ACTA MATHEMATICA VIETNAMICA
Volume 37, Number 4, 2012, pp. 563–577

REALISING UNSTABLE MODULES AS THE COHOMOLOGY OF SPACES AND MAPPING SPACES

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Abstract. This report discusses the question whether or not a given unstable module is the mod-p cohomology of a space. One first discusses shortly the Hopf invariant 1 problem and the Kervaire invariant 1 problem and gives their relations to homotopy theory and geometric topology. Then one describes some more qualitative results, emphasizing the use of the space $\mathrm{map}(B\mathbb{Z}/p, X)$ and of the structure of the category of unstable modules.

1. Introduction

Let p be a prime number, in all the sequel H^*X will denote the mod p singular cohomology of the topological space X. All spaces X will be supposed p-complete and connected.

The singular mod-p cohomology is endowed with various structures:

- it is a graded $\mathbb{F}_p$-algebra, commutative in the graded sense,
- it is naturally a module over the algebra of stable cohomology operations which is known as the mod p Steenrod algebra and denoted by $\mathcal{A}_p$.

The Steenrod algebra is generated by elements Sq^i of degree $i > 0$ if $p = 2$, β and P^i of degree 1 and $2i(p-1) > 0$ if $p > 2$. These elements satisfy the Adem relations.

In the mod-2 case the relations write

$$Sq^a Sq^b = \sum_0^{[a/2]} \binom{b-t-1}{a-2t} Sq^{a+b-t} Sq^t.$$

There are two types of relations for $p > 2$

$$P^a P^b = \sum_0^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$

Received August 3, 2012.

1991 *Mathematics Subject Classification.* 55S10, 55S35.

Key words and phrases. unstable modules, unstable algebras, Postnikov systems.

The first author is supported by the SFB/TR45 and the ERC Advanced Grant 226257, the second author by the DFG Emmy Noether-Nachwuchsgruppe “Arithmetik über endlich erzeugten Körpern”.

The second author is supported by the LIA Formath Vietnam and the UMR 7539 CNRS, Université Paris 13 .

for $a, b > 0$, and

$$P^a \beta P^b = \sum_0^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t} P^t + \sum_0^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t} \beta P^t$$

for $a, b > 0$.

An easy consequence of the relations is

Theorem 1.1. *The elements Sq^{2^i} if $p = 2$; and β and P^{p^i} if $p > 2$ form a minimal set of multiplicative generators.*

There is more structure. The cohomology is an unstable module, which means that for any cohomology class x

- $Sq^i(x) = 0$ if $i > |x|$ if $p = 2$,
- $\beta^\epsilon P^i(x) = 0$ if $\epsilon + 2i > |x|$, if $p > 2$.

One denotes by $\mathcal{U}$ the abelian category of unstable modules.

As an example consider $H^*B\mathbb{Z}/2 \cong \mathbb{F}_2[u]$, $|u| = 1$; $H^*B\mathbb{Z}/p \cong E(t) \otimes \mathbb{F}_p[x]$, $|t| = 1$ and $|x| = 2$.

One has $Sq^1(u) = u^2$; resp. $\beta(t) = u$ and $P^1(x) = x^p$.

These relations, the Cartan formula that gives the action on products, the restriction axiom which tells that

- $Sq^d = x^2$ if $|x| = d$ $(p = 2)$;
- $P^i x = x^p$ if $|x| = 2i$ $(p > 2)$

and the instability completely determine the action.

The definition of the suspension of an unstable module is central in the theory. This is motivated by the suspension theorem for the cohomology of ΣX:

$$\tilde{H}^* \Sigma X \cong \Sigma \tilde{H}^* X$$

with ΣM defined by:

- $(\Sigma M)^n \cong M^{n-1}$,
- for any $\theta \in \mathcal{A}_p$, $\theta(\Sigma x) = \Sigma \theta(x)$

or

$$\Sigma M \cong M \otimes \Sigma \mathbb{F}_p.$$

The category of algebras that are unstable modules, and such that the above properties relating the two structures hold is called the category of unstable algebras and denoted by $\mathcal{K}$.

