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FOURIER AND HERZ ALGEBRAS OF A COMPACT TENSOR HYPERGROUP

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Abstract. We study the basic properties of the Fourier and Herz spaces $A(K)$ and $A_p(K)$ of a compact hypergroup K and associate them with subspaces of the Cartesian product of matrix algebras on the dual hypergroup. When K is a tensor hypergroup we show that $A(K)$ is a regular Banach algebra whose spectrum is K. We also compute some of the corresponding multiplier algebras.

1. Introduction

Fourier algebra of a locally compact group is defined in 60's by Pier Eymard [Ey] and since then, it plays a crucial role in harmonic analysis of topological groups [HR],[Pi]. Fourier algebra is a generalization of the algebra of all Fourier transforms of absolutely integrable functions on a locally compact abelian group. In the non-abelian case, positive definite functions are used to define this algebra. Positive definite functions are introduced and studied much earlier (see for instance [Go]).

Topological hypergroups are generalizations of topological groups, motivated by Physical applications [BK]. The harmonic analysis of these structures (and in particular, positive definite functions on them) is widely studied [BH], [La1], [La2], [Vo]. These functions show a pathological behavior on hypergroups. In contrast with the group case [Ey], the product of positive definite functions on a topological hypergroup need not be positive definite. Hence, for hypergroups (even in the compact case) we only have a Fourier space [Vr]. Indeed, there is a finite hypergroup with three elements on which the Banach algebra norm condition fails even for characters [Vr].

The hypergroups with property (P) are defined as the class of hypergroups for which the product of two positive definite functions is again positive definite (see for instance [Vo]). If this is the case, there is an equivalent norm which makes the Fourier space a Banach algebra [AM].

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The authors introduced and studied the (n-)tensor hypergroups in [AM] and showed that hypergroups with property (P) are exactly 2-tensor hypergroups. The Fourier space of tensor hypergroups are Banach algebras and share some of important properties of group Fourier algebras.

In this paper, we further study Fourier algebras of compact tensor hypergroups and show that they are natural, regular Banach function algebras. We also calculate the linear conjugate of these algebras. Similar results are obtained for the Herz algebra of a compact hypergroup. Finally, multipliers of function algebras on a compact hypergroup are studied along the lines of [HR] and in particular, the conjugate spaces of Fourier and Herz algebras of a compact hypergroup is calculated as a multiplier algebra.

2. FOURIER AND HERZ SPACES

Let K be a compact hypergroup and $\hat{K}$ denote the set of equivalence classes of all continuous irreducible representations of K. For $\pi \in \hat{K}$, let $\{\xi_i^\pi\}_{i=1}^{d_\pi}$ be an orthonormal basis for the corresponding (finite dimensional) Hilbert space H_π and put

$$\pi_{i,j}(x) = \langle \pi(x)\xi_i^\pi, \xi_j^\pi \rangle \quad (1 \le i, j \le d_\pi).$$

Define the conjugation operator D_π on H_π by

$$D_\pi(\sum_{i=1}^{d_\pi} \alpha_i \xi_i^\pi) = \sum_{i=1}^{d_\pi} \bar{\alpha}_i \xi_i^\pi,$$

and put $\bar{\pi} = D_\pi \pi D_\pi$. Let $Trig_\pi(K) = \text{span}\{\pi_{i,j} : 1 \le i, j \le d_\pi\}$ and $Trig(K) = \text{span}\{\pi_{i,j} : \pi \in \hat{K}, 1 \le i, j \le d_\pi\}$. Then $\dim Trig_\pi(K) = d_\pi^2$ and there is $k_\pi \ge d_\pi$ such that

$$\int_K \pi_{i,j} \bar{\sigma}_{r,s} dm = k_\pi^{-1} \delta_{\pi,\sigma} \delta_{i,r} \delta_{j,s} \quad (\pi, \sigma \in \hat{K}) \text{ [Vr, 2.6]}.$$

Also $\{k_\pi^{\frac{1}{2}} \pi_{i,j} : \pi \in \hat{K}, 1 \le i, j \le d_\pi\}$ is an orthonormal basis of $L^2(K)$ and

$$Trig(K) = \oplus_{\pi \in \hat{K}} Trig_\pi(K) \text{ [Vr, 2.7]}.$$

In particular, $Trig(K)$ is norm dense in both $C(K)$ and $L^2(K)$ [Vr, 2.13, 2.9].

For each $f \in L^2(K)$ we have the Fourier series expansion

$$f = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} k_\pi \langle f, \pi_{i,j} \rangle \pi_{i,j}$$

where the series converges in L^2-norm.

Consider the $*$-algebra

$$\mathcal{E}(\hat{K}) := \prod_{\pi \in \hat{K}} B(H_\pi)$$

with coordinatewise operations. For $f = (f_\pi) \in \mathcal{E}(\hat{K})$ and $1 \le p < \infty$ put

$$\|f\|_p := \big(\sum_{\pi \in \hat{K}} k_\pi \|f_\pi\|_p^p \big)^{\frac{1}{p}}, \quad \|f\|_\infty := \sup_{\pi \in \hat{K}} \|f_\pi\|_\infty,$$

