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MINIMUM L_1-NORM ESTIMATION FOR MIXED FRACTIONAL ORNSTEIN-UHLENBECK TYPE PROCESS

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Abstract. In the present paper, the asymptotic properties of the minimum L_1-norm estimator of the drift parameter for mixed fractional Ornstein-Uhlenbeck type process satisfying a linear stochastic differential equation driven by a mixed fractional Brownian motion are obtained.

1. Introduction

Let $W^H = \{W_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst index H defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a continuous Gaussian process with the following properties:

(I) $W_0^H = 0$, $\mathbb{P}$-almost surely;

(II) $\mathbb{E}W_t^H = 0$, $\mathbb{E}W_t^H W_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H})$ for all $s, t \geq 0$;

(III) the increments of W^H are stationary and self-similar with order H, and the trajectories of W^H are almost surely continuous and not differentiable.

Note that the standard Brownian motion W is a fractional Brownian motion with Hurst index $H = 1/2$. Let us take a and b as two real constants such that $(a, b) \neq (0, 0)$.

Definition 1.1. *A mixed fractional Brownian motion (MFBM) of parameters a, b, and H is a process $Z^H = \{Z_t^H(a,b); t \geq 0\} = \{Z_t^H; t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by*

$$\forall t \in \mathbb{R}_+, \quad Z_t^H = Z_t^H(a,b) = aW_t + bW_t^H, \tag{1.1}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion, and $(W_t^H)_{t\in\mathbb{R}_+}$ is a fractional Brownian motion of Hurst parameter H, and processes $(W_t)_{t\in\mathbb{R}_+}$ and $(W_t^H)_{t\in\mathbb{R}_+}$ are independent.

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This process has been introduced by Cheridito [1] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process

$$X_t^H(a,b) = X_0^H(a,b)\exp\Big(\nu t + \sigma Z_t^H(a,b)\Big), \tag{1.2}$$

where ν, σ are constants, a is a strictly positive constant, $b = 1$, and $Z^H(a,b)$ is a MFBM of parameters a, b, and H.

On account of the possibility of long run non-periodic statistical dependence in stock price returns, it is necessary to study the properties of MFBM. Zili [16] obtained some general stochastic properties of the mixed fractional Brownian motion and treat the Hölder continuity of the sample paths and α-differentiability of the trajectories of MFBM. The author et al. [6] generalized Zili's works from MFBM to fractional mixed fractional Brownian motion.

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modeling in wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao [13]. There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence. Le Breton [5] studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. The author and Wang [7] obtained the Large deviation inequalities for MLE and Bayes estimator in SDE with fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton [4] studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = X_0 + \theta \int_0^t X_s ds + \sigma W_t^H, \quad t \geq 0.$$

In the present paper, our aim is to obtain the minimum L_1-norm estimates of the drift parameter of a fraction Ornstein-Uhlenbeck type process and investigate the asymptotic properties of such estimators following the work of Prakasa Rao [14].

2. Preliminaries and elementary lemmas

From Zili [16], we know that MFBM is a mixed-self-similar process: $\{Z_{ht}^H(a,b)\} =^d \{Z_t^H(ah^{1/2}, bh^H)\}$, where $h > 0$ is a constant and the notation $\{X_t\} =^d \{Y_t\}$ means that the $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ have the same law. For any process X

denote by X^* the supremum process: $X_t^* = \sup_{s\le t}|X_s|$. Therefore even more is true: from the self-similarity it follows for the supremum process Z^* that $Z_{ht}^{H*}(a,b) =^d Z_t^{H*}(ah^{1/2}, bh^H)$. For any $p>0$ we have then the following result by self-similarity:

Lemma 2.1. *Let $T>0$ be a constant and Z a MFBM with parameters a,b,H. Then for any $p>0$,*

$$\mathbb{E}(Z_T^{H*}(a,b))^p = \mathbb{E}(Z_1^{H*}(aT^{1/2}, bT^H))^p = \mathbb{E}(\sup_{t\le 1}|aT^{1/2}W_t + bT^H W_t^H|)^p. \tag{2.1}$$

