|u_t(\theta)\|^2 d\theta\\ &\le 2(1+re^{cr})e^{-ct}\int_{-\infty}^{t} e^{cs}\|g(s)\|^2 ds + (1+r)M_2 = R_H^2(t).\end{aligned}$$

Then, for any bounded set $D \subset M_H^2$, one easily deduces that

$$U(t,\tau)D \subset \mathcal{B}_H(t) = B_{M_H^2}(0, R_H(t)),$$

for all $\tau \le \hat{\tau}(t, D) - r$. Thus $U(t,\tau)$ has a family of bounded pullback absorbing sets in M_H^2. □

Lemma 3.7. *Assume that* $(H1)-(H5)$ *hold. Then the solution of* (1.1) *satisfies*

$$\begin{aligned}
&\|u(t)\|^2 + \|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx \\
&\leq M_{20}\Bigg(\left(1+(t-\tau)+\frac{1}{t-\tau}\right)e^{-c(t-\tau)}\|u^0\|^2 \\
&+\left(1+(t-\tau)+\frac{1}{t-\tau}\right)e^{-c(t-r-\tau)}\|\varphi\|_{L_H^2}^2 \\
&+\left(1+\frac{1}{t-\tau}\right)+\left(1+\frac{1}{t-\tau}\right)e^{-ct}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds \\
&+\left(1+\frac{1}{t-\tau}\right)e^{-ct}\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2 dyds\Bigg),
\end{aligned}$$

for all $t \geq \tau$, *where* $\mathcal{F}(u) = \int_0^u f(\xi)d\xi$ *is the primitive of* f.

Proof. Multiplying (1.1) by $u(t) + \dot{u}(t)$ then integrating over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_V^2 + \frac{1}{2}\|u(t)\|^2 + \int_\Omega \mathcal{F}(u(t))dx\right) + \|u(t)\|_V^2 + \int_\Omega f(u(t))u(t)dx \\
&= \langle F(u_t), u(t)\rangle + \langle F(u_t), \dot{u}(t)\rangle + \langle g(t), \dot{u}(t)\rangle + \langle g(t), u(t)\rangle - \|\dot{u}(t)\|^2.
\end{aligned}$$

We have

$$\|u(t)\|_V^2 \geq \frac{1}{2}\|u(t)\|_V^2 + \frac{\lambda_1}{2}\|u(t)\|^2.$$

From condition (1.2), there exist $M_3, M_4 > 0$ such that

$$M_3(|u|^p - 1) \leq \mathcal{F}(u) \leq M_4(|u|^p + 1), \tag{3.6}$$

and using (1.2) once again, we have

$$\int_\Omega f(u(t))u(t)dx \geq \frac{C_1}{M_4}\int_\Omega \mathcal{F}(u(t))dx - (C_0 + C_1)|\Omega|.$$

Using condition (1.5), we get

$$\begin{aligned}
\langle F(u_t), u(t)\rangle &= \frac{4k_1}{\lambda_1}\langle F(u_t), \frac{\lambda_1}{4k_1}u(t)\rangle \\
&\leq \frac{4k_1}{\lambda_1}\left(k_1\frac{\lambda_1^2}{16k_1^2}\|u(t)\|^2 + k_2\int_{-r}^0 \|u(t+\theta)\|^2 d\theta + k_3\right) \\
&\leq \frac{\lambda_1}{4}\|u(t)\|^2 + \frac{4k_1k_2}{\lambda_1}\int_{-r}^0 \|u(t+\theta)\|^2 d\theta + \frac{4k_1k_3}{\lambda_1}.
\end{aligned}$$

Similarly,

$$\langle F(u_t), \dot{u}(t)\rangle \leq \frac{1}{2}\|\dot{u}(t)\|^2 + 2k_1k_2\int_{-r}^0 \|u(t+\theta)\|^2 d\theta + 2k_1k_3.$$

Using the Cauchy inequality, we get

- $\langle g(t), \dot{u}(t)\rangle \leq \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|\dot{u}(t)\|^2;$

- $\langle g(t), u(t)\rangle \leq \dfrac{1}{\lambda_1}\|g(t)\|^2 + \dfrac{\lambda_1}{4}\|u(t)\|^2 \leq \dfrac{1}{\lambda_1}\|g(t)\|^2 + \dfrac{1}{4}\|u(t)\|_V^2.$