where the right hand side norms are operator norms as in [HR, D.37, 36(e)]. Define $\mathcal{E}_p(\hat{K})$, $\mathcal{E}_\infty(\hat{K})$, and $\mathcal{E}_0(\hat{K})$ as in [HR, 28.24]. These are Banach spaces with isometric involution [HR, 28.25], [BH]. Also $\mathcal{E}(\hat{K})$ is a C^*-algebra [HR, 28.26]. For each $\mu \in M(K)$, define $\hat{\mu} \in \mathcal{E}_\infty(\hat{K})$ by $\hat{\mu}(\pi) = \bar{\pi}(\mu)$, then $\mu \mapsto \hat{\mu}$ is a norm-decreasing $*$-isomorphism of $M(K)$ into $\mathcal{E}_\infty(\hat{K})$. Similarly one can define a norm-decreasing $*$-isomorphism $f \mapsto \hat{f}$ of $L^1(K)$ onto a dense subalgebra of $\mathcal{E}_0(\hat{K})$ [Vr, 3.2, 3.3]. Also there is an isometric isomorphism $g \mapsto \hat{g}$ of $L^2(K)$ onto $\mathcal{E}_2(\hat{K})$. Each $g \in L^2(K)$ has a Fourier expansion

$$g = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} k_\pi \langle \hat{g}(\pi)\xi_i^\pi, \xi_j^\pi \rangle \pi_{i,j},$$

where the series converges in L^2-norm [Vr, 3.4].

For $\mu \in M(K)$ and $\pi \in \hat{K}$, we set $a_\pi = \bar{\pi}(\mu)^*$, and write

$$\mu \approx \sum_{\pi \in \hat{K}} k_\pi \mathrm{tr}(a_\pi \pi).$$

If $\mu = f dm$, where $f \in L^1(K)$, then we write

$$f \approx \sum_{\pi \in \hat{K}} k_\pi \mathrm{tr}(a_\pi \pi).$$

If moreover $\sum_{\pi \in \hat{K}} k_\pi \|a_\pi\|_1 < \infty$, we write $f \in A(K)$ and put

$$\|f\|_A = \sum_{\pi \in \hat{K}} k_\pi \|\hat{f}(\pi)\|_1.$$

$A(K)$ is a Banach space with respect to this norm, and $f \mapsto \hat{f}$ is an isometric isomorphism of $A(K)$ onto $\mathcal{E}_1(\hat{K})$. Also for each $f \in A(K)$ with $f \simeq \sum_{\pi \in \hat{K}} k_\pi \mathrm{tr}(a_\pi \pi)$ we have

$$f(x) = \sum_{\pi \in \hat{K}} k_\pi \mathrm{tr}(a_\pi \pi(x)),$$

m-a.e. [Vr, 4.2]. If moreover f is positive definite, we have

$$f(e) = \|f\|_u := \sum_{\pi \in \hat{K}} k_\pi \mathrm{tr}(\hat{f}(\pi)),$$

where the series converges absolutely [Vr, 4.4]. If we denote the set of all continuous positive definite functions on K by $P(K)$, then $f \in P(K)$ if and only if $f \in A(K)$ and each operator $\hat{f}(\pi)$ is positive definite [Vr, 4.6] and $A(K) = \mathrm{span}(P(K)) = L^2(K) * L^2(K)$ [Vr, 4.8, 4.9].

$A(K)$ is a regular Banach algebra with convolution product and if K is a tensor hypergroup (a hypergroup with property (P)) then $A(K)$ is also a Banach algebra with pointwise product (under an equivalent norm) [AM].

For each $\pi \in \hat{K}$, let M_π denote the algebra of all complex $d_\pi \times d_\pi$ matrices with norm $\|T\| = k_\pi \|T\|_1$, where the norm on the right hand side is the trace class norm $\|T\|_1 = \text{trace}(T^*T)^{\frac{1}{2}}$. Consider the map $j : A(K)\hat{\otimes}A(K) \to A(K \times K)$ defined by norm-decreasing injective maps $j_{\pi\rho} : M_\pi\hat{\otimes}M_\rho \to M_{\pi\rho}$. Then j is injective (but not an isometry). We are guessing that if $\{k_\pi : \pi \in \hat{K}\}$ is bounded above, then j is surjective (compare with [Jo, 2.4]).

Next consider the convolution product $\gamma : A(K)\hat{\otimes}A(K) \to A(K)$ and the quotient map $q : A(K)\hat{\otimes}A(K) \to A(K)\hat{\otimes}A(K)/\ker(\gamma)$ and put $A_\gamma(K) = \text{Im}(\gamma)$ endowed with the norm

$$\|\gamma(F)\|_\gamma := \|q(F)\| \quad \big(F \in A(K)\hat{\otimes}A(K)\big),$$

where the right hand side is in the quotient norm.

Lemma 2.1. *For each $f \in L^1(K)$,*

(i)*$f \in A_\gamma(K)$ if and only if $\sum_{\pi\in\hat{K}} k_\pi^2\|\hat{f}(\pi)\|_1 < \infty$.*

(ii) *If $f \in A_\gamma(K)$, $\|f\|_\gamma = \sum_{\pi\in\hat{K}} k_\pi^2\|\hat{f}(\pi)\|_1$.*

(iii) *If $\phi : \hat{K} \to \cup_{\pi\in\hat{K}}M_\pi$ and $\phi(\pi) \in M_\pi$, for each $\pi \in \hat{K}$ and $\sum_{\pi\in\hat{K}} k_\pi^2\|\hat{f}(\pi)\|_1 < \infty$, then there is $f \in A_\gamma(K)$ such that $\hat{f} = \phi$.*