It is a very classical question in homotopy theory to ask whether or not a certain unstable $\mathcal{A}_p$-algebra K is the mod p cohomology of a space. If K is the cohomology of a space one also would like to read off the structure of the cohomology some homotopical properties of the associated space. This could be turned -using the Kan-Thurston theorem- into a (discrete) group theoretic question, however very little can be said from this point of view.

Characterising unstable modules or algebras that are the cohomology of a space is out of reach in full generality. But there are interesting types of questions which are accessible:

- Questions about specific modules. One gives two very classical examples in the second section and another one later.
- Necessary conditions for a module to be possibly the cohomology of a space, this will be done in the last sections.

There are two ways (at least) to get results on these questions. The first one is to use deeper structure in singular cohomology, for example secondary operations which give factorisations of primary operations, or to use extraordinary cohomology theories.

A second approach is to consider mapping spaces. More precisely let K be an unstable module, assume it is the reduced cohomology of a space X. Then consider the mapping spaces $\mathrm{map}(S, X)$, pointed or not, and get restrictions by looking at the cohomology of the mapping space. One option for the source space S is to choose S^n. In this case one considers the space of pointed maps. If $n = 1$, as tool to compute the cohomology of ΩX, one has at hand the Eilenberg-Moore spectral sequence. More generally for any n there is a generalisation of the previous one induced by the Goodwillie-Arone tower. The first case is studied in [25], the second one in [18]. A short description of results is given at the end of the next section.

Another option for S is the classifying space $B\mathbb{Z}/p$. In this case a theorem of Jean Lannes computes the cohomology of the mapping space. This will be described in more details in the last sections.

This talk was presented at the Vietnam Institute of Advanced Studies in Mathematics (VIASM) yearly meeting on August 2012. The second author thanks the VIASM, and in particular Ngo Bao Chau and Le Tuan Hoa, as well as the staff of the Institute, for their invitation and warm hospitality.

2. The Hopf invariant and the Kervaire invariant

The Hopf invariant 1 problem is a very famous example of the following question. Let f be a map $S^{2n-1} \to S^n$, consider the cone C_f (homotopy cofiber). Here are two examples: the self-map of S^1, $z \mapsto z^2$ whose cone is $\mathbb{R}P^2$, the Hopf map $S^3 \to S^3/S^1 \cong S^2$ whose cone is $\mathbb{C}P^2$.

The reduced cohomology of C_f is of dimension 1 in degree n and $2n$, trivial elsewhere. Denote by g_n (resp. g_{2n}) a generator in degree n (resp. $2n$). The Hopf invariant of f is defined (up to a sign) by the equation

$$g_n^2 = H(f) g_{2n}.$$

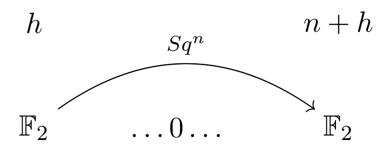
One can work either with integral cohomology or with mod-p cohomology. If n is odd one works with mod-2 cohomology.

The two examples above have both Hopf invariant 1. The question is to decide for which values of n $H(f)$ can take the value 1.

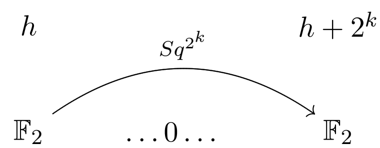
Consider the case of mod-2 singular cohomology. The restriction axiom allows to rewrite the equation as

$$Sq^n g_n = H(f) g_{2n}.$$

So one can reformulate the Hopf invariant 1 question as follows. Let h be a given integer, suspend the map $h - n$ times. Does there exists a 2-cells space, with one cell in degree h a second one in degree $n + h$ related by the operation Sq^n



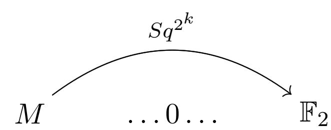
Doing that one modifies the question by going to the stable homotopy world. Because of 1.1 for such a complex to exist n must be a power of 2. So the problem reduces to look at complexes as:



This is a non trivial element in the group $\mathrm{E}xt^{1,2^k}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ which is denoted by h_k. If one works with the Adams spectral sequence for spheres the question is which elements h_k of the first line of the E_2-term are infinite cycles.

