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ON SOME ASYMPTOTIC PROPERTIES OF FINITELY GENERATED MODULES

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Dedicated to Professor Tran Duc Van on the occasion of his sixtieth birthday

Abstract. Let I, J be a pair of ideals of a Noetherian local ring R. The purpose of this paper is to study asymptotic properties such as the asymptotic stability of the depths in dimension $> k$, $\operatorname{depth}_k(I, M_n)$ and $\operatorname{depth}_k(I, M_n/M_{n+1})$ of an ideal I with respect to a J-filtration (M_n) of submodules of a finitely generated module M; or the asymptotic behaviour of sets of associated prime ideals or of attached prime ideals of certain local cohomology modules determined by M and I, J.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring, I, J two ideals of R, and M a finitely generated $R-$module. In 1979, M. Brodmann [2] had proved that the sets $\operatorname{Ass}_R(J^nM/J^{n+1}M)$ and $\operatorname{Ass}_R(M/J^nM)$ are stable for large n. Hence he showed in [3] that the integers $\operatorname{depth}(I, J^nM/J^{n+1}M)$ and $\operatorname{depth}(I, M/J^nM)$ take constant values for large n. Recently, in [5], M. Brodmann and L. T. Nhan introduced the notion of $M-$sequence in dimension $> k$. They showed that if $\dim M/IM > k$, then each M-sequence in dimension $> k$ in I can be extended to a maximal one and all maximal $M-$sequences in dimension $> k$ in I have the same length. This common length is denoted by $\operatorname{depth}_k(I, M)$. Then, in [7] we proved that the integers $\operatorname{depth}_k(I, J^nM/J^{n+1}M)$ and $\operatorname{depth}_k(I, M/J^nM)$ take constant values for large n. In 2005, J. Herzog and T. Hibi [9] denoted the eventual values of $\operatorname{depth}(\mathfrak{m}, J^n)$, $\operatorname{depth}(\mathfrak{m}, J^n/J^{n+1})$, and $\operatorname{depth}(\mathfrak{m}, R/J^n)$ by $\lim_{n\to\infty} \operatorname{depth} J^n$, $\lim_{n\to\infty} \operatorname{depth} J^n/J^{n+1}$, and $\lim_{n\to\infty} \operatorname{depth} R/J^n$, respectively, and they showed that

$$\lim_{n\to\infty} \operatorname{depth} R/J^n \leq \lim_{n\to\infty} \operatorname{depth} J^n/J^{n+1} = \lim_{n\to\infty} \operatorname{depth} J^n - 1.$$

The first result of this paper is to generalize Herzog and Hibi's theorem to the depth in dimension $> k$ in I with respect to a stable $J-$filtration (M_n). Next, in 1990, C. Huneke [11, Problem 4] asked whether the set of associated primes of

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$H^i_I(M)$ is finite for all finitely generated modules M and all ideals I. Although, M. Katzman [12] had given an example of finitely generated modules having some local cohomology with infinite associated prime ideals, the problem is still true in many situations. Hence the question of when the set $\mathrm{Ass}_R(H^i_I(M))$ is finite has been studied by many authors (see [4], [7], [10], [17],...). It is well known that, in general, local cohomology modules $H^i_I(M)$ are not finitely generated. The least integer i for which $H^i_I(M)$ is not finitely generated is called the finiteness dimension of M with respect to I, and is denoted by $f_I(M)$. Then, M. Brodmann and L. Faghani had proved in [4] that $\mathrm{Ass}_R(H^i_I(M))$ is finite for all $i \leq f_I(M)$. Therefore, the sets $\mathrm{Ass}_R(H^1_I(J^nM/J^{n+1}M))$ and $\mathrm{Ass}_R(H^1_I(M/J^nM))$ are finite for all pairs of ideals I, J. Then, our next results are concerning with the following questions:

Question 1: Are these sets stable for enough large n?
In fact, we will show that the set $\mathrm{Ass}_R(H^1_I(J^nM/J^{n+1}M))$ is stable for enough large n, but $\mathrm{Ass}_R(H^1_I(M/J^nM))$ is not. It is somehow strange to us, because many asymptotic properties, which hold true on $J^nM/J^{n+1}M$, are also true on M/J^nM.

