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STABILITY OF FUNCTIONAL EQUATIONS OF SEVERAL VARIABLES WHICH ARE ADDITIVE OR QUADRATIC IN EACH VARIABLES IN FRÉCHET SPACES

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Abstract. In this paper, we establish the Hyers–Ulam–Rassias stability of the system of functional equations

$$\begin{cases} f(x_1,x_2,...,x_{i_r-1},a+b,x_{i_r+1},...,x_n) = f(x_1,x_2,...,x_{i_r-1},a,x_{i_r+1},...,x_n) \\ +f(x_1,x_2,...,x_{i_r-1},b,x_{i_r+1},...,x_n), \\ \\ f(x_1,x_2,...,x_{j_s-1},a+b,x_{j_s+1},...,x_n) + f(x_1,x_2,...,x_{j_s-1},a-b,x_{j_s+1},...,x_n) \\ = 2f(x_1,x_2,...,x_{j_s-1},a,x_{j_s+1},...,x_n) + 2f(x_1,x_2,...,x_{j_s-1},b,x_{j_s+1},...,x_n) \end{cases}$$

in Fréchet spaces, where $1 \leq i_1 < i_2 < ... < i_k < n$, $1 < j_1 < j_2 < ... < j_{n-k}, \{1,2,..,n\} = \{i_1,i_2,...,i_k\} \cup \{j_1,j_2,...,j_{n-k}\}$.

1. Introduction

A topological vector space X is a Fréchet space if it satisfies the following three properties:

a) it is complete as a uniform space,

b) it is locally convex,

c) its topology can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x,y) = d(x+a,y+a)$ for all a,x,y in X. This means that a subset U of X is open if and only if for every u in U there exists an $\varepsilon > 0$ such that $\{v : d(u,v) < \varepsilon\}$ is a subset of U. Note that there is no natural notion of distance between two points of a Fréchet space: many different translation-invariant metrics may induce the same topology.

The vector space $C^\infty([0,1])$ of all infinitely often differentiable functions $f : [0,1] \to \mathbb{R}$ becomes a Fréchet space with the seminorms $\|f\|_k = sup\{|f^{(k)}(x)| : x \in [0,1]\}$ for every integer $k \geq 0$. Here, $f^{(k)}$ denotes the k-th derivative of f, and $f^{(0)} = f$.

Fréchet spaces are studied because even though their topological structure is more complicated due to the lack of a norm, many important results in functional analysis, like the open mapping theorem and the Banach-Steinhaus theorem, still hold.

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The stability problem of functional equations originated from a question of Ulam [17] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. This phenomenon of stability is called the Hyers–Ulam stability.

In 1978, Th. M. Rassias [15] proved the following theorem.

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [7] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations (see also [6, 8, 12, 18] and [19]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \tag{1.1}$$

is related to symmetric bi-additive function [1, 2, 3, 6, 11, 12, 13]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [1, 13]). The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \tag{1.2}$$

A Hyers–Ulam–Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, where A is a normed space and B is a Banach space (see [4, 5, 9, 14, 16]).

Let $n > 1$ and let $1 \leq i_1 < i_2 < ... < i_k < n$, $1 < j_1 < j_2 < ... < j_{n-k}, \{1, 2, .., n\} = \{i_1, i_2, ..., i_k\} \cup \{j_1, j_2, \dots, j_{n-k}\}$. Let G, Y be an additive group and a Fréchet space, respectively. In this paper we investigate the Hyers–Ulam–Rassias stability of the system of functional equations

(1.3)

$$\begin{cases} f(x_1, x_2, ..., x_{i_r-1}, a+b, x_{i_r+1}, ..., x_n) = f(x_1, x_2, ..., x_{i_r-1}, a, x_{i_r+1}, ..., x_n) \\ +f(x_1, x_2, ..., x_{i_r-1}, b, x_{i_r+1}, ..., x_n), \\ f(x_1, x_2, ..., x_{j_s-1}, a+b, x_{j_s+1}, ..., x_n) + f(x_1, x_2, ..., x_{j_s-1}, a-b, x_{j_s+1}, ..., x_n) \\ = 2f(x_1, x_2, ..., x_{j_s-1}, a, x_{j_s+1}, ..., x_n) + 2f(x_1, x_2, ..., x_{j_s-1}, b, x_{j_s+1}, ..., x_n) \end{cases}$$

where f is a mapping from G^n into Y.