The problem was solved by John Frank Adams using secondary operations in mod 2 cohomology in a celebrated paper [1], the only values of k for which this holds are $0, 1, 2, 3$. The case of an odd prime was done by Arunas Liulevicius. Later Adams and Michael Atiyah gave an "unstable proof" based on Adams operations in K-theory [2]. This proof works for $p = 2$ and also for $p > 2$ but is slightly more difficult in this last case. Some interesting generalisations are to be found in [13].

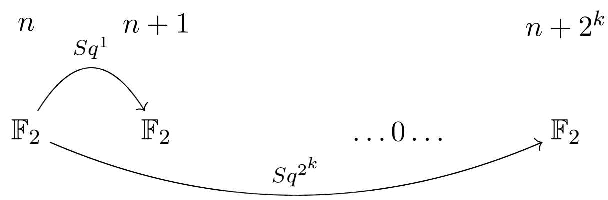
In fact Adams theorem is stronger, it applies (for example) to modules which look like



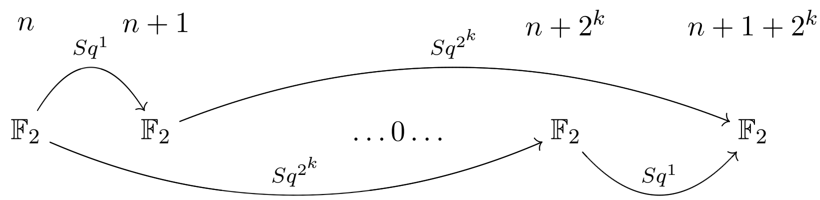
where M in an unstable module which is possibly non trivial only between degrees h and $h + \ell$, and ℓ is small compared to 2^k. This shows that it cannot be the cohomology of a space. But, there is a very important restriction: it is that the operation Sq^1 acts trivially.

The Hopf invariant 1 problem is strongly linked to geometry: for which values of k does there exist a Lie group structure (or a somewhat weaker structure, *e.g.* H-space structure) on the sphere S^k? Outside of S^0 the sphere needs to be of odd dimension by elementary differential geometry. The answer is that the only possible values are 1, 3, 7.

The Kervaire invariant 1 problem is a case not covered by Adams' theorem. It is (roughly) equivalent to the existence of complexes as shown below. They are not in the range of application of Adams' theorem because of the action of Sq^1



or



The geometric question is the following. Given a stably framed manifold the Kervaire invariant is an obstruction to do surgery and to make it cobordant (as stably framed manifold) to a sphere with an exotic differentiable structure. William Browder [3] shows this is equivalent to decide whether or not h_j^2 on the second line of the E_2-term of the Adams spectral sequence for spheres is an infinite cycle.

Thus, such examples of manifolds can only occur in dimension $2^j - 2$; examples of stably framed manifolds that are not cobordant to exotic spheres are $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ with the framing induced by the Lie group structure, or the octonions.

Such complexes are known to exist if $k = 0, 1, 2, 3, 4, 5, 6$. They do not exist if $n > 7$ after the recent (stable) work of Michael Hill, Michael Hopkins and Douglas Ravenel [15]. Their proof depends on a $Z/8Z$-equivariant cohomology theory linked to a Lubin-Tate formal $\mathbb{Z}_2[\zeta_8]$-module, $\mathbb{Z}_2$ being the 2-adic integers, ζ_8 a primitive 8-root of unity and $\pi = \zeta_8 - 1$, whose logarithm writes:

$$\log_F(x) = x + \sum_{k>0} \frac{x^{2^k}}{\pi^k}.$$

The case $n = 7$ remains unsolved.

Let M_1 and M_2 be two unstable modules. Assume M_i is the reduced cohomology of a space X_i, and that one is given a map $f\colon \Sigma^k X_2 \to X_1$ that induces the trivial map in cohomology. Then the long exact sequence splits and the cohomology of the cone of f is an element in $\mathrm{E}xt^1_{\mathcal{U}}(M_1, \Sigma^{k+1} M_2)$. The most famous examples have been described above as $\tilde{H}^*\mathbb{R}P^2$ and $\tilde{H}^*\mathbb{C}P^2$.