It is well-known that the modules $H^i_{\mathfrak{m}}(M)$ are Artinian for all i. Therefore the set $\mathrm{Att}_R(H^i_{\mathfrak{m}}(M))$ of attached prime ideals of $H^i_{\mathfrak{m}}(M)$ is a finite set. Now, if we use N_n to denote one of the three following $R-$modules I^nM, $I^nM/I^{n+1}M$ and M/I^nM, it is clear that $\mathrm{Att}_R(H^0_{\mathfrak{m}}(N_n))$ is stable for enough large n. Moreover, by I. G. Macdonald and R. Y. Sharp [14, Theorem 2.2], the set $\mathrm{Att}_R(H^d_{\mathfrak{m}}(N_n))$ is stable for enough large n, where $d = \dim N_n$. So, it is natural to ask the following question:

Question 2: Is the set $\mathrm{Att}_R(H^i_{\mathfrak{m}}(N_n))$ for all i stable for enough large n?
Unfortunately, we will see in this paper that the answer for this question is not affirmative in general.

Our paper is divided into 3 sections. In the next section, we prove a generalization of Herzog and Hibi's theorem. Section 3 is devoted to give answers for the two questions above.

2. The generalized depth of a filtration

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, I, J two ideals of R, and M a finitely generated $R-$module. First of all, we recall the definition of a generalization of regular sequence which was first given by Brodmann and Nhan.

Definition 2.1. ([5, Definition 2.1]) Let $k \geq -1$ be an integer and $x_1, ..., x_r \in R$ a sequence. We say that $x_1, ..., x_r$ is an *$M-$sequence in dimension* $> k$ if $x_i \notin \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1, ..., x_{i-1})M)$, $\dim(R/\mathfrak{p}) > k$ and for all $i = 1, ..., r$.

It is easy to see that $x_1, ..., x_r$ is an $M-$sequence in dimension > -1 if and only if it is a regular sequence of M; $x_1, ..., x_r$ is an $M-$sequence in dimension > 0 if and only if it is a filter regular sequence of M introduced by N. T. Cuong, P. Schenzel, and N. V. Trung in [8]; and $x_1, ..., x_r$ is an $M-$sequence in dimension

> 1 if and only if it is a generalized regular sequence of M introduced by Nhan in [17].

If $\dim(M/IM) > k$, then any $M-$sequence in dimension $> k$ in I can be extended to a maximal one, and all maximal $M-$ sequences in dimension $> k$ in I have same length. Then, in [7], we denoted this common length by $\mathrm{depth}_k(I, M)$. If $\dim(M/IM) \leq k$, for every positive integer r we can choose an $M-$sequence in dimension $> k$ in I of length r, in this case we set $\mathrm{depth}_k(I, M) = +\infty$.

We use the convention that $\inf(\emptyset) = +\infty$ and $\sup(\emptyset) = -\infty$. Then we have the following lemma.

Lemma 2.2. ([7, Lemma 2.3]) *Let $k \geq -1$ be an integer. Then*

$$\mathrm{depth}_k(I, M) = \inf\{i| \dim(\mathrm{Ext}^i_R(R/I, M)) > k\}.$$

Let $\underline{x} = x_1, ..., x_t$ be a system of generators of the ideal I and $H_i(\underline{x}; M)$ the i-th Koszul homology module of the Koszul complex $K_\bullet(\underline{x}; M)$ of M with respect to $\underline{x}$. It should be noted that $\mathrm{depth}_{-1}(I, M)$ is the usual depth of M in I, $\mathrm{depth}(I, M)$, and it is determined by

$$\mathrm{depth}(I, M) = t - \sup\{i|H_i(\underline{x}; M) \neq 0\}.$$

To prove the main result in this section, we need to extend the above formula to $\mathrm{depth}_k(I, M)$ as follows.