Let $n \geq 2, k \geq 1$. It is easy to see that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, ..., x_n) = (\sum_{j=1}^{k} x_j) + (\sum_{i=k+1}^{n} x_i^2),$$

is additive on variables $x_1, x_2, ..., x_k$ and is quadratic on variables $x_{k+1}, x_{k+2}, ..., x_n$. It follows that f satisfies (1.3). As another example, let A be a normed algebra. Let $k > 1, n = 2k$. Define $g : A^n \to A$ by

$$g(a_1, a_2, ...a_n) = \|a_1\|^2 a_2 \|a_3\|^2 a_4 ... \|a_{2k-1}\|^2 a_{2k}.$$

It is easy to see that g is additive on even variables and is quadratic on odd variables. Then g satisfies (1.3).

2. Main theorem

In this section, we investigate the generalized Hyers–Ulam–Rassias stability problem of system of functional equations (1.3).

Throughout this section, G and Y will be an additive group and a real Fréchet space with metric d, respectively. Let $n > 1$ and let $1 \leq i_1 < i_2 < ... < i_k < n$, $1 < j_1 < j_2 < ... < j_{n-k}, \{1, 2, .., n\} = \{i_1, i_2, ..., i_k\} \cup \{j_1, j_2, ..., j_{n-k}\}$. Let $f : G^n \to Y$ be a mapping then we define $\Delta_f^{j_s}, D_f^{i_r} : G^{n+1} \to \mathbb{R}$ by

$$\begin{aligned} D_f^{i_r}(x_1, x_2 ..., x_{i_r-1}, a, b, x_{i_r+1}, ..., x_n) &= d(f(x_1, x_2, ..., x_{i_r-1}, a+b, x_{i_r+1}, ..., x_n) \\ &, f(x_1, x_2, ..., x_{i_r-1}, a, x_{i_r+1}, ..., x_n) + f(x_1, x_2, ..., x_{i_r-1}, b, x_{i_r+1}, ..., x_n)) \\ \Delta_f^{j_s}(x_1, x_2 ..., x_{j_s-1}, a, b, x_{j_s+1}, ..., x_n) &= d(f(x_1, x_2, ..., x_{j_s-1}, a+b, x_{j_s+1}, ..., x_n) \\ &+ f(x_1, x_2, ..., x_{j_s-1}, a-b, x_{j_s+1}, ..., x_n), 2f(x_1, x_2, ..., x_{j_s-1}, a, x_{j_s+1}, ..., x_n) \\ &+ 2f(x_1, x_2, ..., x_{j_s-1}, b, x_{j_s+1}, ..., x_n)) \end{aligned}$$

for all $x_1, x_2, ..., x_n, a, b \in G$. Let $\phi, \psi : G^{n+1} \to [0, \infty)$ be mappings. Then for every $u \in \mathbb{N}, x_1, ..., x_n \in G$, we define

$$\phi_{i_r}(u, (x_1, x_2 ..., x_n)) := \phi(w_1, w_2, ..., w_{n+1}),$$

where

$$w_i = \begin{cases} 2^u x_i, & i < i_r; \\ 2^{u-1} x_r, & i = i_r; \\ 2^{u-1} x_{r'-1}, & i = i_{r'} + 1, r' \in \{r, r+1, ..., k\}; \\ 2^u x_{i-1}, & i = j_s + 1 > i_r + 1; \end{cases}$$

and define

$$\psi_{j_s}(u, (x_1, x_2 ..., x_n)) := \psi(z_1, z_2, ..., z_{n+1})$$

where

$$z_i = \begin{cases} 2^{u-1} x_i, & i \leq j_s; \\ 2^u x_{i-1}, & i = j_{s'} + 1, s' \in \{s+1, s+2, ..., n-k\}; \\ 2^{u-1} x_{i-1}, & i = j_s + 1; \\ 2^{u-1} x_{i-1}, & i = i_r + 1 > j_s + 1. \end{cases}$$