Proof. Consider the product map $\gamma_\pi : M_\pi\hat{\otimes}M_\pi \to M_\pi$ and let $q_\pi : M_\pi\hat{\otimes}M_\pi \to M_\pi\hat{\otimes}M_\pi/\ker(\gamma_\pi)$ be the corresponding quotient map, then $\gamma_\pi = \gamma_\pi' \circ q_\pi$, for some surjective isometry $\gamma_\pi' : M_\pi\hat{\otimes}M_\pi/\ker(\gamma_\pi) \to M_\pi$, with respect to the trace class norm on M_π [Jo, 2.5]. Now $A_\gamma(K) = \text{Im}(\gamma) = \oplus\text{Im}(\gamma_\pi)$ and the result follows from [Jo, 2.3] and the fact that $A(K) \cong \mathcal{E}_1(\hat{K}) \cong \bigoplus_{\pi\in\hat{K}} M_\pi$, where the sum is the ℓ^1-direct sum of Banach spaces. □

Lemma 2.2. *Each norm-closed, translation and conjugation invariant subspace I of $L^2(K)$ is a two-sided ideal with respect to convolution.*

Proof. For each $f, g, \phi \in L^2(K)$,

$$\int_K (f * g)\phi dm = \int_K g(\bar{f} * \phi)dm \quad [\text{Je}, 6.2D].$$

If $\phi \in I^\perp$ then for each $f \in I$ and $x \in K$, ${}_x\bar{f} \in I$ and so $f * \phi(x) = \int_K {}_x\bar{f}\phi dm = 0$. Hence $\int_K (f * g)\phi dm = 0$, and so $f * g \in (I^\perp)^\perp = \bar{I} = I$. □

Proposition 2.3. *If K is a tensor hypergroup (a hypergroup with property (P)) then $A(K)$ and $A_\gamma(K)$ are Banach algebras with pointwise product (under an equivalent norm).*

Proof. The first assertion is proved in [AM]. It is known that $A_\gamma(K)$ is a regular Banach algebra with convolution product [Vr]. A minimal two-sided ideal of $A_\gamma(K)$ is of the form

$$J_\pi = \{f \in A_\gamma(K) : \hat{f}(\rho) = 0 \ (\rho \neq \pi)\}.$$

For each $f \in A_\gamma(K)$ we have

$$\hat{f} = \sum_{\pi\in\hat{K}} \hat{f}(\pi)I_\pi,$$

where I_π is the identity matrix in M_π. Consider $\phi : \hat{K} \to \cup_{\pi\in\hat{K}} M_\pi$ defined by $\phi_\pi(\rho) = I_\pi \delta_{\pi\rho}$. Then by Lemma 2.1 (iii), there is $f_\pi \in A_\gamma(K)$ such that $\hat{f}_\pi = \phi_\pi$. Then $f_\pi \in J_\pi$ and

$$f = \sum_{\pi\in\hat{K}} \hat{f}(\pi) f_\pi = \sum_{\pi\in\hat{K}} \lambda_\pi f_\pi^{'},$$

where $\lambda_\pi = k_\pi^2 \|\hat{f}(\pi)\|_1 / \|f\|_\gamma$ and $f_\pi^{'} = \lambda_\pi^{-1} \hat{f}(\pi) f_\pi$, if $\lambda_\pi \neq 0$ and $f_\pi^{'} = 0$, otherwise. Hence $f_\pi^{'} \in J - \pi$ and f is a limit of convex combinations of elements of J_π's. Therefore we only need to show that for each $\pi, \rho \in \hat{K}$, $f \in J_\pi$ and $g \in J_\rho$, we have $\|fg\|_\gamma \leq \|f\|_\gamma \|g\|_\gamma$. Let $\gamma_{\pi\rho} : J_\pi \otimes J_\rho \to L^2(K)$ be the convolution product and $I = \mathrm{Im}(\gamma_{\pi\rho})$. Then $\mathrm{dim} I \leq k_\pi^2 k_\rho^2$ and I is clearly conjugation closed (i.e. $\bar{f} \in I$, whenever $f \in I$). Also I is translation invariant, since ${}_x(f * g) = ({}_x f) * g$ and $({}_x f)\hat{}(\rho) = \rho(\bar{x}) \hat{f}(\rho)$, for each $x \in K$, $\rho \in \hat{K}$, and $f, g \in L^2(K)$. Therefore, by Lemma 2.2, I is a finite dimensional two-sided ideal of $A_\gamma(K)$, and so $I = \oplus_{\sigma\in S} J_\sigma$, for some finite subset S of $\hat{K}$. Also $\sum_{\sigma\in S} k_\sigma^2 = \mathrm{dim} I \leq k_\pi^2 k_\rho^2$. Therefore

$$\begin{aligned}\|fg\|_\gamma &= \sum_{\sigma\in S} k_\sigma^2 \|(fg)\hat{}(\sigma)\|_1 \leq k_\pi k_\rho \sum_{\sigma\in S} k_\sigma \|(fg)\hat{}(\sigma)\|_1 \\ &= k_\pi k_\rho \|fg\| \leq k_\pi k_\rho \|f\| \|g\| = k_\pi^2 k_\rho^2 \|\hat{f}(\pi\|_1 \|\hat{g}(\rho)\|_1 = \|f\|_\gamma \|g\|_\gamma.\end{aligned}$$

□

Theorem 2.4. *If K is a tensor hypergroup, the spectrum of the commutative Banach algebra $A(K)$ is K. In particular, $A(K)$ is a natural, regular Banach function algebra.*