The value of (2.1) is not known to us. However it is fortunately that we have the classical Burkholder-Davis-Gundy (B-D-G) inequalities for the standard Brownian motion:

B-D-G Inequalities (Brownian case) *For any stopping time τ with respect to the filtration generated by the Brownian motion W and $p>0$ we have*

$$c(p)\mathbb{E}(\tau^{p/2}) \le \mathbb{E}((W_\tau^*)^p) \le C(p)\mathbb{E}(\tau^{p/2}), \tag{2.2}$$

where the constants $c(p)$, $C(p)>0$ depend only upon the parameter p.

Recall that Novikov and Valkeila [9] gave the following bounds for fractional Brownian motion.

Theorem NV *Let τ be a stopping time with respect to the filtration generated by the fraction Brownian motion W^H. Then for any $p\ge 0$ and $H\in(1/2,1)$ we have*

$$c(p,H)\mathbb{E}(\tau^{pH}) \le \mathbb{E}((W_\tau^{H*})^p) \le C(p,H)\mathbb{E}(\tau^{pH}), \tag{2.3}$$

and for any $p>0$ and $H\in(0,1/2)$ we have

$$c(p,H)\mathbb{E}(\tau^{pH}) \le \mathbb{E}((W_\tau^{H*})^p), \tag{2.4}$$

where the constants $c(p,H)$, $C(p,H)>0$ depend only upon the parameters p,H.

From Theorem NV and B-D-G inequality it is easy to obtain the upper bound of (2.1).

Let us consider a stochastic process $\{X_t, t\ge 0\}$ defined by the stochastic integral equation

$$X_t = x_0 + \theta\int_0^t X_s ds + \varepsilon Z_t^H, \quad X_0 = x_0, \quad 0\le t\le T, \tag{2.5}$$

where θ is an unknown drift parameter and Z_t^H is a MFBM.

By the Hölder continuity of the fractional Brownian motion B^H, the integrals below can be defined by integration by parts, where the singularities of the kernel do not cause problems (see Norros et al., [8] Lemma 2.1).

Put $K(t,s) := (c/C)s^{-r}(t-s)^{-r}$ for $s\in(0,t)$ and $K(t,s)=0$ for $s>t$, where $r = H - 1/2$,

$$C := \sqrt{\frac{H}{(H-1/2)B(H-1/2, 2-2H)}}$$

and

$$c := \frac{1}{B(H+1/2, 3/2-H)},$$

where the beta coefficient $B(\mu,\nu)$ for $\mu,\nu > 0$ is defined by

$$B(\mu,\nu) := \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

From Proposition 2.1 in Norros et al. [8], we have

Proposition 2.2. *For $H \in (0,1)$ define M by*

$$M_t = \int_0^t K(t,s)dW_s^H.$$

Then M is a Gaussian martingale with variance $\langle M\rangle_t = (C^2/4H^2(2-2H))t^{2-2H}$ with respect to filtration $\mathcal{F}_t = \sigma(W_s^H : s \le t)$.

For further discussions of fractional Brownian motion, the readers are referred to Gripenberg and Norros [3], Kleptsyna et al. [4] and Norros et al. [8].

3. Main results

In the section, we will give the minimum L_1-norm estimation and its asymptotic properties which are analogous with Prakasa Rao [14].

3.1. **Minimum L_1-norm estimation.** We now discuss the problem of estimation of the unknown parameter θ based on the observation of mixed fractional Ornstein-Uhlenbeck type process $X = \{X_t, 0 \le t \le T\}$ satisfying the stochastic differential equation

$$dX_t = \theta X_t dt + \varepsilon dZ_t^H, \quad X_0 = x_0, \quad 0 \le t \le T, \tag{3.1}$$

for a fixed time T where $\theta \in \Theta \subset \mathbb{R}$ and study its asymptotic properties as $\varepsilon \to 0$.