Put

$$\begin{aligned}
\Psi(t) &= \|u(t)\|^2 + \|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx,\\
\gamma &= \min\left\{\frac{\lambda_1}{2}, \frac{1}{2}, \frac{C_1}{M_4}\right\},\\
M_5 &= 2(C_0 + C_1)|\Omega| + \frac{8k_1k_3}{\lambda_1} + 4k_1k_3,\\
M_6 &= \frac{8k_1k_2}{\lambda_1} + 4k_1k_2,\\
M_7 &= 1 + \frac{2}{\lambda_1},
\end{aligned}$$

from the above estimates we get

$$\frac{d}{dt}\Psi(t) + \gamma\Psi(t) \leq M_5 + M_6\int_{-r}^0 \|u_t(\theta)\|^2 d\theta + M_7\|g(t)\|^2.$$

Hence

$$\begin{aligned}
\frac{d}{dt}\left((t-\tau)e^{ct}\Psi(t)\right) \leq{}& [1 + (c-\gamma)(t-\tau)]e^{ct}\Psi(t) + M_5(t-\tau)e^{ct}\\
&+ M_6(t-\tau)\int_{-r}^0 e^{ct}\|u_t(\theta)\|^2 d\theta + M_7(t-\tau)e^{ct}\|g(t)\|^2.
\end{aligned}$$

Integrating from τ to t and using (3.5), we get

$$\begin{aligned}
(t-\tau)e^{ct}\Psi(t) \leq{}& [1 + (c-\gamma)(t-\tau)]\int_\tau^t e^{cs}\Psi(s)ds + \frac{M_5}{c}(t-\tau)e^{ct}\\
&+ M_6(t-\tau)re^{cr}\int_\tau^t e^{cs}\|u(s)\|^2 ds + M_6(t-\tau)re^{c(\tau+r)}\|\varphi\|_{L_H^2}^2\\
&+ M_7(t-\tau)\int_\tau^t e^{cs}\|g(s)\|^2 ds.
\end{aligned} \tag{3.7}$$

Now, we will derive some estimates on $\int_\tau^t e^{cs}\|u(s)\|^2 ds$ and $\int_\tau^t e^{cs}\Psi(s)ds$. Multiplying (3.3) by e^{ct}, we get

$$e^{ct}\|u(t)\|^2 \leq e^{c\tau}\|u^0\|^2 + 2k_2re^{c(\tau+r)}\|\varphi\|_{L_H^2}^2 + M_2e^{ct} + \int_{-\infty}^t e^{cs}\|g(s)\|ds.$$

and integrating from τ to t, we obtain

$$\begin{aligned}
\int_\tau^t e^{cs}\|u(s)\|^2 ds \leq{}& (t-\tau)e^{c\tau}\left(\|u^0\|^2 + 2k_2re^{cr}\|\varphi\|_{L_H^2}^2\right) + \frac{M_2}{c}e^{ct}\\
&+ \int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2 dyds.
\end{aligned} \tag{3.8}$$

Using (3.4) and the fact that $\lambda_1\|u(t)\|^2 \leq \|u(t)\|_V^2$, we have

$$\begin{aligned}&\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|_V^2 + 2C_1\|u(t)\|_{L^p(\Omega)}^p \\ &\leq 2C_0|\Omega| + 2k_1\|u(t)\|^2 + 2k_2\int_{-r}^0 \|u_t(\theta)\|^2 d\theta + 2k_3 + \frac{1}{\lambda_1}\|g(t)\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\left(e^{ct}\|u(t)\|^2\right) + e^{ct}\left(\|u(t)\|_V^2 + 2C_1\|u(t)\|_{L^p(\Omega)}^p\right) \\ &\leq 2(C_0|\Omega| + k_3)e^{ct} + (c+2k_1)e^{ct}\|u(t)\|^2 + 2k_2\int_{-r}^0 e^{ct}\|u_t(\theta)\|^2 d\theta + \frac{1}{\lambda_1}e^{ct}\|g(t)\|^2.\end{aligned}$$

Integrating from τ to t, we have

$$\begin{aligned}&e^{ct}\|u(t)\|^2 - e^{c\tau}\|u(\tau)\|^2 + \int_\tau^t e^{cs}\left(\|u(s)\|_V^2 + 2C_1\|u(s)\|_{L^p(\Omega)}^p\right) ds \\ &\leq M_8 e^{ct} + (M_9 - 1)\int_\tau^t e^{cs}\|u(s)\|^2 ds + 2k_2 r e^{c(r+\tau)}\|\varphi\|_{L_H^2}^2 + \frac{1}{\lambda_1}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds,\end{aligned}$$

where $M_8 = \dfrac{2(C_0|\Omega| + k_3)}{c}$, $M_9 = c + 2k_1 + 2k_2 r e^{cr} + 1$.