Proof. Using notations of [Da], let $x \in K$ and S be a closed subset of K and put

$$J(S) = \{f \in A(K) : \mathrm{supp}(f) \cap S = \varnothing\} \quad I(S) = \{f \in A(K) : f(S) \subseteq \{0\}\},$$

and put $J_x = J(\{x\})$ and $M_x = I(\{x\})$. We follow the idea of [Da, Thm. 4.5.31]. Let $0 \neq f \in M_x$ and choose $0 < \varepsilon < \|f\|_\infty$. Put $f_a(x) = f(x * a)$, for $a, x \in K$ and

$$W = \{a \in K : \|f_{\bar{a}} - f\|_2 \leq \varepsilon\}.$$

This is a compact neighborhood of $e \in K$. Choose $K_1 \subseteq K$ so that

$$\int_{K\backslash K_1} |f(x * \bar{t})|^2 dm(t) < \varepsilon$$

and put $V_1 = (K\backslash K_1) \cup \{0\}$ and $V = W \cap \bar{W} \cap V_1$, and define

$$g(t) = f(t)\chi_V(\bar{x} * t), \quad u = \frac{1}{m(V)}\chi_V \quad (t \in K),$$

and $h = (f - g) * \check{u} \in A(K)$, then

$$\begin{aligned}h(x) &= \int_K \big(f(t) - f(t)\chi_V(\bar{x} * t)\big)\check{u}(\bar{x} * t) dm(t) \\ &= \frac{1}{m(V)} \int_K \big(f(t)\chi_V(\bar{x} * t) - f(t)\chi_V^2(\bar{x} * t) dm(t) = 0,\end{aligned}$$

that is $h \in J_x$. On the other hand,

$$\|g\|_2^2 = \int_K |f(t)|^2 \chi_V(\bar{x} * t) dm(t) = \int_V |f(x * \bar{t})|^2 dm(t) < \varepsilon.$$

Next, $\|u\|_2 = \frac{1}{m(V)^{\frac{1}{2}}}$ and

$$\begin{aligned}\|f - f * \check{u}\|_2^2 &= \int_K |f(t) - f * \check{u}(t)|^2 dm(t) \\ &\leq \frac{1}{m(V)^2} \int_K \big(\int_V |f(t) - f(t * \bar{y})| dm(y) \big)^2 dm(t) \\ &= \frac{1}{m(V)} \int_K \int_V |f(t) - f(t * \bar{y})|^2 dm(y) dm(t) \\ &= \frac{1}{m(V)} \int_V \|f_{\bar{y}} - f\| dm(y) \leq \varepsilon,\end{aligned}$$

hence

$$\begin{aligned}\|f - h\|_2 &\leq \|f - f * \check{u}\|_2 + \|g\|_2 \|\check{u}\|_2 \\ &\leq \varepsilon(1 + \frac{1}{m(V)^{\frac{1}{2}}}).\end{aligned}$$

Therefore J_x is dense in M_x and the result follows from [Da, 4.1.32]. □

For $1 \leq p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$ and consider the vector space

$$A_p(K) = \{h \in C_0(K) : h = \sum_{k=1}^{\infty} f_k * g_k, f_k \in L^p, g_k \in L^q, \sum_{k=1}^{\infty} \|f_k\|_p . \|g_k\|_q < \infty\}$$

with norm

$$\|h\|_{A_p} = \inf\{\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_q : h = \sum_{k=1}^{\infty} f_k * g_k\}.$$

Note that if $h = \sum_{k=1}^{\infty} f_k * g_k$, then the sum converges uniformly on K and $\|h\|_\infty \leq \|h\|_{A_p}$.

Lemma 2.5. *$A_p(K)$ is a Banach space and Banach $L^1(K)$-module.*

Proof. Let $\{h_n\}$ be a Cauchy sequence in A_p, then $h_n \to h$ in $C_0(K)$. Choose a subsequence such that $\sum_j \|h_{n_{j+1}} - h_{n_j}\|_{A_p} < \infty$ and put $h_{n_{j+1}} - h_{n_j} = \sum_k f_k^j * g_k^j$ with $\sum_k \|f_k^j\|_p \|\tilde{g}_k^j\|_q < \|h_{n_{j+1}} - h_{n_j}\|_{A_p} + 2^{-j}$, then $h = h_{n_1} + \sum_j (h_{n_{j+1}} - h_{n_j}) = h_{n_1} + \sum_j \sum_k f_k^j * g_k^j \in A_p$. Next let $f \in L^1$ and $h \in A_p$. For $\varepsilon > 0$, write $h = \sum_{k=1}^{\infty} f_k * g_k$ with $\sum_k \|f_k\|_p \|g_k\|_q < \|h\|_{A_p} + \varepsilon$. Since $\|h - \sum_{k=1}^n f_k * g_k\|_\infty \to 0$, we get $\|f * h - \sum_{k=1}^n f * f_k * g_k\|_\infty \to 0$, so $f * h = \sum_{k=1}^{\infty} f * f_k * g_k$. Also

$$\sum_k \|f * f_k\|_p \|\tilde{g}_k\|_q \leq \|f\|_1 \sum_k \|f_k\|_p \|g_k\|_q \leq \|f\|_1 (\|h\|_{A_p} + \varepsilon),$$

hence $f * h \in A_p$ and $\|f * h\|_{A_p} \leq \|f\|_1 \|h\|_{A_p}$. □

Proposition 2.6. *We have the isometric isomorphisms $A_1(K) \simeq C_0(K)$ and $A_2(K) \simeq A(K)$.*