Proposition 2.3. *Suppose that I is generated by $\underline{x} = x_1, ..., x_t$. Then*

$$\mathrm{depth}_k(I, M) = t - \sup\{i| \dim(H_i(\underline{x}; M)) > k\}.$$

Proof. If $\dim(M/IM) \leq k$, then $\mathrm{depth}_k(I, M) = +\infty$. On the other hand, it follows by $IH_i(\underline{x}; M) = 0$ that $\dim(H_i(\underline{x}; M)) \leq \dim(M/IM) \leq k$ for all i. Hence $t - \sup\{i| \dim(H_i(\underline{x}; M)) > k\} = t - \sup(\emptyset) = +\infty$. Therefore, the equality is true in this case. If $\dim(M/IM) > k$, then $r = \mathrm{depth}_k(I, M) < +\infty$. By the same argument in the proof for the usual depth ($k = -1$), we can show by induction on r that the equality is also true in this case. □

Lemma 2.4. *Let M, N, P be $R-$modules and $f : M \to N$, $g : N \to P$ homomorphisms. Assume that* $\dim(\mathrm{Ker}\, f) \leq k$ *and* $\dim(\mathrm{Ker}\, g) \leq k$, *then* $\dim(\mathrm{Ker}(g \circ f)) \leq k$.

Proof. Set $K = \mathrm{Ker}(g \circ f)$. Then we have the following exact sequence

$$0 \to \mathrm{Ker}\, f \to K \to f(K) \to 0.$$

It follows by the assumption that $\dim(\mathrm{Ker}\, f) \leq k$ and $\dim(f(K)) \leq \dim(\mathrm{Ker}\, g) \leq k$. Hence $\dim(K) \leq k$, as required. □

Let $\mathcal{R} = \oplus_{n \geq 0} R_n$ be a finitely generated standard graded algebra over $R_0 = R$, and $\mathcal{M} = \oplus_{n \geq 0} M_n$ a finitely generated graded $\mathcal{R}-$module. Then, we have the following result.

Lemma 2.5. ([7, Theorem 1.1]) *Let $k \geq -1$ be an integer. Then,* $\mathrm{depth}_k(I, M_n)$ *takes a constant value for large n.*

We denote the constant value of $\operatorname{depth}_k(I, M_n)$ by $\lim\limits_{n\to\infty} \operatorname{depth}_k(I, M_n)$.

Recall that a chain $M = M_0 \supseteq M_1 \supseteq ... \supseteq M_n \supseteq ...$, where M_n are submodules of M, is called a filtration of M, and denoted by (M_n). It is called an $J-$filtration of M if $JM_n \subseteq M_{n+1}$ for all n, and a stable $J-$filtration of M if there exists n_0 such that $JM_n = M_{n+1}$ for all $n \geq n_0$.

By Artin-Rees Lemma we can prove the following lemma.

Lemma 2.6. *Let (M_n) be a stable $J-$filtration of M and $J \subseteq I$. Assume that $JM_n = M_{n+1}$ for all $n \geq n_0$ and I is generated by $\underline{x} = x_1, ..., x_t$. Then, for all i, the natural homomorphism $H_i(\underline{x}; M_n) \to H_i(\underline{x}; M_{n_0})$ is zero for enough large n.*

Proof. Consider the following commutative diagram with exact rows and exact columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
... & \longrightarrow & K_{i+1}(\underline{x}) \otimes M_{n+n_0} & \xrightarrow{d_{i+1}} & K_i(\underline{x}) \otimes M_{n+n_0} & \xrightarrow{d_i} & K_{i-1}(\underline{x}) \otimes M_{n+n_0} & \longrightarrow & ... \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
... & \longrightarrow & K_{i+1}(\underline{x}) \otimes M_{n_0} & \xrightarrow{\delta_{i+1}} & K_i(\underline{x}) \otimes M_{n_0} & \xrightarrow{\delta_i} & K_{i-1}(\underline{x}) \otimes M_{n_0} & \longrightarrow & ....
\end{array}$$

We have

$$\operatorname{Ker} d_i = \operatorname{Ker} \delta_i \cap (K_i(\underline{x}) \otimes M_{n+n_0}) = \operatorname{Ker} \delta_i \cap J^n(K_i(\underline{x}) \otimes M_{n_0}).$$