This construction can be generalised as follows. Suppose given a map $f\colon X_2 \to X_1$ and assume it can be factored as a composition of n-maps g_i, $1 \le i \le n$, inducing the trivial map in reduced cohomology. One says that f has Adams filtration at least n. Splicing together the extensions obtained from the maps g_i

one gets an element in $Ext^n_{\mathcal{U}}(\tilde{H}^* X_1, \tilde{\Sigma}^n H^* X_2)$. This is a way to construct the Adams spectral sequence.

The preceding examples, as well as others, give evidences for the "Local Realisation Conjecture" (LCR) done in a slightly more restricted form by Nick Kuhn [17].

Conjecture 2.1. *Let M_1 and M_2 be two finite unstable modules. Let k be an integer that is large enough. Then, for any h, any non-trivial extension*

$$E \in \mathrm{Ext}^1_{\mathcal{U}}(\Sigma^h M_1, \Sigma^{h+k} M_2)$$

is not the cohomology of a space.

As such elements are common for h large enough this implies the existence of differentials in the Adams spectral sequence computing homotopy classes of stable maps from X_2 to X_1.

Consider a certain unstable module M, assume it is the reduced cohomology of a space X, and then consider mapping spaces $\mathrm{map}(S, X)$, may be pointed. One can get restrictions on M by looking at the the cohomology of the mapping space. One option for the space S is to choose S^n. In this case one considers the space of pointed maps. If $n = 1$ one uses the Eilenberg-Moore spectral sequence to evaluate the cohomology of the space of pointed loops. More generally for any n one uses the generalisation induced by the Goodwillie-Arone tower. The first case is studied in [25], the second one in [18]. It is possible to use these tools because of their nice behaviour with respect to the action of the Steenrod algebra.

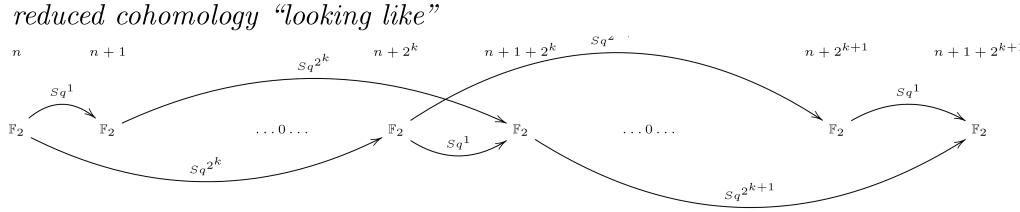
For example the Eilenberg-Moore spectral sequence has E_2-term

$$\mathrm{T}or^{-i,*}_{H^*X}(\mathbb{F}_p, \mathbb{F}_p)$$

which as $\mathcal{A}_p$-module is well understood. One gets contradiction by showing that certain Adem relations cannot be satisfied in the cohomology of the (iterated)-loop space.

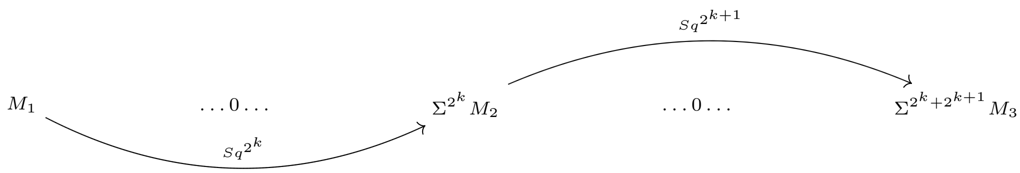
One will not describe the results in details here, but here are some:

Theorem 2.2. *Let k be large enough. There does not exist a complex having as reduced cohomology "looking like"*



More generally, one would like to have a result as described informally below (at $p = 2$):

Conjecture 2.3. *Let M_1, M_2, M_3 be given finite modules. Let k be large enough. As soon as there exists $x \in M_1$ such that $Sq^{2^{k+1}} Sq^{2^k} x \neq 0$ there does not exist a* *complex which has as reduced cohomology "looking like" the following:*



A lot of special cases are known, and the result looks to be within to reach.

One can also consider the space of all maps with $S = B\mathbb{Z}/p$. In this case the Bousfield-Kan spectral sequence for the cohomology of the mapping space degenerates because of the properties of $H^*B\mathbb{Z}/p$ as an object of the categories $\mathcal{U}$ and $\mathcal{K}$. This is what one is going to do, and show how information about the algebraic structure of the category $\mathcal{U}$ allows to get substantial results.