Using (3.8), we have

$$\begin{aligned}&\int_\tau^t e^{cs}\left(\|u(s)\|^2 + \|u(s)\|_V^2 + 2C_1\|u(s)\|_{L^p(\Omega)}^p\right) ds \\ &\leq e^{c\tau}\|u^0\|^2 + M_8 e^{ct} + 2k_2 r e^{c(r+\tau)}\|\varphi\|_{L_H^2}^2 \\ &\quad + \frac{1}{\lambda_1}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds + M_9\int_\tau^t e^{cs}\|u(s)\|^2 ds \\ &\leq [1 + M_9(t-\tau)]\left(e^{c\tau}\|u^0\|^2 + 2k_2 r e^{c(r+\tau)}\|\varphi\|_{L_H^2}^2\right) + \left(M_8 + \frac{M_2M_9}{c}\right)e^{ct} \\ &\quad + \frac{1}{\lambda_1}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds + M_9\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2 dy ds.\end{aligned}$$

Since

$$\|u(t)\|^2 + \|u(t)\|_V^2 + 2C_1\|u(t)\|_{L^p(\Omega)}^p \geq \gamma\Psi(t) - 2C_1|\Omega|,$$

we have

$$\begin{aligned}&\int_\tau^t e^{cs}\Psi(s) ds \\ &\leq \frac{1 + M_9(t-\tau)}{\gamma}\left(e^{c\tau}\|u^0\|^2 + 2k_2 r e^{c(r+\tau)}\|\varphi\|_{L_H^2}^2\right) + \left(\frac{M_8}{\gamma} + \frac{M_2M_9 + 2C_1|\Omega|}{c\gamma}\right)e^{ct} \\ &\quad + \frac{1}{\gamma\lambda_1}\int_{-\infty}^t e^{cs}\|g(s)\|^2 ds + \frac{M_9}{\gamma}\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2 dy ds.\end{aligned} \tag{3.9}$$

Combine (3.8) and (3.9) with (3.7), we get

$$
\begin{aligned}
&(t-\tau)e^{ct}\Psi(t) \le \left[\frac{1}{\gamma} + \frac{M_9+c-\gamma}{\gamma}(t-\tau) + \left(\frac{M_9(c-\gamma)}{\gamma} + M_6re^{cr}\right)(t-\tau)^2\right] \\
&\times e^{c\tau}\|u^0\|^2 + \left[\frac{2k_2r}{\gamma} + \frac{2k_2r(c+M_9-\gamma)+M_6r\gamma}{\gamma}(t-\tau)\right. \\
&+ \left.\frac{2k_2r(M_6\gamma re^{cr}+M_9(c-\gamma))}{\gamma}(t-\tau)^2\right] e^{c(\tau+r)}\|\varphi\|^2_{L^2_H} \\
&+ \left[\frac{2C_1|\Omega|+cM_8+M_2M_9}{c\gamma}\right. \\
&+ \left.\frac{(2C_1|\Omega|+cM_8+M_2M_9)(c-\gamma)+\gamma(M_5+M_2M_6re^{cr})}{c\gamma}(t-\tau)\right] e^{ct} \\
&+ \left[\frac{1}{\gamma\lambda_1} + \left(\frac{c-\gamma}{\gamma\lambda_1} + M_7\right)(t-\tau)\right]\int_{-\infty}^t e^{cs}\|g(s)\|^2ds \\
&+ \left[\frac{M_9}{\gamma} + \left(\frac{(c-\gamma)M_9}{\gamma} + M_6re^{cr}\right)(t-\tau)\right]\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2dyds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Psi(t) \le\ & \left[\frac{1}{\gamma(t-\tau)} + \frac{M_9+c-\gamma}{\gamma} + M_{10}(t-\tau)\right] e^{-c(t-\tau)}\|u^0\|^2 \\
&+ \left[\frac{M_{11}}{t-\tau} + M_{12} + M_{13}(t-\tau)\right] e^{-c(t-\tau-r)}\|\varphi\|^2_{L^2_H} + \left[\frac{M_{14}}{t-\tau} + M_{15}\right] \\
&+ \left[\frac{M_{16}}{t-\tau} + M_{17}\right] e^{-ct}\int_{-\infty}^t e^{cs}\|g(s)\|^2ds \\
&+ \left[\frac{M_{18}}{t-\tau} + M_{19}\right] e^{-ct}\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2dyds.
\end{aligned}
$$