By Artin-Rees Lemma there exists an integer k such that

$$\operatorname{Ker} d_i = J^{n-k}(\operatorname{Ker} \delta_i \cap J^{n-1}(K_i(\underline{x}) \otimes M_{n_0})) \subseteq J^{n-k} \operatorname{Ker} \delta_i$$

for all $n \geq k$. Since $J \subseteq I$, $JH_i(\underline{x}; M_{n_0}) = 0$. Hence $J^{n-k} \operatorname{Ker} \delta_i \subseteq \operatorname{Im} \delta_{i+1}$. Therefore, the natural homomorphism $H_i(\underline{x}; M_{n+n_0}) \to H_i(\underline{x}; M_{n_0})$ is zero for all $n \geq k$. □

The following theorem is the main result of this section.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I, J \subset R$ two ideals, M a finitely generated $R-$module, and (M_n) a stable $J-$filtration of M. Then for all $k \geq -1$*

(i) *There exist the limits*

$$\lim_{n\to\infty} \operatorname{depth}_k(I, M_n), \lim_{n\to\infty} \operatorname{depth}_k(I, M_n/M_{n+1}), \text{ and} \lim_{n\to\infty} \operatorname{depth}_k(I, M/M_n).$$

(ii) *We always have the inequalities*

$$\lim_{n\to\infty} \operatorname{depth}_k(I, M/M_n) \leq \lim_{n\to\infty} \operatorname{depth}_k(I, M_n/M_{n+1})$$

and

$$\lim_{n\to\infty} \operatorname{depth}_k(I, M_n) - 1 \leq \lim_{n\to\infty} \operatorname{depth}_k(I, M_n/M_{n+1}).$$

(iii) *If $J \subseteq I$, then*

$$\lim_{n\to\infty} \operatorname{depth}_k(I, M_n/M_{n+1}) = \lim_{n\to\infty} \operatorname{depth}_k(I, M_n) - 1.$$

Proof. (i) We set $\mathcal{R}(J)=\oplus_{n\geq 0}J^n$, $\mathcal{R}(M)=\oplus_{n\geq 0}M_n$, and $G(M)=\oplus_{n\geq 0}M_n/M_{n+1}$. Since (M_n) is a stable J–filtration and by [1, Lemma 10.8], $\mathcal{R}(M)$ and $G(M)$ are finitely generated graded $\mathcal{R}(J)-$modules. By Lemma 2.5we get that $\operatorname{depth}_k(I,M_n)$ and $\operatorname{depth}_k(I,M_n/M_{n+1})$ are stable for large n. Assume that $r=\operatorname{depth}_k(I,M_n)$ and $s=\operatorname{depth}_k(I,M_n/M_{n+1})$ for all $n\geq a$. We will prove the stability of $\operatorname{depth}_k(I,M/M_n)$. For each $n\geq a$, we set $f(n)=\operatorname{depth}_k(I,M/M_n)$. By the short exact sequence

$$0\to M_n/M_{n+1}\to M/M_{n+1}\to M/M_n\to 0,$$

we get the long exact

$$\begin{aligned}...\to \operatorname{Ext}^{i-1}_R(R/I,M/M_n)\to \operatorname{Ext}^{i}_R(R/I,M_n/M_{n+1})\to \operatorname{Ext}^{i}_R(R/I,M/M_{n+1})\\ \to \operatorname{Ext}^{i}_R(R/I,M/M_n)\to ...\end{aligned}$$

Therefore, we obtain by Lemma 2.2 that $\dim(\operatorname{Ext}^{i}_R(R/I,M_n/M_{n+1}))\leq k$ for all $i<s$, and $\dim(\operatorname{Ext}^{i}_R(R/I,M/M_n))\leq k$ for all $i<f(n)$. It follows that $\dim(\operatorname{Ext}^{i}_R(R/I,M/M_{n+1}))\leq k$ for all $i<\min\{s,f(n)\}$. Hence $f(n+1)\geq\min\{s,f(n)\}$ for all $n\geq a$. By a similar argument, we can also show that $s\geq\min\{f(n+1),f(n)+1\}$. Consider two cases: Firstly, there exists $n_0\geq a$ such that $f(n_0)>s$ then $f(n_0+1)=s$. Hence $f(n)=s$ for all $n\geq n_0+1$. Secondly, $f(n)\leq s$ for all $n\geq a$. In this case, $f(n+1)\geq f(n)$ for all $n\geq a$. It follows for all cases that $f(n)$ is an increasing function and bounded by s. Hence $f(n)$ is stable for enough large n.