The results obtained using this approach are of a more qualitative nature. As an example the following was conjectured by Kuhn [17] .

Theorem 2.4 (G. Gaudens, L. Schwartz)**.** *Let X be a space such that H^*X is finitely generated as an $\mathcal{A}_p$-module. Then H^*X is finite.*

This result is also a consequence of 3.4 which is going to be given in the next section.

In the next section one describes m-cones and discuss some results that motivates interest for spaces so that any element in $\tilde{H}^*X$ is nilpotent (one will say by abuse that the cohomology is nilpotent). In section 4 one describes a first filtration of the category $\mathcal{U}$, a second one is described in section 5.

3. m-cones and finite Postnikov systems

This section is a short digression before the description of the algebraic structure of the category $\mathcal{U}$. It also gives some motivations to consider spaces with nilpotent cohomology. It shows that such spaces naturally occur in topology as m-cones. In [10] Yves Félix, Stephen Halperin, Jean-Michel Lemaire, Jean-Claude Thomas get some informations about the topological structure by looking at the homology of the loop space. This uses rather different techniques than the ones considered later in this report.

There are two, dual to some extent, ways to construct spaces in homotopy theory. The first one is by attaching cells. One says that a space is 0-cone if it is contractible, an m-cone, $m \geq 1$, is the homotopy cofiber (the cone) of any continuous map from a space A to an $(m-1)$-cone, an m-cone has cup-length less than $m+1$. This means that any $(m+1)$-fold product of cohomology classes of positive degree is trivial. In particular any element of positive degree is nilpotent. The following theorem [10] gives restrictions on the cohomology of an n-cone.

Theorem 3.1 (Y. Félix, S. Halperin, J.-M. Lemaire, J.-C.Claude Thomas)**.** *If X is 1-connected and the homology is finite dimensional in each degree then*

$$depth(H_*(\Omega X; \mathbb{F}_p)) \leq cat(X).$$

The depth of a graded connected k-algebra R (possibly infinity) is the largest n such that $Ext^i_R(k, R) = \{0\}$, $i < n$, $cat(X)$ denotes the Lusternick-Schnirelman category of X. This is the minimum number of elements of covering of X by contractible subspaces.

Here is the second way to construct spaces: a 1-Postnikov system or GEM (generalized Eilenberg-Mac Lane space) is a product (may be infinite) of usual Eilenberg-Mac Lane $K(\pi, n)$-spaces. An m-Postnikov system is the homotopy fiber of an $(m-1)$-Postnikov system into a GEM. The m-th Postnikov tower $P_m(X)$ of a space X is a particular case of an m-system.

Corollary 3.2. *Let X be a 1-connected m-cone, assume that the cohomology is finite dimensional in any degree. Then the p-localisation of X is never a finite p-local Postnikov system.*

This is to be compared with [20]:

Theorem 3.3 (J. Lannes, L. Schwartz)**.** *Let $P_n(X)$ be a 1-connected n-Postnikov tower such that $H^*P_n(X)$ is non-trivial. Then the reduced cohomology $\tilde{H}^*P_nX$ contains a non nilpotent element.*

A finite 1-connected Postnikov tower is never an m-cone, because the cup length of an m-cone is bounded by $m+1$. Nevertheless, Jiang Dong Hua [14] has shown there exists a 3-stage Postnikov system with nilpotent cohomology.

The following result applies to m-cones and more generally to spaces with nilpotent cohomogy. The proof uses the Krull filtration on the category $\mathcal{U}$ (see section 4). Here $Q(K) = \mathrm{Ker}(\epsilon)/\mathrm{Ker}(\epsilon)^2$ denotes as usual the indecomposable functor of an augmented unstable algebra $\epsilon\colon K \to \mathbb{F}_p$.

Theorem 3.4 (G. Gaudens, Nguyen T. Cuong, L. Schwartz)**.** *Let X be an m-cone, for some m. If $QH^*X \in \mathcal{U}_n$, then $QH^*X \in \mathcal{U}_0$.*

The proof depends on the algebraic structure of the category $\mathcal{U}$, and as said above, on the cohomology of mapping spaces.