This completes the proof. □

Lemma 3.8. *Assume that* $(H1)-(H5)$ *hold. Then the process* $U(t,\tau)$ *associated to* (1.1) *has a family of pullback absorbing sets* $\{\mathcal{B}_V(t)\}_{t\in\mathbb{R}}$ *in the space* M^2_V.

Proof. Let

$$
\begin{aligned}
R_2(t) = 2M_{20}&\left(1 + e^{-ct}\int_{-\infty}^t e^{cs}\|g(s)\|^2ds + e^{-ct}\int_{-\infty}^t\int_{-\infty}^s e^{cy}\|g(y)\|^2dyds\right) \\
&< +\infty,
\end{aligned}
$$

then from Lemma 3.7, there exists $\hat\tau = \hat\tau(t,u^0,\varphi) \le t$ such that

$$\|u(t)\|^2_V \le R_2(t), \tag{3.10}$$

$$\|u_t(\theta)\|^2_V \le e^{cr}R_2(t), \tag{3.11}$$

$$\|u_t(\theta)\|^2_V + 2M_3\|u(t)\|^p_{L^p(\Omega)} \le 2M_3|\Omega| + e^{cr}R_2(t), \tag{3.12}$$

for all $\tau \leq \hat{\tau} - r$ and $\theta \in (-r, 0)$. So, we have

$$\|U(t,\tau)(u^0,\varphi)\|^2_{M^2_V} = \|u(t)\|^2_V + \int_{-r}^{0} \|u_t(\theta)\|^2_V d\theta \leq (1 + re^{cr})R_2(t) = R^2_V(t).$$

Then, for any bounded set $D \subset M^2_V$, one easily deduces that

$$U(t,\tau)D \subset \mathcal{B}_V(t) = B_{M^2_V}(0, R_V(t)),$$

for all $\tau \leq \hat{\tau}(t, D) - r$. This completes the proof. □

Theorem 3.9. *Under assumptions* $(H1) - (H5)$*, the process* $U(t,\tau)$ *associated to* (1.1) *has a pullback attractor* $\hat{A} = \{A(t)\}_{t\in\mathbb{R}}$ *in the space* M^2_H.

Proof. Due to the result of Lemma 3.3, $U(t,\tau)$ is a continuous process on M^2_H. Therefore, by Theorem 2.3, we need to prove that there exists a family of compact pullback absorbing sets in M^2_H. From Lemma 3.7, $U(t,\tau)$ has a family of pullback absorbing sets $\{\mathcal{B}_V(t)\}$ in M^2_V. Let

$$\mathcal{B}(t) = \bigcup_{\tau \leq \hat{\tau}(t,\mathcal{B}_V) - r} U(t,\tau)\mathcal{B}_V(t).$$

It is easy to see that $\{\mathcal{B}(t)\}$ is a pullback absorbing in M^2_H for $U(t,\tau)$. We now show that $\mathcal{B}(t)$ is precompact in M^2_H. Let Π_1 and Π_2 are canonical projectors on M^2_H, i.e. $\Pi_1 : (u^0,\varphi) \mapsto u^0$ and $\Pi_2 : (u^0,\varphi) \mapsto \varphi$. One observes that $\Pi_1\mathcal{B}(t)$ is bounded in V and then it is precompact in H. It remains to prove that $\Pi_2\mathcal{B}(t)$ is precompact in L^2_H.