Proof. By definition, $A_1(K) \subseteq C_0(K)$ and $\|h\|_\infty \leq \|h\|_{A_p}$, for each $h \in A_1$. Conversely, since $C_0(K)$ is a Banach left L^1-module and $L^1(K)$ has a bounded approximate identity [GM], it follows from Cohen Factorization Theorem [HR, 32.22] that for each $h \in C_0(K)$ and $\delta > 0$, there are $f \in L^1(K)$ and $g \in C_0(K)$ with $\|f\|_1 \leq 1$, $\|g - h\|_\infty \leq \delta$, and $h = f * g$. Hence $h \in A_1(K)$ and $\|h\|_{A_1} \leq \|f\|_1\|g\|_\infty \leq \|g\|_\infty \leq \|h\|_\infty + \delta$.

Next let $h \in A_2(K)$ and $\varepsilon > 0$. Write $h = \sum_{k=1}^{\infty} f_k * g_k$ with $\sum_k \|f_k\|_2\|g_k\|_2 < \|h\|_{A_2} + \varepsilon$. Then $\|f_k * g_k\|_A \leq \|f_k\|_2\|g\|_2$ and so $\{\sum_{k=1}^{n} f_k * g_k\}$ is a Cauchy sequence in $A(K)$ which converges to h (since $\|.\|_\infty \leq \|.\|_A$). Hence $h \in A(K)$ and

$$\|h\|_A \leq \sum_{k=1}^{\infty} \|f_k * g_k\|_A \leq \sum_{k=1}^{\infty} \|f_k\|_2\|g_k\|_2 < \|h\|_{A_2} + \varepsilon.$$

Hence $\|h\|_A \leq \|h\|_{A_2}$ and $A_2(K) \subseteq A(K)$. Conversely, if $h \in A(K)$ then $h \in A_2(K)$ and

$$\|h\|_{A_2} \leq \inf\{\|f\|_2\|g\|_2 : f, g \in L^2, h = f * g\} = \|h\|_A,$$

Hence $A(K) \subseteq A_2(K)$. □

For $f, g : K \to \mathbb{C}$ define

$$\Gamma(f)(x,y) = f(x * \bar{y}), \;\; f \otimes g(x,y) = f(x)g(y) \quad (x, y \in K).$$

Define $\Gamma_1 : L^2(K)\hat{\otimes}L^2(K) \to A(K)$ by

$$\Gamma_1(\phi)(x) = \int_K \phi(x * y, y)dm(y) \quad (\phi \in L^2(K)\hat{\otimes}L^2(K)).$$

Then

$$\Gamma_1(f \otimes g)(x) = \int_K f(x * y)g(y)dm(y) = f * \check{g}(x),$$

for each $x \in K$ and $f, g \in L^2(K)$. On the other hand,

$$\begin{aligned}
\Gamma(f * \check{g})(x,y) = f * \check{g}(x * \bar{y}) &= \int_K \int_K f(\bar{u})\check{g}(u * t)dm(u)d(\delta_x * \delta_{\bar{y}})(t) \\
&= \int_K \int_K f(\bar{u})\check{g}_u(t)dm(u)d(\delta_x * \delta_{\bar{y}})(t) = \int_K f(\bar{u})_{\bar{u}}\check{g}(x * \bar{y})dm(u) \\
&= \int_K f(\bar{u})_{\bar{u}}\check{g}_{\bar{y}}(x)dm(u) = \int_K f(\bar{u})\check{g}_{\bar{y}}(u * x)dm(u) \\
&= \int_K f(x * u)\check{g}_{\bar{y}}(\bar{u})dm(u) = \int_K f(x * u)\check{g}(\bar{u} * \bar{y})dm(u) \\
&= \int_K f(x * u)g(y * u)dm(u) = \int_K f_u(x)g_u(y)dm(u),
\end{aligned}$$

in particular,

$$\begin{aligned}\Gamma(f * \check{g})(h \otimes k)(x,y) &= \int_K (hf_u)(x)(kg_u)(y)dm(u) \\ &= \int_K (hf_u) \otimes (kg_u)(x,y)dm(u),\end{aligned}$$

for each $f, g, h, k \in L^2(K)$. Hence $\Gamma(f * \check{g})(h \otimes k) \in L^2(K)\hat{\otimes}L^2(K))$ with

$$\|\Gamma(f * \check{g})(h \otimes k)\|_\gamma \leq \|f\|_2\|g\|_2\|h\|_2\|k\|_2.$$

Next let $\phi \in L^2(K)\hat{\otimes}L^2(K)$, and, given $\varepsilon > 0$, find a presentation $\phi = \sum_{n=1}^\infty h_n \otimes k_n$ with $\sum_{n=1}^\infty \|h_n\|_2\|k_n\|_2 < \|\phi\|_\gamma + \varepsilon$. Then $\psi = \sum_{n=1}^\infty \Gamma(f * \check{g})h_n \otimes k_n$ converges in $L^2(K)\hat{\otimes}L^2(K)$ and $\psi = \Gamma(f * \check{g})\phi$ and $\|\psi\|_\gamma \leq \|f\|_2\|g\|g\|_2(\|\phi\|_\gamma + \varepsilon)$. Hence $\|\Gamma(f * \check{g})\phi\|_\gamma \leq \|f\|_2\|g\|_2\|\phi\|_\gamma$. Summing up

Proposition 2.7. *$A(K) \simeq (L^2(K)\hat{\otimes}L^2(K))/ker(\Gamma_1)$, as Banach spaces.*

Next let

$$M(L^2(K)) = \{T \in B(L^2(K)) : T(f * g) = Tf * g\}$$

and let $PM(K)$ be the smallest ultra-weakly closed subspace of $M(L^2(K))$ containing all operators $L_f : L^2(K) \to L^2(K)$ defined by $L_f(g) = f*\check{g}$ for $f \in L^1(K)$ and $g \in L^2(K)$.