Theorem 2.1. *Let $\phi, \psi : G^{n+1} \to [0, \infty)$ be mappings satisfying*

$$\sum_{u=0}^{\infty} \frac{\phi_{i_r}(u, (x_1, x_2, ..., x_n)) + \psi_{j_s}(u, (x_1, x_2, ..., x_n))}{(2^k 4^{n-k})^u} < \infty \tag{2.1}$$

for all $x_1, x_2, ..., x_n \in G$, and

$$\lim_u \frac{\psi(2^u x_1, 2^u x_2, ..., 2^u x_{n+1}) + \phi(2^u x_1, 2^u x_2, ..., 2^u x_{n+1})}{(2^k 4^{n-k})^u} = 0 \tag{2.2}$$

for all $x_1, x_2, ..., x_n, x_{n+1} \in G$. Suppose $f : G^n \to Y$ is a mapping such that

$$D_f^{i_r}(x_1, x_2 ..., x_{i_r - 1}, a, b, x_{i_r + 1}, ..., x_n) \leq \phi(x_1, x_2, ..., x_{i_r - 1}, a, b, x_{i_r + 1}, ..., x_n), \tag{2.3}$$

$$\Delta_f^{j_s}(x_1, x_2 ..., x_{j_s - 1}, a, b, x_{j_s + 1}, ..., x_n) \leq \psi(x_1, x_2, ..., x_{j_s - 1}, a, b, x_{j_s + 1}, ..., x_n) \tag{2.4}$$

for all $x_1, x_2, ..., x_n, a, b \in G, r \in \{1, 2, ..., k\}, s \in \{1, 2, ..., n-k\}$. Then there exists a unique mapping $T : G^n \to Y$ satisfying (1.3) and

$$\begin{aligned} d(f(x_1, x_2, ..., x_n), T(x_1, x_2, ..., x_n)) \leq \sum_{u=0}^{\infty} \frac{1}{(2^k 4^{n-k})^u} \Big[&\sum_{r=1}^{k} \frac{2^{r-1}}{2^k} \phi_{i_r}(u, (x_1, x_2 ..., x_n)) \\ &+ \sum_{s=1}^{n-k} \frac{4^{s-1}}{2^k} \psi_{j_s}(u, (x_1, x_2 ..., x_n)) \Big] \end{aligned} \tag{2.5}$$

for all $x_1, x_2, ..., x_n \in G$.

Proof. Replacing b by a in (2.3) we get

$$\begin{aligned} d(f(x_1, x_2, ..., x_{i_r - 1}, 2a, x_{i_r + 1}, ..., x_n), &2f(x_1, x_2, ..., x_{i_r - 1}, a, x_{i_r + 1}, ..., x_n)) \\ &\leq \phi(x_1, x_2, ..., x_{i_r - 1}, a, x_{i_r + 1}, ..., x_n). \end{aligned} \tag{2.6}$$

Replacing b by a in (2.4) we obtain

(2.7)
$$d(f(x_1,x_2,...,x_{j_s-1},2a,x_{j_s+1},...,x_n),4f(x_1,x_2,...,x_{j_s-1},a,x_{j_s+1},...,x_n)) \\ \leq \psi(x_1,x_2,...,x_{j_s-1},a,x_{j_s+1},...,x_n).$$

It follows from (2.6) that

(2.8)
$$\begin{aligned} &d\big(f(x_1,x_2,...,x_{i_1-1},2x_{i_1},x_{i_1+1},...,x_{i_2-1},2x_{i_2},x_{i_2+1},...,x_{i_k-1},2x_{i_k},x_{i_k+1},...,x_n), \\ &\quad 2^k f(x_1,x_2,...,x_n)\big) \\ &\leq \sum_{r=1}^{k} 2^{k-r}\phi\big((x_1,x_2,...,x_{i_1-1},2x_{i_1},x_{i_1+1},...,x_{i_2-1},2x_{i_2},x_{i_2+1},..., \\ &\qquad x_{i_r-1},2x_{i_r},x_{i_r+1},...,x_n)\big) \\ &= \sum_{r=1}^{k} 2^{k-r}\phi_{i_r}(1,(x_1,x_2,...,x_n)). \end{aligned}$$