Let $\{u^n_t\}_{n=1}^{\infty} \subset \Pi_2\mathcal{B}(t)$. For a given $t > \tau + r$, (3.12) ensures that $u^n(t+\theta)$, $\theta \in (-r, 0)$ belongs to a bounded set in $V \cap L^p(\Omega)$. It follows that u^n belongs to a bounded set in $L^2(t-r, t; V \cap L^p(\Omega))$.

By rewriting the equation in (1.1) as

$$\dot{u}^n(t) = F(u^n_t) + g(t) - Au^n(t) - f(u^n(t)),$$

we obtain that $\dot{u}^n$ belongs to a bounded set in

$$L^2(t-r, t; V') + L^{p'}(t-r, t; L^{p'}(\Omega)) \subset L^{p'}(t-r, t; V' + L^{p'}(\Omega)).$$

Using the Aubin-Lions lemma [17], we conclude that u^n belongs to a compact set in $L^2(t-r, t; L^2(\Omega))$, or equivalently, $\{u_t \in \Pi_2\mathcal{B}(t)\}$ is precompact in L^2_H. The proof is complete. □

Remark 3.10. In the case $p = 2$, i.e., $f(u) = du$ $(d > 0)$ as in [15], since we no longer have Lemma 3.1, the conditions of the external force g should be changed as follows:

$$\int_{-\infty}^{0} e^{\lambda_1 s}\|g(s)\|^2 ds < +\infty \text{ and } \int_{-\infty}^{0}\int_{-\infty}^{s} e^{\lambda_1 y}\|g(y)\|^2 dy ds < +\infty,$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator A. Using the above arguments, one can show that if g satisfies the above conditions and $\lambda_1 + d > k_1 + k_2 r$, then there exists a pullback attractor in the space M^2_H for the process $U(t,\tau)$.

4. Existence of a uniform attractor

In this section, instead of $(H5)$, we assume that the external force g satisfies

- $(H5bis)$ g is a translation bounded function in $L^2_{loc}(\mathbb{R};H)$, that is, $g \in L^2_{loc}(\mathbb{R};H)$ such that

$$\|g\|^2_{L^2_b} = \|g\|^2_{L^2_b(\mathbb{R};H)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s)\|^2 ds < +\infty.$$

Denote by $L^2_b(\mathbb{R};H)$ the space of all translation bounded functions and by $\mathcal{H}_w(g)$ the closure of the set $\{g(s+\cdot) \mid s \in \mathbb{R}\}$ in $L^{2,w}_{loc}(\mathbb{R};H)$ with the weak topology. It is well-known [11, 18] that $\mathcal{H}_w(g)$ is weakly compact, and if $g_0 \in \mathcal{H}_w(g)$ then $g_0 \in L^2_b(\mathbb{R};H)$ and $\|g_0\|^2_{L^2_b} \le \|g\|^2_{L^2_b}$. Therefore, for any $g_0 \in \mathcal{H}_w(g)$,

$$\begin{aligned} e^{-ct}\int_{-\infty}^t e^{cs}\|g_0(s)\|^2 ds &= \int_{-\infty}^t e^{-c(t-s)}\|g_0(s)\|^2 ds \\ &= \sum_{k=0}^{\infty} \int_{t-k-1}^{t-k} e^{-c(t-s)}\|g_0(s)\|^2 ds \\ &\le \sum_{k=0}^{\infty} e^{-ck}\int_{t-k-1}^{t-k} \|g_0(s)\|^2 ds \\ &\le \|g_0\|^2_{L^2_b}\sum_{k=0}^{\infty} e^{-ck} = \frac{1}{1-e^{-c}}\|g_0\|^2_{L^2_b} \\ &\le \frac{1}{1-e^{-c}}\|g\|^2_{L^2_b}. \end{aligned}$$

It is evident that $(H5bis)$ implies $(H5)$, so we can use all the results obtained in Section 3.

Consider the corresponding family of equations:

$$(4.1) \qquad \begin{cases} \dfrac{d}{dt}u(t) + Au(t) + f(u(t)) = F(u_t) + g_0(t), \\ u(\tau) = u^0, \quad u(\tau+\theta) = \varphi(\theta), \qquad \theta \in (-r,0). \end{cases}$$

Assume conditions $(H2)-(H4)$ hold. Then for any $g_0 \in \mathcal{H}_w(g)$ and $(u^0,\varphi) \in M^2_H$, $\tau \in \mathbb{R}$ are given, Theorem 3.2 implies that there exists a unique weak solution $u(.) = u(.;\tau,(u^0,\varphi),g_0)$ of Problem (4.1).