Lemma 2.8. *To each $F \in A(K)^*$ there corresponds a unique $F' \in PM(K)$ with*

$$\langle F'(g), f\rangle = \langle f * \check{g}, F\rangle.$$

The mapping $\theta : (A(K)^*, w^*) \to (PM(K), u\text{-}weak)$, $F \mapsto F'$ is an isometric isomorphism.

Proof. We follow the argument of [Pi,10.B]. Given $f, g \in L^2(K)$, let $\theta_g(f) = \langle f * \check{g}, F\rangle$, then

$$|\langle f * \check{g}, F\rangle| \leq \|F\|\|f * \check{g}\|_A \leq \|F\|\|f\|_2\|g\|_2,$$

so $\theta_g \in L^2(K)^* \simeq L^2(K)$ with $\|\theta_g\| \leq \|F\|\|g\|_2$, and $F' \in B(L^2(K))$ is defined by $F'(g) = \theta_g$ and $\|F'\| \leq \|F\|$. On the other hand, for the absolutely convergent series $u = \sum_{n=1}^\infty f_n * \check{g}_n$ we have

$$F(u) = \sum_{n=1}^\infty F(f_n * \check{g}_n) = \sum_{n=1}^\infty \langle F'(g_n), f_n\rangle.$$

Hence $|F(u)| \leq \|F'\| \sum_{n=1}^\infty \|f_n\|_2\|g_n\|_2$ and so $|F(u)| \leq \|F'\|\|u\|_A$, for each $u \in A(K)$. Hence $\|F\| \leq \|F'\|$. To show that θ is surjective, let us first note that each $\mu \in M(K)$ defines

$$F_\mu(u) = \int_K u d\mu \quad (u \in A(K)),$$

and for $f, g \in L^2(K)$,

$$\begin{aligned}\langle F_\mu'(g), f\rangle &= F_\mu(f * \check{g}) = \int_K f * \check{g}(x) d\mu(x) \\ &= \int_K \int_K f(y) g(\bar{x} * y) dm(y) d\mu(x) \quad [\text{Je}, 5.5.A] \\ &= \int_K f(y) \int_K g(\bar{x} * y) d\mu(x) dm(y) \\ &= \langle L_\mu g, f\rangle \quad [\text{Je}, 4.2].\end{aligned}$$

Hence $F_\mu' = L_\mu$. In particular, range of θ includes all convolution operators L_f, with $f \in L^1(K)$. Now the corresponding set of functionals consisting of F_f, with $f \in L^1(K)$ separates the points of $A(K)$ and is dense in $A(K)^*$, so the map is onto. □

Note that $A(K)^*$ is the double conjugate of the hypergroup C^*-algebra $C^*(K)$ and so it inherits the von Neumann algebra structure of $C^*(K)^{**}$. By the above proposition, $PM(K)$ could be considered as a von Neumann algebra. Also $\Gamma_1 : L^2(K)\hat{\otimes}L^2(K) \to L^2(K) * L^2(K)$ is an isometry, and so $A(K) = L^2(K) * L^2(K)$, as already shown in [Vr].

3. MULTIPLIERS OF FOURIER AND HERZ SPACES

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathcal{E}(\hat{K})$. An element $f \in \mathcal{E}(\hat{K})$ is called an $(\mathcal{A}, \mathcal{B})$-multiplier if $f\mathcal{A} \subseteq \mathcal{B}$ [HR, 35.1]. For $\mathcal{B} \subseteq M(K)$, we have $\hat{\mathcal{B}} \subseteq \mathcal{E}(\hat{K})$ and an $(\mathcal{A}, \hat{\mathcal{B}})$-multiplier is simply called an $(\mathcal{A}, \mathcal{B})$-multiplier. The set of all multipliers in both cases is denoted by $M(\mathcal{A}, \mathcal{B})$. Same abbreviation is used when $\mathcal{B}$ is a subset of function spaces such as $L^p(K)$ or $C_0(K)$. To relate the new spaces $\mathcal{E}_2(\hat{K})$ to $A(K)$ and $A_\gamma(K)$ defined in the previous section, we refer the reader to [HR, 28.32(v), 34.5-7] and [Vr, 4.2, 4.11].