(ii) By the proof above, it also implies that

$$\lim_{n\to\infty}\operatorname{depth}_k(I,M/M_n)\leq s=\lim_{n\to\infty}\operatorname{depth}_k(I,M_n/M_{n+1}).$$

On the other hand, by the short exact sequence $0\to M_{n+1}\to M_n\to M_n/M_{n+1}\to 0$ we get the long exact

$$...\to \operatorname{Ext}^{i}_R(R/I,M_n)\to \operatorname{Ext}^{i}_R(R/I,M_n/M_{n+1})\to \operatorname{Ext}^{i+1}_R(R/I,M_{n+1})\to ....$$

From this exact sequence and by Lemma 2.2, we have

$$\lim_{n\to\infty}\operatorname{depth}_k(I,M_n)-1\leq\lim_{n\to\infty}\operatorname{depth}_k(I,M_n/M_{n+1})$$

as required.

(iii) Let $r=\lim_{n\to\infty}\operatorname{depth}_k(I,M_n)$ and $s=\lim_{n\to\infty}\operatorname{depth}_k(I,M_n/M_{n+1})$. We need only to show that $s\leq r-1$ when $J\subseteq I$. In fact, suppose that $s>r-1$. We can choose n_0 such that $JM_n=M_{n+1}$, $r=\operatorname{depth}_k(I,M_n)$, and $s=\operatorname{depth}_k(I,M_n/M_{n+1})$ for all $n\geq n_0$. Assume that I is generated by $\underline{x}=x_1,...,x_t$. Then by Proposition 2.3,

$$t-r=\sup\{i|\dim(H_i(\underline{x};M_n))>k\},$$

and

$$t-s=\sup\{i|\dim(H_i(\underline{x};M_n/M_{n+1}))>k\}.$$

By our assumption, $t-r \geq t-s$. Therefore, for all $i > t-r$ we get the inequalities $\dim(H_i(\underline{x}; M_n)) \leq k$ and $\dim(H_i(\underline{x}; M_n/M_{n+1})) \leq k$. By using again the short exact $0 \to M_{n+1} \to M_n \to M_n/M_{n+1} \to 0$, we get the long exact

$$... \to H_{t-r+1}(\underline{x}; M_n/M_{n+1}) \to H_{t-r}(\underline{x}; M_{n+1}) \overset{g_n}{\to} H_{t-r}(\underline{x}; M_n) \to ....$$

Since $\dim(H_{t-r+1}(\underline{x}; M_n/M_{n+1})) \leq k$, we have $\dim(\operatorname{Ker} g_n) \leq k$. Therefore, by Lemma 2.4, we get

$$\dim(\operatorname{Ker}(g_{n_0} \circ g_{n_0+1} \circ \ldots \circ g_{n_0+l})) \leq k$$

for all $l > 0$. By Lemma 2.6, the homomorphism $H_{t-r}(\underline{x}; M_{n_0+l}) \to H_{t-r}(\underline{x}; M_{n_0})$ is zero for enough large l. Hence $\dim(H_{t-r}(\underline{x}; M_n)) \leq k$ for enough large n, which is a contradiction with the choice of r. Thus $s = r-1$, and the proof of the theorem is complete. □

3. Associated and attached primes of local cohomology modules

Before giving answers to the questions mentioned in the introduction, we need some preliminary results. A sequence $x_1, \ldots, x_r \in I$ is called an *$I-$filter regular sequence of M* if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M) \setminus \mathrm{V}(I)$ and all $i = 1, \ldots, r$. It should be mentioned that the notion of $I-$ filter regular sequence of M is a generalization of the concept of filter regular sequence of M defined by N. T. Cuong, P. Schenzel, and N. V. Trung in [8].