We thus can define a process $U_{g_0}(.,.) : M^2_H \to M^2_H$ in the product space as

$$U_{g_0}(t,\tau)(u^0,\varphi) = (u(t;\tau,(u^0,\varphi),g_0), u_t(.,\tau,(u^0,\varphi),g_0)), \quad \forall (u^0,\varphi) \in M^2_H,\ t \ge \tau,$$

and the corresponding family of processes as $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$.

Lemma 4.1. *Assume that $(H1)-(H4)$ and $(H5bis)$ hold. Then the family of processes $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$ is $(M^2_H \times \mathcal{H}_w(g), M^2_H)$-continuous.*

proof The proof follows the same lines in the proof of Lemma 4.4 in [15], so it is omitted here.

Lemma 4.2. *Assume that* $(H1)-(H4)$ *and* $(H5bis)$ *hold. Then there exists a bounded uniformly absorbing set* $\mathcal{B}_1$ *in* M_H^2 *for the family of processes* $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$.

Proof. By Lemma 3.5, we have

$$\begin{aligned}\|u(t)\|^2 &\leq e^{-c(t-\tau)}\|u^0\|^2 + 2k_2 r e^{-c(t-\tau-r)}\|\varphi\|^2_{L^2_H} + M_2 + e^{-ct}\int_{-\infty}^{t} e^{cs}\|g_0(s)\|^2 ds \\ &\leq e^{-c(t-\tau)}\|u^0\|^2 + 2k_2 r e^{-c(t-\tau-r)}\|\varphi\|^2_{L^2_H} + M_2 + \frac{1}{1-e^{-c}}\|g\|^2_{L^2_b}.\end{aligned}$$

Denote $\rho_1 = \rho_1(g) = 2M_2 + \frac{2}{1-e^{-c}}\|g\|^2_{L^2_b}$. Given $D \in \mathcal{B}(M_H^2)$, there exists $\hat{\tau} = \hat{\tau}(D) > \tau$ such that for all $t \geq \hat{\tau} + r$, $(u^0, \varphi) \in D$, $g_0 \in \mathcal{H}_w(g)$, we have

$$\|u(t)\|^2 \leq \rho_1(g), \tag{4.2}$$

$$\|u_t\|^2_{L^2_H} = \int_{-r}^{0} \|u(t+\theta)\|^2 d\theta \leq r\rho_1(g). \tag{4.3}$$

Hence, it is obvious that

$$\|U_{g_0}(t,\tau)(u^0,\varphi)\|^2_{M_H^2} = \|u(t)\|^2 + \|u_t\|^2_{L^2_H} \leq (1+r)\rho_1(g) = \rho_H^2(g).$$

This means that the closed ball $\mathcal{B}_1 = B_{M_H^2}(0, \rho_H(g))$ forms a uniformly absorbing set for the mappings $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$. □

Lemma 4.3. *Under the assumptions of Lemma 4.2, there exists a bounded uniformly absorbing set* $\mathcal{B}_2$ *in* M_V^2 *for the family of processes* $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$.

Proof. Let $u(t) = U_{g_0}(t,\tau)(u^0,\varphi)$. We will prove that

$$\|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx \leq \rho_2 = \rho_2(g),$$

for all $t \geq \hat{\tau} + r + 1$ by using the uniform Gronwall lemma.

First, multiplying the first equation in (4.1) by $\dot{u}(t)$, we get

$$\begin{aligned}&\|\dot{u}(t)\|^2 + \frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx\right) \\ &\leq \frac{1}{2}\|\dot{u}(t)\|^2 + 2k_1k_2\|u_t\|^2_{L^2_H} + 2k_1k_3 + \frac{1}{2}\|g_0(t)\|^2 + \frac{1}{2}\|\dot{u}(t)\|^2,\end{aligned}$$

and therefore

$$\frac{d}{dt}\left(\|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx\right) \leq 4k_1k_2\|u_t\|^2_{L^2_H} + 4k_1k_3 + \|g_0(t)\|^2. \tag{4.4}$$