4. The Krull filtration on $\mathcal{U}$

The category of unstable modules $\mathcal{U}$, as any abelian category, has a natural filtration: the Krull filtration, by thick subcategories stable under colimits

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots \subset \mathcal{U}.$$

Because of the degree filtration on any unstable module the simple objects in $\mathcal{U}$ are the modules $\Sigma^n\mathbb{F}_p$.

The subcategory $\mathcal{U}_0$ is the largest thick sub-category generated by simple objects and stable under colimits. It is the subcategory of locally finite modules. An unstable module is locally finite if the span over $\mathcal{A}_p$ of any $x \in M$ is finite. Indeed this last condition is obviously satisfied when taking extensions and is preserved by colimits.

Having defined by induction $\mathcal{U}_n$ one defines $\mathcal{U}_{n+1}$ as follows. One first introduces the quotient category $\mathcal{U}/\mathcal{U}_n$ whose objects are the same of those of $\mathcal{U}$ but where morphisms in $\mathcal{U}$ that have kernel and cokernel in $\mathcal{U}_0$ are formally inverted. Then $(\mathcal{U}/\mathcal{U}_n)_0$ is defined as above and $\mathcal{U}_{n+1}$ is the pre-image of this subcategory in $\mathcal{U}$ *via* the canonical projection functor. As said above this construction works for any abelian category. One refers to [11] for details. This induces a filtration on any unstable module M, one has [24]:

Theorem 4.1. *Let $M \in \mathcal{U}$ and $K_n(M)$ be the largest sub-object of M that is in $\mathcal{U}_n$, then*

$$M = \cup_n K_n(M).$$

As examples one has

- $\Sigma^k F(n) \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, the unstable modules $F(n)$ are the canonical generators of $\mathcal{U}$, generated in degree n by ι_n and $\mathbb{F}_p$-basis $P^I \iota_n$, I an admissible multi-index of excess less than n;
- $H^*B\mathbb{Z}/2 \cong \mathbb{F}_2[u]$, does not belong to $\mathcal{U}_n$ for any n but,
- $H^*B\mathbb{Z}/2$ is a Hopf algebra and the n-th step of the primitive filtration $P_nH^*B\mathbb{Z}/2$ is in $\mathcal{U}_n$. The analogous result holds for $H^*\mathbb{Z}/p$.

There is a characterisation of the Krull filtration in terms of a functor introduced by Lannes and denoted by T.

Definition 4.2. The functor $T\colon \mathcal{U} \to \mathcal{U}$ is left adjoint to the functor $M \mapsto H^*B\mathbb{Z}/p \otimes M$. As the unstable module splits up as the direct sum $\mathbb{F}_p \oplus \tilde{H}^*B\mathbb{Z}/p$. The functor T is isomorphic to the direct sum of the identity functor and of the functor $\bar{T}$ left adjoint of $M \mapsto \tilde{H}^*B\mathbb{Z}/p \otimes M$.

Direct computation shows that $T(\Sigma^n\mathbb{F}_p)$ is isomorphic to $\Sigma^n\mathbb{F}_p$. The functor T has wonderful properties. It generalizes directly to a functor T_V, V an elementary abelian p-group, left adjoint to $M \mapsto H^*BV \otimes M$, $T_V \cong T^{dim(V)}$. Below are the main properties of T_V, [19], [24].

Theorem 4.3 (J. Lannes)**.** *The functor T_V commutes with colimits (as a left adjoint). It is exact. Moreover there is a canonical isomorphism*

$$T_V(M_1 \otimes M_2) \cong T_V(M_1) \otimes T_V(M_2).$$

A special case of the last property, if $M_1 = \Sigma\mathbb{F}_p$ writes as

$$T_V(\Sigma M) \cong \Sigma T_V(M).$$

One gets [24]

Theorem 4.4. *The following two conditions are equivalent:*

- $M \in \mathcal{U}_n$,
- $\bar{T}^{n+1}(M) = \{0\}$.