Lemma 3.1. ([16, 3.4]) *If $x_1, \ldots, x_r$ is an $I-$filter regular sequence of M, then we have*

$$H^j_I(M) = \begin{cases} H^j_{(x_1,\ldots,x_r)}(M) & \text{if } j < r, \\ H^{j-r}_I(H^r_{(x_1,\ldots,x_r)}(M)) & \text{if } j \geq r. \end{cases}$$

Below, we recall a counter-example of M. Katzman which plays an important role in the rest of this paper.

Lemma 3.2. ([12, Corollary 1.3]) *Let $S = k[x, y, s, t, u, v]$ be the polynomial ring of six variables over a field k and $f = sx^2v^2 - (s+t)xyuv + ty^2u^2$. Denote by T the localization of S/fS at the irrelevant maximal ideal $\mathfrak{m} = (x, y, s, t, u, v)$. Then, the set $\operatorname{Ass}_T(H^2_{(u,v)T}(T))$ is infinite.*

The following result is a complete answer for Question 1.

Theorem 3.3. *The following statements are true.*

(i) *If M is a finitely generated R-module and I, J are ideals of R, then the set $\operatorname{Ass}_R(H^1_I(J^nM/J^{n+1}M))$ is stable for enough large n.*

(ii) *There exist a finitely generated $R-$module M and ideals I, J of R such that $\operatorname{Ass}_R(H^1_I(M/J^nM))$ is not stable for large n.*

Proof. The statement (i) is a special case of the following result.

Proposition 3.4. *Let $\mathcal{R} = \bigoplus_{n\geq 0} R_n$ be a finitely generated standard graded algebra over R_0 with $R_0 = R$ and $\mathcal{M} = \bigoplus_{n\geq 0} M_n$ a finitely generated graded $\mathcal{R}-$module. Let $I \subseteq R$ be an ideal. Then, $\operatorname{Ass}_R(H^1_I(M_n))$ is stable for large n.*

Proof. We know that $\bigoplus_{n\geq 0} M_n/\Gamma_I(M_n)$ is a finitely generated graded $\mathcal{R}-$module and $H^i_I(M_n/\Gamma_I(M_n)) \cong H^i_I(M_n)$ for all $i>0$ and for all n. Hence we can replace M_n by $M_n/\Gamma_I(M_n)$. So we can assume that $\operatorname{depth}(I, M_n) > 0$ for all n. Let r be the eventual value of $\operatorname{depth}(I, M_n)$. Then $r \geq 1$. It follows by [7, Theorem 3.2] that $\operatorname{Ass}_R(H^i_I(M_n))$ is stable for large n for all $i \leq r$. □

Proof of (ii) of Theorem 3.3. We consider the local ring T and the elements u and v as in Lemma 3.2. Then $\operatorname{Ass}_T(H^2_{(u,v)}(T))$ is infinite. Let a, b be an $(u,v)T$-filter regular sequence of T. Then we prove that the set $\operatorname{Ass}_R(H^1_I(M/J^nM))$ is not asymptotically stable, where $M = T$, $I = (b)$ and $J = (a)$. To do this, we need only to show that the set $\bigcup\limits_n \operatorname{Ass}_T(H^1_{(b)}(T/a^nT))$ is an infinite set. Indeed, since

$$H^2_{(u,v)}(T) = H^0_{(u,v)}(H^2_{(a,b)}(T)) = H^0_{(u,v)}(H^1_{(a,b)}(H^1_{(a)}(T))),$$