Let $x_t(\theta)$ be the solution of the above differential equation with $\varepsilon = 0$. It is obvious that

$$x_t(\theta) = x_0 e^{\theta t}, \quad 0 \le t \le T. \tag{3.2}$$

Let

$$S_T(\theta) = \int_0^T |X_t - x_t(\theta)|dt. \tag{3.3}$$

We define θ_ε^* to be a *minimum L_1-norm estimator* if there exists a measurable selection θ_ε^* such that

$$S_T(\theta_\varepsilon^*) = \inf_{\theta\in\Theta} S_T(\theta). \tag{3.4}$$

Conditions for the existence of a measurable selection are given in Lemma 3.1.2 in Prakasa Rao [12]. We assume that there exists a measurable selection θ_ε^* satisfying the above equation.

3.2. **Consistency.** Let θ_0 denote the true parameter and for any $\delta > 0$ define

$$g(\delta) = \inf_{|\theta-\theta_0|>\delta} \int_0^T |x_t(\theta) - x_t(\theta_0)|dt. \tag{3.5}$$

Then it is easy to see that $g(\delta) > 0$ for any $\delta > 0$.

Theorem 3.1. *For any $p > 0$, there exist constants $C_1(p,H)$, $C_2(p) > 0$ such that for every $\delta > 0$,*

$$\begin{aligned} \mathbb{P}_{\theta_0}^{(\varepsilon)}(|\theta_\varepsilon^* - \theta_0| > \delta) \leq& 2^p \varepsilon^p T^p (g(\delta))^{-p} e^{p|\theta_0 T|}(C_1(p,H)T^{Hp} + C_2(p)T^{p/2}) \\ =& O((g(\delta)^{-p})\varepsilon^p). \end{aligned} \tag{3.6}$$

Proof. Let $\|\cdot\|$ denote the L_1-norm. Then
(3.7)

$$\begin{aligned} \mathbb{P}_{\theta_0}^{(\varepsilon)}(|\theta_\varepsilon^* - \theta_0| > \delta) =& \mathbb{P}_{\theta_0}^{(\varepsilon)}(\inf_{|\theta-\theta_0|\leq\delta} \|X - x(\theta)\| > \inf_{|\theta-\theta_0|>\delta} \|X - x(\theta)\| > \delta) \\ \leq& \mathbb{P}_{\theta_0}^{(\varepsilon)}(\inf_{|\theta-\theta_0|\leq\delta} (\|X - x(\theta_0)\| + \|x(\theta_0) - x(\theta)\|) \\ & > \inf_{|\theta-\theta_0|>\delta} (\|x(\theta_0) - x(\theta)\| - \|X - x(\theta_0)\|) > \delta) \\ =& \mathbb{P}_{\theta_0}^{(\varepsilon)}(\|X - x(\theta_0)\| > g(\delta)/2). \end{aligned}$$

Since $x_t(\theta) = x_0 e^{\theta t}$ and from (3.1), we have

$$\begin{aligned} X_t - x_t(\theta_0) =& x_0 + \theta_0 \int_0^t X_s ds + \varepsilon Z_t - x_t(\theta) \\ =& \theta_0 \int_0^t (X_s - x_s(\theta_0))ds + \varepsilon Z_t. \end{aligned} \tag{3.8}$$

Let $V_t = |X_t - x_t(\theta_0)|$. The above relation implies that

$$V_t = |X_t - x_t(\theta_0)| \leq |\theta_0| \int_0^t V_s ds + \varepsilon |Z_t|.$$

By the Gronwall's inequality, we have

$$\sup_{0\leq t\leq T} |V_t| \leq \varepsilon e^{|\theta_0 T|} \sup_{0\leq t\leq T} |Z_t|.$$