By (2.5), we have the relation

$$\begin{aligned} &d\big(f(x_1,x_2,...,x_{j_1-1},2x_{j_1},x_{j_1+1},...,x_{j_2-1},2x_{j_2},x_{j_2+1},...,x_{j_{n-k}-1},2x_{j_{n-k}}, \\ &\qquad x_{j_{n-k}+1},...,x_n),4^{n-k}f(x_1,x_2,...,x_n)\big) \\ &\leq \sum_{s=1}^{n-k} 4^{s-1}\psi(x_1,x_2,...,x_{j_s-1},2x_{j_s},x_{j_s+1},...,x_{j_{s+1}-1},2x_{j_{s+1}}, \\ &\qquad x_{j_{s+1}+1},...,x_{j_{n-k}-1},2x_{j_{n-k}},x_{j_{n-k}+1},...,x_n) \\ &= \sum_{s=1}^{n-k} 4^{s-1}\psi_{j_s}(1,(x_1,x_2,...,x_n)). \end{aligned} \tag{2.9}$$

Now, combine (2.8) and (2.9) by use of the triangle inequality to get

$$\begin{aligned} &d(f(2x_1,2x_2,...,2x_n),2^k4^{n-k}f(x_1,x_2,...,x_n)) \\ &\leq d(f(2x_1,2x_2,...,2x_n),4^{n-k}f(x_1,x_2,...,x_{i_1-1},2x_{i_1},x_{i_1+1},...,x_{i_2-1}, \\ &\qquad 2x_{i_2},x_{i_2+1},...,x_{i_k-1},2x_{i_k},x_{i_k+1},...,x_n)) \\ &\quad + d(4^{n-k}f(x_1,x_2,...,x_{i_1-1},2x_{i_1},x_{i_1+1},...,x_{i_2-1}, \\ &\qquad 2x_{i_2},x_{i_2+1},,...,x_{i_k-1},2x_{i_k},x_{i_k+1},...,x_n),2^k4^{n-k}f(x_1,x_2,...,x_n)) \\ &\leq \sum_{r=1}^{k} 2^{k-r}\phi_{i_r}(1,(x_1,x_2,...,x_n) + \sum_{s=1}^{n-k} 4^{s-1}\psi_{j_s}(1,(x_1,x_2,...,x_n)). \end{aligned}$$

Hence, we have

$$\begin{aligned}
&d(\frac{1}{2^k4^{n-k}}f(2x_1,2x_2,...,2x_n),f(x_1,x_2,...,x_n)) \\
&\leq\sum_{r=1}^{k}\frac{2^{r-1}}{2^k}\phi_{i_r}(1,(x_1,x_2,...,x_n)+\sum_{s=1}^{n-k}\frac{4^{s-n+k-1}}{2^k}\psi_{j_s}(1,(x_1,x_2,...,x_n))
\end{aligned} \tag{2.10}$$

for all $x_1,x_2,...,x_n\in G$. Now, proceed in this way to prove by induction on m that

$$\begin{aligned}
&d(\frac{1}{(2^k4^{n-k})^m}f(2^mx_1,2^mx_2,...,2^mx_n),f(x_1,x_2,...,x_n)) \\
&\leq\sum_{u=0}^{m-1}\frac{1}{(2^k4^{n-k})^u}[\sum_{r=1}^{k}\frac{2^{r-1}}{2^k}\phi_{i_r}(u+1,(x_1,x_2,...,x_n)) \\
&\quad+\sum_{s=1}^{n-k}\frac{4^{s-n+k-1}}{2^k}\psi_{j_s}(u+1,(x_1,x_2,...,x_n))].
\end{aligned} \tag{2.11}$$