$\operatorname{Ass}_T(H^1_{(a,b)}(H^1_{(a)}(T)))$ is an infinite set. On the other hand, since

$$H^1_{(a,b)}(H^1_{(a)}(T)) = H^1_{(b)}(H^1_{(a)}(T)) = H^1_{(b)}(\varinjlim_n (T/a^nT)) = \varinjlim_n H^1_{(b)}(T/a^nT),$$

$$\operatorname{Ass}_T(H^1_{(a,b)}(H^1_{(a)}(T))) = \operatorname{Ass}_T(\varinjlim_n H^1_{(b)}(T/a^nT)) \subseteq \bigcup_n \operatorname{Ass}_T(H^1_{(b)}(T/a^nT)).$$

Therefore, $\bigcup\limits_n \operatorname{Ass}_T(H^1_{(b)}(T/a^nT))$ is an infinite set. □

In [13], I. G. Macdonald introduced the theory of secondary representation of a module, which is in some sense dual to the theory of primary decomposition. Following [13], any Artinian $R-$module A has a minimal secondary representation $A = A_1 + ... + A_r$, where A_i is $\mathfrak{p}_i-$secondary. The set $\{\mathfrak{p}_1, ..., \mathfrak{p}_r\}$ is independent of the choice of a minimal secondary representation of A, and it is denoted by $\operatorname{Att}_R(A)$.
We use N_n to denote one of the three following $R-$modules $I^nM, I^nM/I^{n+1}M$ and M/I^nM. It is well known that $H^i_{\mathfrak{m}}(N_n)$ is an Artinian $R-$module for all i. Hence the set $\operatorname{Att}_R(H^i_{\mathfrak{m}}(N_n))$ is finite. Moreover, for $i=0$ or $i=d$, where $d = \dim(N_n)$, the set $H^i_{\mathfrak{m}}(N_n)$ is stable for large n. However, for i being an arbitrary integer, this property is not true in general. We have the following theorem which is a negative answer for Question 2.

Theorem 3.5. *The following statements are true.*

(i) *Let $(T, \mathfrak{m})$ be the local ring as in Lemma 3.2 and $I = (u,v)T$. Then the sets $\operatorname{Att}_T(H^3_{\mathfrak{m}}(T/I^n))$ and $\operatorname{Att}_T(H^4_{\mathfrak{m}}(I^n))$ are not stable for enough large n.*
(ii) *There exist a finitely generated module M over a local ring $(R, \mathfrak{m})$ and an ideal J of R such that $\operatorname{Att}_R(H^i_{\mathfrak{m}}(J^nM/J^{n+1}M))$ is not asymptotically stable for some i.*

Before proving Theorem 3.5, we recall some properties of Matlis duality. We denote by $E = E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. For each $R-$module N, the Matlis dual $D(N)$ of N is defined by $D(N) = \mathrm{Hom}_R(N, E)$. Suppose that N is a Noetherian $R-$module, it follows by Matlis duality that $D(N)$ is an Artinian $R-$module, and $\mathrm{Att}_R(D(N)) = \mathrm{Ass}_R(N)$. Moreover, if R is a Gorenstein local ring of dimension n and N is a finitely generated $R-$module, by Local Duality Theorem we have $H^{n-i}_{\mathfrak{m}}(N) \cong D(\mathrm{Ext}^i_R(N, R))$ for all $i \in \mathbb{Z}$.

The following result is well-known and will be used in sequel.

Lemma 3.6. ([15, Proposition 4.1]) *Let $f : R \to R'$ be a homomorphism of rings and A an Artinian $R'-$module. Then we have*

$$\mathrm{Att}_R(A) = \{\mathfrak{p} \cap R | \mathfrak{p} \in \mathrm{Att}_{R'}(A)\}.$$

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. (i). We consider the local ring $(T, \mathfrak{m})$, and $I = (u, v)T$ as in Lemma 3.2. Since $(T, \mathfrak{m})$ is a local Gorenstein ring of dimension 5, it follows from Local Duality Theorem that

$$H^3_{\mathfrak{m}}(T/I^n) \cong \mathrm{Hom}_T(\mathrm{Ext}^2_T(T/I^n, T), E).$$

Therefore,

$$\mathrm{Att}_T(H^3_{\mathfrak{m}}(T/I^n)) = \mathrm{Ass}_T(\mathrm{Ext}^2_T(T/I^n, T)).$$