Hence, from (3.7) and applying Lemma 2.1, B-D-G inequalities and Theorem NV, we have

$$\begin{aligned} \mathbb{P}_{\theta_0}^{(\varepsilon)}(|\theta_\varepsilon^* - \theta_0| > \delta) \leq& \mathbb{P}_{\theta_0}^{(\varepsilon)}(\|X - x(\theta_0)\| > g(\delta)/2) \\ \leq& \mathbb{P}_{\theta_0}^{(\varepsilon)}(Z_T^* > \frac{e^{-|\theta_0 t|}g(\delta)}{2\varepsilon T}) \\ \leq& 2^p \varepsilon^p T^p (g(\delta))^{-p} e^{p|\theta_0 T|}(C_1(p,H)T^{Hp} + C_2(p)T^{p/2}) \\ =& O((g(\delta)^{-p})\varepsilon^p). \end{aligned}$$

□

Remark 3.2. As a consequence of the above theorem, we obtain that θ_ε^* converges in probability to θ_0 under $\mathbb{P}_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Furthermore the rate of convergence is of the order $O(\varepsilon^p)$ for every $p > 0$.

3.3. **Asymptotic distribution.** One can check that

$$X_t = e^{\theta_0 t}(x_0 + \int_0^t e^{-\theta_0 s}\varepsilon dZ_s) \tag{3.9}$$

or equivalently

$$X_t - x_t(\theta_0) = \varepsilon e^{\theta_0 T}\int_0^t e^{-\theta_0 s}\varepsilon dZ_s.$$

Let

$$Y_t = e^{\theta_0 t}\int_0^t e^{-\theta_0 s}dZ_s = ae^{\theta_0 t}\int_0^t e^{-\theta_0 s}dW_s + be^{\theta_0 t}\int_0^t e^{-\theta_0 s}dW_s^H =: aY_t^1 + bY_t^2.$$

Then Y is a Gaussian process and can be interpreted as the "derivative" of the process X with respect to ε. From the discussions in Prakasa Rao [14] (or cf. Kleptsyna et al. [4] and Gripenberg and Norros [3]), we know that for any $h \geq 0$,

$$Cov(Y_t^2, Y_{t+h}^2) = e^{2\theta_0 t + \theta_0 h}\gamma_H(t),$$

where

$$\gamma_H(t) = H(2H-1)\int_0^t\int_0^t e^{-\theta_0(u+v)}|u-v|^{2H-2}dudv.$$

And it is obvious that

$$Cov(Y_t^1, Y_{t+h}^1) = e^{2\theta_0 t + \theta_0 h}\int_0^t e^{-2\theta_0 s}ds.$$

Hence Y is a zero mean Gaussian. Let

$$\xi = \arg\inf_{-\infty<u<\infty}\int_0^T |Y_t - utx_0e^{\theta_0 t}|dt.$$

Note that it is not clear what the distribution of ξ is.

Theorem 3.3. *The random variable $u^* = \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ converges in probability to a random variable whose probability distribution is the same as that of ξ under $\mathbb{P}_{\theta_0}$.*

Proof. Let $x_t'(\theta) = x_0te^{\theta t}$ and let

$$N_\varepsilon(u) = \|Y - \varepsilon^{-1}(x(\theta_0 + \varepsilon u) - x(\theta_0))\|$$

and

$$N_0(u) = \|Y - ux'(\theta_0)\|.$$

Furthermore, let

$$A_\varepsilon = \{\omega : |\theta_\varepsilon^* - \theta_0| < \delta_\varepsilon\}, \quad \delta_\varepsilon = \varepsilon^\tau, \quad \tau \in (1/2, 1) \;\; L_\varepsilon = \varepsilon^{\tau-1}.$$

Note that the random variable u^* satisfies the equation

$$N_\varepsilon(u_\varepsilon^*) = \inf_{|u|<L_\varepsilon} Z_\varepsilon(u), \quad \omega \in A_\varepsilon.$$