In order to show that the functions

$$T_m(x_1,x_2,...,x_n)=\frac{1}{(2^k4^{n-k})^m}f(2^mx_1,2^mx_2,...,2^mx_n)$$

form a convergent sequence, we use Cauchy convergence criterion. Indeed, replace x_i by $2^lx_i(1\leq i\leq n)$ in (2.11) and divide the result by $(2^k4^{n-k})^l$, where l is an arbitrary positive integer, we find that

$$\begin{aligned}
&d(\frac{1}{(2^k4^{n-k})^{m+l}}f(2^{m+l}x_1,2^{m+l}x_2,...,2^{m+l}x_n),f(x_1,x_2,...,x_n)) \\
&\leq\sum_{u=l}^{m+l-1}\frac{1}{(2^k4^{n-k})^u}[\sum_{r=1}^{k}\frac{2^{r-1}}{2^k}\phi_{i_r}(u,(x_1,x_2,...,x_n) \\
&\quad+\sum_{s=1}^{n-k}\frac{4^{s-n+k-1}}{2^k}\psi_{j_s}(u,(x_1,x_2,...,x_n))].
\end{aligned}$$

It follows from (2.1) that $\{T_m(x_1,x_2,...,x_n)\}$ is a Cauchy sequence in Y. On the other hand, Y is complete. Then $T(x_1,x_2,...,x_n):=\lim_m T_m(x_1,x_2,...,x_n)$ exists for all $x_1,x_2,...,x_n\in G$. It follows from (2.2) and (2.3) that

$$\begin{aligned}
&D_T^{i_r}(x_1,x_2...,x_{i_r-1},a,b,x_{i_r+1},...,x_n) \\
&\quad=\lim_m\frac{1}{(2^k4^{n-k})^m}D_f^{i_r}(2^mx_1,2^mx_2...,2^mx_{i_r-1},2^ma,2^mb,2^mx_{i_r+1},...,2^mx_n) \\
&\quad\leq\lim_m\frac{1}{(2^k4^{n-k})^m}\phi(2^mx_1,2^mx_2,...,2^mx_{i_r-1},2^ma,2^mb,2^mx_{i_r+1},...,2^mx_n)=0
\end{aligned}$$

for all $r \in \{1, 2, ..., k\}$. Similarly, by (2.2) and (2.4), we have

$$\begin{aligned}
&\Delta_T^{j_s}(x_1, x_2..., x_{j_s-1}, a, b, x_{j_s+1}, ..., x_n)\\
&= \lim_m \frac{1}{(2^k 4^{n-k})^m} \Delta_f^{j_s}(2^m x_1, 2^m x_2..., 2^m x_{j_s-1}, 2^m a, 2^m b, 2^m x_{j_s+1}, ..., 2^m x_n)\\
&\leq \lim_m \frac{1}{(2^k 4^{n-k})^m} \psi(2^m x_1, 2^m x_2, ..., 2^m x_{j_s-1}, 2^m a, 2^m b, 2^m x_{j_s+1}, ..., 2^m x_n) = 0
\end{aligned}$$

for all $s \in \{1, 2, ..., n-k\}$. This means that T satisfies (1.3). It remains to show that T is unique. Suppose that there exists another mapping $T' : G^n \to Y$ which satisfies (1.3) and (2.5). Since

$$(2^k 4^{n-k})^m T(x_1, x_2, ..., x_n) = T(2^m x_1, 2^m x_2, ..., 2^m x_n),$$

$$(2^k 4^{n-k})^m T'(x_1, x_2, ..., x_n) = T'(2^m x_1, 2^m x_2, ..., 2^m x_n)$$

for all $m \in \mathbb{N}, x_1, x_2, ..., x_n \in G$, we conclude that

$$\begin{aligned}
&d(T(x_1, x_2, ..., x_n), T'(x_1, x_2, ..., x_n))\\
&= \frac{1}{(2^k 4^{n-k})^m} d(T(2^m x_1, 2^m x_2, ..., 2^m x_n), T'(2^m x_1, 2^m x_2, ..., 2^m x_n))\\
&= \frac{1}{(2^k 4^{n-k})^m} d(T(2^m x_1, 2^m x_2, ..., 2^m x_n), f(2^m x_1, 2^m x_2, ..., 2^m x_n))\\
&\quad + \frac{1}{(2^k 4^{n-k})^m} d(f(2^m x_1, 2^m x_2, ..., 2^m x_n), T'(2^m x_1, 2^m x_2, ..., 2^m x_n))\\
&\leq \frac{2}{(2^k 4^{n-k})^m} \sum_{u=0}^{\infty} \frac{1}{(2^k 4^{n-k})^u} [\sum_{r=1}^{k} \frac{2^{r-1}}{2^k} \phi_{i_r}(u+1, (2^m x_1, 2^m x_2..., 2^m x_n))\\
&\quad + \sum_{s=1}^{n-k} \frac{4^{s-1}}{2^k} \psi_{j_s}(u+1, (2^m x_1, 2^m x_2..., 2^m x_n))].
\end{aligned}$$