Hence

$$\begin{aligned}\bigcup_{n\geq 0} \mathrm{Att}_T(H^3_{\mathfrak{m}}(T/I^n)) &= \bigcup_{n\geq 0} \mathrm{Ass}_T(\mathrm{Ext}^2_T(T/I^n, T))\\ &\supseteq \mathrm{Ass}_T(\varinjlim_n \mathrm{Ext}^2_T(T/I^n, T))\\ &= \mathrm{Ass}_T(H^2_I(T)).\end{aligned}$$

Since $\mathrm{Ass}_T(H^2_I(T))$ is an infinite set by Lemma 3.2, the set $\bigcup_{n\geq 0} \mathrm{Att}_T(H^3_{\mathfrak{m}}(T/I^n))$ is infinite, and the first conclusion of (i) follows.

Now, from the short exact sequence

$$0 \to I^n \to T \to T/I^n \to 0$$

we get that $\mathrm{Ext}^1_T(I^n, T) \cong \mathrm{Ext}^2_T(T/I^n, T)$. Therefore, the set

$$\begin{aligned}\bigcup_{n\geq 0} \mathrm{Att}_T(H^4_{\mathfrak{m}}(I^n)) &= \bigcup_{n\geq 0} \mathrm{Ass}_T(\mathrm{Ext}^1_T(I^n, T))\\ &= \bigcup_{n\geq 0} \mathrm{Ass}_T(\mathrm{Ext}^2_T(T/I^n, T))\end{aligned}$$

is infinite, and so the set $\mathrm{Att}_T(H^4_{\mathfrak{m}}(I^n))$ cannot be stable for enough large n.

(ii). With $(T, \mathfrak{m})$, and $I = (u, v)T$ as in the proof of (i), we consider the Rees ring $\mathcal{R} = \oplus_{n\geq 0} I^n$. Set $\mathcal{R}_+ = \oplus_{n>0} I^n$ the irrelevant ideal and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ the maximal homogeneous ideal of $\mathcal{R}$. We consider the local ring $R = \mathcal{R}_{\mathfrak{M}}$, $J = (\mathcal{R}_+)R$ and the finitely generated R-module $M = R$. For convenience, we also denote by $\mathfrak{M}$ the maximal ideal and by $\mathcal{R}_+$ the ideal $(\mathcal{R}_+)R$ of R. Then we prove that the set $\mathrm{Att}_R(H^4_{\mathfrak{M}}(J^n M/J^{n+1}M))$ is not asymptotically stable. Since

$\mathcal{R}_+^n/\mathcal{R}_+^{n+1}$ is annihilated by $\mathcal{R}_+$, the Independence Theorem of local cohomology implies the following R-isomorphisms

$$\begin{aligned} H^4_{\mathfrak{M}}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}) &= H^4_{mR+\mathcal{R}_+}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}) \\ &\cong H^4_{\mathfrak{m}R}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}) \\ &\cong H^4_{\mathfrak{m}}(I^n). \end{aligned}$$

It follows by Lemma 3.6 that

$$\begin{aligned} \mathrm{Att}_T(H^4_{\mathfrak{m}}(I^n)) &= \mathrm{Att}_T(H^4_{\mathfrak{m}R}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1})) \\ &= \{P \cap T | P \in \mathrm{Att}_R(H^4_{\mathfrak{M}}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}))\}. \end{aligned}$$

Therefore,

$$\bigcup_{n\geq 0} \mathrm{Att}_T(H^4_{\mathfrak{m}}(I^n)) = \{P \cap T | P \in \bigcup_{n\geq 0} \mathrm{Att}_{\mathcal{R}}(H^4_{\mathfrak{M}}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}))\}.$$

Since $\bigcup_{n\geq 0} \mathrm{Att}_T(H^4_{\mathfrak{m}}(I^n))$ is an infinite set by the proof of (i), the set $\bigcup_{n\geq 0} \mathrm{Att}_{\mathcal{R}}(H^4_{\mathfrak{M}}(\mathcal{R}_+^n/\mathcal{R}_+^{n+1}))$ is infinite, and so the statement (iii) follows.

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