From Equation (4.1), using (1.5), (1.2) and the Cauchy inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|_V^2 + 2C_1\|u(t)\|_{L^p(\Omega)}^p \\
&\leq 2C_0|\Omega| + 2k_1\|u(t)\|^2 + 2k_2\int_{-r}^0 \|u_t(\theta)\|^2 d\theta + 2k_3 + \frac{1}{\lambda_1}\|g_0(t)\|^2 \\
&\leq 2C_0|\Omega| + 2k_1\rho_1 + 2k_2 r\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g_0(t)\|^2.
\end{aligned}$$

Integrating from t to $t+1$ (with $t \geq \hat{\tau} + r$), we have

$$\begin{aligned}
&\|u(t+1)\|^2 - \|u(t)\|^2 + \int_t^{t+1}\left(\|u(s)\|_V^2 + \frac{2C_1}{M_4}\int_\Omega \mathcal{F}(u(s))dx\right)ds \\
&\leq 2(C_0+C_1)|\Omega| + 2k_1\rho_1 + 2k_2 r\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g_0\|_{L_b^2}^2 \\
&\leq 2(C_0+C_1)|\Omega| + 2k_1\rho_1 + 2k_2 r\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g\|_{L_b^2}^2.
\end{aligned}$$

Since we can take $M_4 \geq C_1$, it follows that

$$\begin{aligned}
&\frac{C_1}{M_4}\int_t^{t+1}\left(\|u(s)\|_V^2 + 2\int_\Omega \mathcal{F}(u(s))dx\right)ds \\
&\leq 2(C_0+C_1)|\Omega| + 2k_1\rho_1 + 2k_2 r\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g\|_{L_b^2}^2 + \|u(t)\|^2 \\
&\leq 2(C_0+C_1)|\Omega| + (1 + 2k_1 + 2k_2 r)\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g\|_{L_b^2}^2.
\end{aligned}$$

Putting

$$I_V = \frac{M_4\left(2(C_0+C_1)|\Omega| + (1 + 2k_1 + 2k_2 r)\rho_1 + 2k_3 + \frac{1}{\lambda_1}\|g\|_{L_b^2}^2\right)}{C_1},$$

we have

$$\int_t^{t+1}\left(\|u(s)\|_V^2 + 2\int_\Omega \mathcal{F}(u(s))dx\right)ds \leq I_V. \tag{4.5}$$

Therefore, we have

$$\int_t^{t+1}\left(4k_1k_2\|u_t\|_{L_H^2}^2 + 4k_1k_3 + \|g_0(s)\|^2\right)ds \leq 4k_1k_2 r\rho_1 + 4k_1k_3 + \|g\|_{L_b^2}^2 = I_h. \tag{4.6}$$

Now, from (4.4)-(4.6), we can apply the uniform Gronwall inequality to obtain

$$\|u(t)\|_V^2 + 2\int_\Omega \mathcal{F}(u(t))dx \leq I_V + I_h = \rho_2, \text{ for all } t \geq \hat{\tau} + r + 1.$$

Using (3.6), we obtain

$$\|u(t)\|_V^2 + 2M_3\|u(t)\|_{L^p(\Omega)}^p \le \rho_2 + 2M_3|\Omega|,$$
$$\|u_t(\theta)\|_V^2 + 2M_3\|u_t(\theta)\|_{L^p(\Omega)}^p \le \rho_2 + 2M_3|\Omega|,$$
$$\|u_t\|_{L_V^2}^2 \le r\rho_2 + 2rM_3|\Omega|,$$

for all $t \ge \hat{\tau} + 2r + 1$ and $\theta \in (-r, 0)$. We arrive at the conclusion that, for any bounded set $D \subset M_V^2$, we have

$$U_{g_0}(t,\tau)D \subset \mathcal{B}_2 = B_{M_V^2}\left(0, \sqrt{(1+r)(\rho_2 + 2M_3|\Omega|)}\right)$$

for t large enough and for all $g_0 \in \mathcal{H}(g)$. Thus, $\{U_{g_0}(t,\tau) \mid g_0 \in \mathcal{H}(g)\}$ has the uniform absorbing set $\mathcal{B}_2$ in M_V^2. □

Theorem 4.4. *Assume that* $(H1) - (H4)$ *and* $(H5bis)$ *hold. Then there exists a uniform attractor* $\mathcal{A}_{\mathcal{H}(g)}$ *in* M_H^2 *for the family of processes* $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$. *Moreover,* $\mathcal{A}_{\mathcal{H}(g)}$ *is compact in* M_H^2*, and*

$$\mathcal{A}_{\mathcal{H}_w(g)} = \bigcup_{g_0 \in \mathcal{H}_w(g)} \mathcal{K}_{g_0}(s) \quad \forall s \in \mathbb{R},$$

where $\mathcal{K}_{g_0}$ *is the kernel of the process* $U_{g_0}(t,\tau)$.