By letting $m \to \infty$ in this inequality, it follows that

$$T(x_1, x_2, ..., x_n) = T'(x_1, x_2, ..., x_n)$$

for all $x_1, x_2, ..., x_n \in G$, which gives the conclusion. □

Theorem 2.2. *Let X be a vector space and Y be a Banach space. Suppose that $\phi, \psi : X^n \to [0, \infty)$ are mappings satisfying*

$$\sum_{u=0}^{\infty} \frac{\phi_{i_r}(u+1, (x_1, x_2, ..., x_n)) + \psi_{j_s}(u+1, (x_1, x_2, ..., x_n))}{(2^k 4^{n-k})^u} < \infty$$

for all $x_1, x_2, ..., x_n \in X$, and

$$\lim_u \frac{\psi(2^u x_1, 2^u x_2, ..., 2^u x_{n+1}) + \phi(2^u x_1, 2^u x_2, ..., 2^u x_{n+1})}{(2^k 4^{n-k})^u} = 0$$

for all $x_1, x_2, ..., x_n, x_{n+1} \in X$. *Suppose* $f : X^n \to Y$ *is a mapping such that*

$$\|f(x_1, x_2, ..., x_{i_r-1}, a+b, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, a, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, b, x_{i_r+1}, ..., x_n)\| \leq \phi(x_1, x_2, ..., x_{i_r-1}, a, b, x_{i_r+1}, ..., x_n),$$

$$\|f(x_1, x_2, ..., x_{j_s-1}, a+b, x_{j_s+1}, ..., x_n) + f(x_1, x_2, ..., x_{j_s-1}, a-b, x_{j_s+1}, ..., x_n) - 2f(x_1, x_2, ..., x_{j_s-1}, a, x_{j_s+1}, ..., x_n) - 2f(x_1, x_2, ..., x_{j_s-1}, b, x_{j_s+1}, ..., x_n)\| \leq \psi(x_1, x_2, ..., x_{j_s-1}, a, b, x_{j_s+1}, ..., x_n)$$

for all $x_1, x_2, ..., x_n, a, b \in X$. *Then there exists a unique mapping* $T : X^n \to Y$ *satisfying (1.3) and*

$$\|f(x_1, x_2, ..., x_n) - T(x_1, x_2, ..., x_n)\| \leq \sum_{u=0}^{\infty} \frac{1}{(2^k 4^{n-k})^u} \Big[\sum_{r=1}^{k} \frac{2^{r-1}}{2^k} \phi_{i_r}(u+1, (x_1, x_2 ..., x_n)) + \sum_{s=1}^{n-k} \frac{4^{s-1}}{2^k} \psi_{j_s}(u+1, (x_1, x_2 ..., x_n))\Big]$$

for all $x_1, x_2, ..., x_n \in X$.

Proof. It follows from Theorem 2.1 by putting $d(a, b) = \|a-b\|$ for all $a, b \in Y$. □

Now we have the following example as an application of the main theorem.

Example 2.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a mapping which satisfies:

$$|f(a+b, x_2) - f(a, x_2) - f(b, x_2)| \leq (|a| + |b| + |x_2|),$$

$$|f(x_1, a+b) + f(x_1, a-b) - 2f(x_1, a) - 2f(x_1, b)| \leq (|a| + |b| + |x_1|)$$

for all $a, b, x_1, x_2 \in \mathbb{R}$. Then there exists a unique mapping $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T_y : \mathbb{R} \to \mathbb{R},\ x \mapsto T(x, y)$ is additive for all $y \in \mathbb{R}$, $T^x : \mathbb{R} \to \mathbb{R},\ y \mapsto T(x, y)$ is quadratic for all $x \in \mathbb{R}$, and that

$$|f(x, y) - T(x, y)| \leq (\frac{5}{7}|x| + \frac{3}{7}|y|)$$

for all $x, y \in \mathbb{R}$.