Lemma 3.1. *Let $1 \le p \le \infty$, $f \in M(\mathcal{A}, \mathcal{B})$, and $T : \mathcal{A} \to \mathcal{B}$ be defined as follows*

(i) *For $\mathcal{A}, \mathcal{B}$ any of $\mathcal{E}_p(\hat{K})$ or $\mathcal{E}_0(\hat{K})$, $T(g) = fg$, for $g \in \mathcal{A}$,*

(ii) *For $\mathcal{A}$ any of $\mathcal{E}_p(\hat{K})$ or $\mathcal{E}_0(\hat{K})$ and $\mathcal{B}$ and of $L^p(K)$, $C(K)$ or $M(K)$, $T(g)\hat{} = fg$, for $g \in \mathcal{A}$,*

(iii) *For $\mathcal{B}$ any of $\mathcal{E}_p(\hat{K})$ or $\mathcal{E}_0(\hat{K})$ and $\mathcal{A}$ and of $L^p(K)$, $C(K)$ or $M(K)$, $T(g) = f\hat{g}$, for $g \in \mathcal{A}$,*

(iv) *For $\mathcal{A}, \mathcal{B}$ any of $L^p(K)$, $C(K)$ or $M(K)$, $T(g)\hat{} = f\hat{g}$, for $g \in \mathcal{A}$, then T is bounded.*

Proof. In all cases the result follows from the closed graph theorem as in [HR, 35.2]. First note that T is well-defined in (i), (ii) by definition, and in (ii), (iv) by the uniqueness of the Fourier-Stieltjes transform. When $\mathcal{A} = \mathcal{E}_p(\hat{K})$ or $\mathcal{E}_0(\hat{K})$, then $\mathcal{A} \subseteq \mathcal{E}_\infty(\hat{K})$, and for each $f \in \mathcal{A}$, $\|f\|_\infty \le \|f\|_\mathcal{A}$, and when $\mathcal{A} = L^p(K)$, $C(K)$ or $M(K)$, then $\mathcal{A} \subseteq M(K)$ and for each $f \in \mathcal{A}$, $\|f\|_\infty \le \|f\|_{M(K)} \le \|f\|_\mathcal{A}$. Writing $\|\hat{f}\|_{\hat{\mathcal{A}}}$ for $\|f\|_\mathcal{A}$, we get $\|h\|_\infty \le \|h\|_{\hat{\mathcal{A}}}$, for each $h \in \hat{\mathcal{A}}$. Hence $\mathcal{A}$ could be regarded as a subspace of $\mathcal{E}_\infty(\hat{K})$. The same observations hold for $\mathcal{B}$. This shows that it is enough to consider only the case $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}_\infty(\hat{K})$ with complete norms $\|.\|_\mathcal{A}$ and $\|.\|_\mathcal{B}$, satisfying $\|f\|_\infty \leq \|f\|_\mathcal{A}$ for each $f \in \mathcal{A}$, and the same for $\mathcal{B}$; and $T : \mathcal{A} \to \mathcal{B}$, defined by $T(g) = fg$, where $g \in \mathcal{A}$ and $f \in M(\mathcal{A}, \mathcal{B})$. Then T is clearly linear and if $g \in \mathcal{A}$, $h \in \mathcal{B}$, and $\{g^n\} \subseteq \mathcal{A}$ is a sequence such that $\|g^n - g\|_\mathcal{A} \to 0$ and $\|T(g^n) - h\|_\mathcal{B} \to 0$, then

$$\|fg^n - h\|_\infty \leq \|fg^n - h\|_\mathcal{B} \to 0,$$

hence $\|f - \pi g^n_\pi - h_\pi\|_\infty \to 0$. On the other hand,

$$\|f_\pi g^n_\pi - f_\pi g_\pi\|_\infty \leq \|fg^n - fg\|_\infty \leq \|f\|_\infty \|g^n - g\|_\mathcal{A} \to 0,$$

for each $\pi \in \hat{K}$. Hence $fg = h$, and T has a closed graph in $\mathcal{A} \times \mathcal{B}$. □

Corollary 3.2. *If $\mathcal{A}, \mathcal{B}$ and their conjugate spaces are any of $C(K)$, $\mathcal{E}_0(\hat{K})$, or $M(K)$, $L^p(K)$,$\mathcal{E}_p(\hat{K})$ $(1 \leq p \leq \infty)$, then $M(\mathcal{A}, \mathcal{B}^*) \cong M(\mathcal{B}, \mathcal{A}^*)$.*

Lemma 3.3. *If $\mu \in M(K)$, $1 < p \leq \infty$ and*

$$\sup\{\|\mu * h\|_p : h \in \mathcal{I}(K), \|h\|_1 \leq 1\} < \infty,$$

then $d\mu = gdm$, for some $g \in L^p(K)$.

Proof. Same as [HR, 35.11] with $\phi = k_\pi \chi_\pi$. □

Proposition 3.4. *For $1 < p \leq \infty$, $M(L^1(K), L^p(K)) = L^p(K)\hat{}$.*

Proof. We clearly have $(L^p)\hat{} \subseteq M(L^1, L^p)$. The reverse inclusion follows from the above lemma as in [HR, 35.12]. □

Theorem 3.5. *$M(L^1(K), \mathcal{E}_1(\hat{K}))) = M(\mathcal{E}_0(\hat{K})), L^\infty(K)) = \mathcal{E}_1(\hat{K})$.*

Proof. It is clear that $\mathcal{E}_1 \subseteq M(\mathcal{E}_0, L^\infty)$. Also $M(L^1, \mathcal{E}_1) = M(L^1, \mathcal{E}_0^*) = M(\mathcal{E}_0, L^\infty)^*$. Let $E \in M(L^1, \mathcal{E}_1)$, then $f \mapsto E\hat{f}$ is a bounded (by k) linear map from L^1 into $\mathcal{E}_1$. By the above proposition, $E = \hat{g}$ for some $g \in L^2$. Let $\{h_\alpha\}$ be an approximate identity for L^1 as in [Vr] and $\Phi \subseteq \hat{K}$ be finite. Then

$$\lim_\alpha \hat{g}(\sigma)\hat{h}_\alpha(\sigma) = \hat{g}(\sigma) \quad (\sigma \in \Phi)$$

and so for large α,

$$\sum_{\sigma \in \Phi} k_\sigma \|E_\sigma\|_1 = \sum_{\sigma \in \Phi} k_\sigma \|\hat{g}(\sigma)\|_1 \leq \sum_{\sigma \in \Phi} k_\sigma \|\hat{g}(\sigma)\hat{h}_\alpha(\sigma)\|_1 + 1$$
$$\leq \|E\hat{h}_\alpha\|_1 + 1 \leq k\|h_\alpha\|_1 + 1 = k + 1.$$

Hence $\|E\|_1 \leq k + 1$ and $E \in \mathcal{E}_1$. □

The following lemma is proved similar by to [HR, 35.16(e)].