Proof. Let us consider the set $\mathcal{B}_2$. This is a bounded uniformly (w.r.t. $g_0 \in \mathcal{H}_w(g)$) absorbing set for $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$.

As in the proof of Theorem 3.9, we can show that $\mathcal{B}$ is precompact in M_H^2.

Since $\mathcal{B}$ is relatively compact in M_H^2, hence $\overline{\mathcal{B}}$, where the closure is taken in M_H^2, is a compact uniformly (w.r.t. $g_0 \in \mathcal{H}_w(g)$) absorbing set in M_H^2 for $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$. By Theorem 2.6, this ensures the existence and structure of the uniform attractor $\mathcal{A}_{\mathcal{H}_w(g)}$ for the family of processes $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$ as stated. □

Remark 4.5. In the case $f(u) = du$ $(d > 0)$ as in [15], using the above arguments one can show that if $\lambda_1 + d > k_1 + k_2 r$, then there exists a uniform attractor in the space M_H^2 for the family of processes $\{U_{g_0}(.,.) \mid g_0 \in \mathcal{H}_w(g)\}$. Thus, in particular, this result improves the recent one in [15]. It is noticed that our approach is different from the one used in [15], and we only require that g is translation bounded, while in [15] the authors assumed that g is translation compact.

5. A relationship between the pullback attractor and the uniform attractor

In this section we assume that the external force g is a translation bounded function. It is proved in Theorem 3.9 that for any $g_0 \in \mathcal{H}_w(g)$, the process $U_{g_0}(t,\tau)$ has a pullback attractor $\hat{\mathcal{A}}_{g_0} = \{A_{g_0}(t) : t \in \mathbb{R}\}$. Moreover, we have

Theorem 5.1. *Under conditions* $(H1)-(H4)$ *and* $(H5bis)$*, for any* $g_0 \in \mathcal{H}_w(g)$*, the process* $\{U_{g_0}(t,\tau)\}$ *has a pullback attractor* $\hat{\mathcal{A}}_{g_0} = \{A_{g_0}(t) : t \in \mathbb{R}\}$*, and*

$$A_{g_0}(s) = \mathcal{K}_{g_0}(s), \qquad \bigcup_{g_0 \in \mathcal{H}_w(g)} A_{g_0}(s) = \mathcal{A}_{\mathcal{H}_w(g)}, \quad \forall s \in \mathbb{R},$$

where $\mathcal{A}_{\mathcal{H}_w(g)}$ *is the uniform attractor of Problem* (1.1)*,* $\mathcal{K}_{g_0}$ *is the kernel of the process* $U_{g_0}(t,\tau)$*.*

Proof. Since $\hat{\mathcal{A}}_{g_0}$ is pullback attracting, and $A_{g_0}(s)$ is compact, we have

$$\mathcal{K}_{g_0}(s) \subset A_{g_0}(s) \quad \text{for any } s \in \mathbb{R}.$$

On the other hand, by the definition of $\mathcal{K}_{g_0}(s)$ and the invariance of $\hat{\mathcal{A}}_{g_0}$, we have

$$A_{g_0}(s) \subset \mathcal{K}_{g_0}(s) \quad \text{for any } s \in \mathbb{R}.$$

So, we have

$$A_{g_0}(s) = \mathcal{K}_{g_0}(s) \quad \text{for any } s \in \mathbb{R}. \tag{5.1}$$

Next, by (5.1) and Theorem 4.4,

$$\mathcal{A}_{\mathcal{H}_w(g)} = \bigcup_{g_0 \in \mathcal{H}_w(g)} \mathcal{K}_{g_0}(s) = \bigcup_{g_0 \in \mathcal{H}_w(g)} A_{g_0}(s), \quad \forall s \in \mathbb{R}.$$

The proof is complete. □

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