3. Applications

We use the main theorem of the paper to investigate the Hyers–Ulam–Rassias stability and Hyers–Ulam stability of system of functional equations (1.3). Moreover, we give an example to show that the conditions of our theorem are necessary.

First, we are going to establish the Hyers–Ulam–Rassias stability problem for system of functional equations (1.3).

Theorem 3.1. *Let $\epsilon > 0, p < k + 2^{n-k}$, and let X, Y be a normed space and a Banach space, respectively. If $f : X^n \to Y$ is a mapping such that*

$$\max\Big\{\|f(x_1, x_2, ..., x_{i_r-1}, a+b, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, a, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, b, x_{i_r+1}, ..., x_n)\|, \|f(x'_1, x'_2, ..., x'_{j_s-1}, a'+b', x'_{j_s+1}, ..., x'_n) + f(x'_1, x'_2, ..., x'_{j_s-1}, a'-b', x'_{j_s+1}, ..., x'_n) - 2f(x_1, x_2, ..., x_{j_s-1}, a', x_{j_s+1}, ..., x_n) - 2f(x_1, x_2, ..., x_{j_s-1}, b', x_{j_s+1}, ..., x_n)\|\Big\}$$
$$\le \epsilon(\min\{(\sum_{i=1}^{n} \|x_i\|^p) + \|a\|^p + \|b\|^p, (\sum_{i=1}^{n} \|x'_i\|^p) + \|a'\|^p + \|b'\|^p\})$$

for all $a, b, a', b', x_1, x'_1, x_2, x'_2, ..., x_n, x'_n \in X$, then there exists a unique mapping $T : X^n \to Y$ satisfying (1.3) and

$$\|f(x_1, x_2, ..., x_n) - T(x_1, x_2, ..., x_n)\| \le \frac{\epsilon}{1 - 2^{p-4}}(\frac{1 + 2^{p-3}}{2}\|x\|^p + \frac{3}{8}\|y\|^p)$$

for all $x_1, x_2, ..., x_n \in X$.

Proof. It follows from Corollary 2.2 by putting

$$\phi(x_1, x_2, ..., x_n) = \psi(x_1, x_2, ..., x_n) = \|x_1\|^p + \|x_2\|^p + ... + \|x_n\|^p$$

for all $x_1, x_2, ..., x_n \in X$. □

For the case $k = 1, n = 2$ and $p = 3$, using an idea from the examples of S. Czwerwik [5] and Z. Gajda [7], we have the following counterexample.

Example 3.2. Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x, y) := \begin{cases} xy^2 & \text{for } |x|, |y| < 1; \\ 1 & \text{otherwise.} \end{cases}$$

Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x, y) := \sum_{n=0}^{\infty} 8^{-n}\phi(2^n x, 2^n y).$$

It is clear that f is continuous and bounded by $\frac{8}{7}$ on $\mathbb{R} \times \mathbb{R}$. We prove that

$$\big|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\big| \le \frac{3 \times 8^3}{7}(|x_1|^3 + |x_2|^3 + |y|^3), \tag{3.1}$$

$$\big|f(y, x_1 + x_2) + f(y, x_1 - x_2) - 2f(y, x_1) - 2f(y, x_2)\big| \le \frac{6 \times 8^3}{7}(|x_1|^3 + |x_2|^3 + |y|^3), \tag{3.2}$$

for all $x, y \in \mathbb{R}$. To see this, if $|x_1|^3 + |x_2|^3 + |y|^3 = 0$ or $|x_1|^3 + |x_2|^3 + |y|^3 \geq \frac{1}{8}$, then

$$\big|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\big| \leq 3\sum_{n=0}^{\infty} 8^{-n} \leq \frac{3 \times 8^2}{7}(|x_1|^3 + |x_2|^3 + |y|^3),$$

$$\big|f(y, x_1 + x_2) + f(y, x_1 - x_2) - 2f(y, x_1) - 2f(y, x_2)\big| \leq 6\sum_{n=0}^{\infty} 8^{-n}$$

$$\leq \frac{6 \times 8^2}{7}(|x_1|^3 + |x_2|^3 + |y|^3).$$

Now suppose that $0 < |x_1|^3 + |x_2|^3 + |y|^3 < \frac{1}{8}$. Then there exists a non-negative integer k such that

$$\frac{1}{8^{k+2}} \leq |x_1|^3 + |x_2|^3 + |y|^3 < \frac{1}{8^{k+1}}. \tag{3.3}$$