Lemma 3.6. *Let $F \in \mathcal{E}(\hat{K})$ and define*

$$L_f(h) = \sum_{\sigma \in \hat{K}} k_\sigma \mathrm{tr}(F_\sigma \hat{h}(\sigma)) \quad (h \in Trig(K)).$$

This is a bounded linear functional on $Trig(K)$ and each bounded linear functional on $Trig(K)$ has this form. Also if $T_F : Trig(K) \to Trig(K)$ is defined by $T_F(h)\hat{} = F\hat{h}$, then the following are equivalent.

(i) *There is $k > 0$ such that $|L_F(h)| \leq k\|h\|_{A_p}$, for each $h \in Trig(K)$,*

(ii) *There is $k > 0$ such that $\|T_F(h)\|_p \leq k\|h\|_p$, for each $h \in Trig(K)$.*

Proof. First note that $h(e) = \sum_{\sigma\in\hat{K}} k_\sigma \mathrm{tr}(\hat{h}(\sigma))$, for $h \in Trig(K)$. Hence

$$L_F(h) = \sum_{\sigma\in\hat{K}} k_\sigma \mathrm{tr}(T_F(h)\hat{}(\sigma)) = T_F(h)(e).$$

Now if (ii) holds, then T_F extends to a linear operator $T_F : L^p(K) \to L^p(K)$ with $\|T_F(f)\|_p \leq k\|f\|_p$, for each $f \in L^p(K)$. Since $Trig(K) * Trig(K) = Trig(K)$, we have $Trig(K) \subseteq A_p(K)$, and given $\varepsilon > 0$, each $h \in Trig(K)$ could be written as $h = \sum_{i=1}^\infty f_i * g_i$, such that $\sum_i \|f_i\|_p\|g_i\|_q < \|h\|_{A_p} + \varepsilon$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence $T_F(h) = \sum_{i=1}^\infty T_F(f_k) * g_k$ and

$$\begin{aligned}
|L_F(h)| = |T_F(h)(e)| &= |\sum_{i=1}^\infty T_F(f_i) * g_i(e)| \\
&\leq \sum_{i=1}^\infty \|T_F(f_i)\|_p\|g_i\|_q \leq \sum_{i=1}^\infty k\|f_i\|_p\|g_i\|_q \\
&\leq k\|h\|_{A_p} + k\varepsilon,
\end{aligned}$$

and (i) follows.

Conversely, if (i) holds and $h \in Trig(K)$, then for each $g \in C(K)$, $h*g \in A_p(K)$ and $\|h * g\|_{A_p} \leq \|h\|_p\|g\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence

$$\begin{aligned}
|\int_K T_F(h)\tilde{g}dm| &= |T_F(h) * g(e)| = |T_F(h * g)(e)| \\
&= |L_F(h * g)| \leq k\| \leq k\|h * g\|_{A_p} \leq k\|h\|_p\|g\|_q.
\end{aligned}$$

Changing g to $\tilde{g}$, we get

$$|\int_K T_F(h)gdm| \leq k\|h\|_p\|g\|_q$$

for each $g \in C(K)$. This implies that $\|T_F(h)\|_p \leq k\|h\|_p$, which is (ii). □

In the above lemma, if $F \in M(L^p, L^p)$, then T_F (respectively, L_F) extends to a bounded linear operator (respectively, functional) on L^p (respectively, A_p) with $\|l_F\| = \|T_F\|$ and $F \mapsto L_F$ is a surjective linear isometry from $M(L^p, L^p)$ to $(A_p)^*$. Indeed, given $L \in (A_p)^*$ and $\sigma \in \hat{K}$, let $\{\xi_1^\sigma, \dots, \xi_{d_\sigma}^\sigma\}$ be an orthonormal basis of $\mathcal{H}_\sigma$ and put $f_{jk}^\sigma = L(u_{kj}^\sigma)$, where u_{kj}^σ are coefficient functions of σ for $1 \leq k, j \leq d_\sigma$. Consider $F_\sigma = [f_{jk}]_{d_\sigma\times d_\sigma} \in \mathcal{B}(\mathcal{H}_\sigma)$, then $(T_F h)\hat{}(\sigma) = F_\sigma\hat{h}(\sigma)$ and $F\hat{h} \in (L^p)\hat{}$, and so $F \in M(L^p, L^p)$ and $L = L_F$. Hence we have proved the following result.

Proposition 3.7. *With the above notations,*
(i) $M(L^p, L^p) = A_p^*$,
(ii) $M(L^2, L^2) = M(\mathcal{E}_2, \mathcal{E}_2) = A_2^* = A^* = PM = \mathcal{E}_\infty$,
(iii) $M(L^1, L^1) = A_1^* = C^* = M$.

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