Define

$$\xi_\varepsilon = \arg \inf_{|u|<L_\varepsilon} Z_0(u).$$

Observe that with probability one,

$$\begin{aligned}\sup_{|u|<L_\varepsilon} \|N_\varepsilon(u) - N_0(u)\| =& |\|Y - ux^{'}(\theta_0) - \frac{1}{2}\varepsilon u^2 x^{''}(\tilde{\theta})\| - \|Y - ux^{'}(\theta_0)\||\\ \le& \frac{\varepsilon}{2} L_\varepsilon^2 \sup_{|\theta-\theta_0|<\delta_\varepsilon} \int_0^T |x^{''}(\theta)| dt\\ \le& C\varepsilon^{2\tau-1},\end{aligned}$$

where $\tilde{\theta} = \theta_0 + \alpha(\theta - \theta_0)$ for some $\alpha \in (0, 1)$. Note that the last term in the above inequality tends to zero as $\varepsilon \to 0$. Furthermore the process $\{Z_0(u), -\infty < u < \infty\}$ has a unique minimum u^* with probability one. This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian [11]. In addition, we can choose the interval $[-L, L]$ such that

$$\mathbb{P}_{\theta_0}^{(\varepsilon)}(u_\varepsilon^* \in (-L, L)) \ge 1 - \beta g(L)^{-p}$$

and

$$\mathbb{P}_{\theta_0}^{(\varepsilon)}(u^* \in (-L, L)) \ge 1 - \beta g(L)^{-p}$$

where $\beta > 0$. Note that $g(L)$ increases as L increases. The processes $Z_\varepsilon(u), u \in [-L, L]$ and $Z_0(u), u \in [-L, L]$ satisfy the Lipschitz conditions and $Z_\varepsilon(u)$ converges uniformly to $Z_0(u)$ over $u \in [-L, L]$. Hence the minimizer of $Z_\varepsilon(\cdot)$ converges to the minimizer of $Z_0(u)$. This completes the proof. □

References

[1] P. Cheridito, Mixed fractional Brownian motion. *Bernoulli* **7**(6) (2001), 913-934.

[2] C. El-Nouty, The fractional mixed fractional Brownian motion. *Statist. Probab. Lett.* **65** (2) (2003), 111-120.

[3] G. Gripenberg and I. Norros, On the prediction of fractional Brownian motion. *J. Appl. Probab.* **33** (1996), 400-410.

[4] M. L. Kleptsyna, A. Le Breton and M. C. Roubaud, Parameter estimation and optimal filtering for fractional type stochastic systems. *Stat. Inference Stoch. Process.* **3** (2000), 173-182.

[5] A. Le Breton, Filtering and parameter estimation in a simple linear model driven by a fractional Brownian motion. *Statist. Probab. Lett.* **38** (1998), 263-274.

[6] Y. Miao, W. Y. Ren and Z. X. Ren, On the fractional mixed fractional Brownian motion. *Appl. Math. Sci. (Ruse)* **2** (33-36) (2008), 1729-1738.

[7] Y. Miao and J. X. Wang, Large deviation inequalities for MLE and Bayes estimator in SDE with fractional Brownian motion. *J. Math. (Wuhan)* **28** (4) (2008), 355-361.

[8] I. Norros, E. Valkeila and J. Virtamo, An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli* **5** (4) (1999), 571-587.

[9] A. Novikov and E. Valkeila, On some maximal inequalities for fractional Brownian motions. *Statist. Probab. Lett.* **44** (1) (1999), 47-54.

[10] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications.* 6th ed., Universitext, Springer, Berlin, 2003.

[11] Yu. Kutoyants and P. Pilibossian, On minimum L_1-norm estimate of the parameter of the Ornstein-Uhlenbeck process. *Statist. Probab. Lett.* **20** (1994), 117-123.

[12] B. L. S. Prakasa Rao, *Asymptotic Theory of Statistical Inference.* Wiley, New York, 1987.

[13] B. L. S. Prakasa Rao, *Statistical Inference for Diffusion Type Processes.* Arnold, London and Oxford University Press, New York, 1999.

[14] B. L. S. Prakasa Rao, Minimum L_1-norm estimation for fractional Ornstein-Uhlenbeck type process. *Theory Probab. Math. Statist.* **71** (2005), 181-189.

[15] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer, Berlin, 1991.

[16] M. Zili, *On the mixed fractional Brownian motion. J. Appl. Math. Stoch. Anal.* (2006), Article ID 32435, pages 1-9.

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