Therefore

$$2^{k-1}|x_1|, 2^{k-1}|x_2|, 2^{k-1}|y|, 2^{k-1}|2x_1 \pm x_2|, 2^{k-1}|x_1 \pm x_2| \in (-1, 1).$$

Hence

$$2^m|x_1|, 2^m|x_2|, 2^m|y|, 2^m|2x_1 \pm x_2|, 2^m|x_1 \pm x_2| \in (-1, 1)$$

for all $m = 0, 1, ..., k - 1$. From the definition of f and (3.3), we have

$$\big|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\big| \leq 3\sum_{n=k}^{\infty} 8^{-n} \leq \frac{3 \times 8^3}{7}(|x_1|^3 + |x_2|^3 + |y|^3),$$

$$\big|f(y, x_1 + x_2) + f(y, x_1 - x_2) - 2f(y, x_1) - 2f(y, x_2)\big| \leq 6\sum_{n=k}^{\infty} 8^{-n}$$

$$\leq \frac{6 \times 8^3}{7}(|x_1|^3 + |x_2|^3 + |y|^3).$$

Therefore f satisfies (3.1) and (3.2). Let $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an additive-quadratic function such that

$$|f(x, y) - Q(x, y)| \leq \beta(|x|^3 + |y|^3)$$

for all $x, y \in \mathbb{R}$, where β is a positive constant. Then it is easy to show that there exists a constant $c \in \mathbb{R}$ such that $Q(x, y) = cxy^4$ for all rational numbers x, y. So we have

$$|f(x, y)| \leq (\beta + |c|)(|x|^3 + |y|^3) \tag{3.4}$$

for all rational numbers x, y. Let m be a positive integer with $m > 2(\beta + |c|)$. If x is a rational number in $(0, 2^{1-m})$, then $2^n x \in (0, 1)$ for all $n = 0, 1, ..., m - 1$. So

$$f(x, x) \geq \sum_{n=0}^{m-1} 8^{-n}\phi(2^n x, 2^n x) = mx^3 > 2(\beta + |c|)x^3$$

which contradicts (3.4).

By Theorem 3.1, we solve the following Hyers–Ulam stability problem for system of functional equations (1.3).

Theorem 3.3. *Let $\epsilon > 0, n > 2$, and let X, Y be a normed space and a Banach space, respectively. If $f : X^n \to Y$ is a mapping such that*

$$\max\big\{\|f(x_1, x_2, ..., x_{i_r-1}, a+b, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, a, x_{i_r+1}, ..., x_n) - f(x_1, x_2, ..., x_{i_r-1}, b, x_{i_r+1}, ..., x_n)\|, \|f(x_1', x_2', ..., x_{j_s-1}', a'+b', x_{j_s+1}', ..., x_n') + f(x_1', x_2', ..., x_{j_s-1}', a'-b', x_{j_s+1}', ..., x_n') - 2f(x_1, x_2, ..., x_{j_s-1}, a', x_{j_s+1}, ..., x_n) - 2f(x_1, x_2, ..., x_{j_s-1}, b', x_{j_s+1}, ..., x_n)\|\big\} \leq \epsilon$$

for all $a, b, a', b', x_1, x_1', x_2, x_2', ..., x_n, x_n' \in X$, then there exists a unique additive-quadratic mapping $T : X^n \to Y$ satisfying (1.3) and

$$\|f(x_1, x_2, ..., x_n) - T(x_1, x_2, ..., x_n)\| \leq (\frac{15\epsilon}{n-2})(\frac{2^{12}}{2^{16}-1})$$

for all $x, y \in X$.

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