It follows that

Corollary 4.5. *If $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$ then $M \otimes N \in \mathcal{U}_{m+n}$.*

There is also a characterisation of objects in $\mathcal{U}_n$ of combinatorial nature for $p = 2$ [27]:

Theorem 4.6. *A finitely generated unstable $\mathcal{A}_2$-module M is in $\mathcal{U}_n$ if and only if its Poincaré series $\Sigma_n a_n t^n$ has the following property. There exists an integer k so that the coefficient a_d is possibly non trivial only for those values of d such that if $\alpha(d-i) \leq n$, for some $0 \leq i \leq k$.*

In this statement $\alpha(k)$ is the number of 1 in the 2-adic expansion of k. This holds in case of finite dimension in any degree. A similar statement holds for $p > 2$.

Let $\mathcal{F}$ be the category of functors from finite dimensional $\mathbb{F}_p$-vector spaces to all vector spaces. Define ([16]), a functor $f\colon \mathcal{U} \to \mathcal{F}$ by (here V is a finite dimensional $\mathbb{F}_p$-vector space)

$$f(M)(V) = \mathrm{H}om_{\mathcal{U}}(M, H^*(BV))^* \cong T_V(M)^0.$$

Let $\mathcal{F}_n$ be the sub-category of polynomial functors of degree less than n. It is defined as follows. Let $F \in \mathcal{F}$, let $\Delta(F) \in \mathcal{F}$ defined by

$$\Delta(F)(V) = \mathrm{Ker}(F(V \oplus \mathbb{F}_p) \to F(V)).$$

Then by definition $F \in \mathcal{F}_n$ if and only if $\Delta^{n+1}(F) = 0$. As an example $V \mapsto V^{\otimes n}$ is in $\mathcal{F}_n$. The followwing holds for any M :

$$\Delta(f(M)) \cong f(\bar{T}(M)).$$

Thus, the diagram commutes:

$$\begin{array}{ccccccc}
\mathcal{U}_0 & \cdots & \mathcal{U}_{n-1} & \hookrightarrow & \mathcal{U}_n & \hookrightarrow & \mathcal{U} \\
\downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\
\mathcal{F}_0 & \cdots & \mathcal{F}_{n-1} & \hookrightarrow & \mathcal{F}_n & \hookrightarrow & F
\end{array}$$

One can now state a conjecture of Kuhn and give results. The first conjecture is as follows.

Theorem 4.7 (Gaudens, Schwartz)**.** *Let X be a space such that $H^*X \in \mathcal{U}_n$ then $H^*X \in \mathcal{U}_0$.*

As said above the following corollary was also conjectured by Kuhn.

Corollary 4.8. *Let X be a space. If H^*X has finitely many generators as unstable module it is finite.*

Indeed, if an unstable module has finitely many generators it is in $\mathcal{U}_n$ for some n, because it is a quotient of a finite direct sum of $F(k)'s$. Then by 4.7 it is in $\mathcal{U}_0$. But a finitely generated locally finite unstable module is finite.

In [17] Kuhn proved the corollary under additional hypothesis, using the Hopf invariant 1 theorem. One key step is a reduction depends on Lannes' mapping space theorem which is going to be described in section 6. In [25] the corollary is proved for $p = 2$ using the Eilenberg-Moore spectral sequence, the argument is claimed to extend to all primes. However it is observed that one has to take care of a differential d_{p-1} in the Eilenberg-Moore spectral sequence. As Gaudens observed the method of [25] does not work without some more hypothesis, alike the triviality of the Bockstein homomorphism, see [28].

For $p = 2$ in [18] Kuhn gives a proof depending on the Goodwillie-Arone spectral sequence. S. Büscher F. Hebestreit, O. Rondig, and M. Stelzer get partial results for $p > 2$, [4].

The theorem is proved now using only the Bott-Samelson theorem and Lannes' mapping space theorem [12].

5. The nilpotent filtration

Above one has considered spaces so that any element in $\tilde{H}^*X$ is nilpotent and introduced the terminology "nilpotent" for the cohomology. The restriction axiom allows to express this in terms of the action of the Steenrod algebra. More precisely (for $p = 2$) it is equivalent to ask that the operation $Sq_0\colon x \mapsto Sq^{|x|}x$ is "nilpotent" on any element. It makes possible to extend this definition to any unstable module.

Definition 5.1. One says that an unstable module M is nilpotent if for any $x \in M$ there exists k such that $Sq_0^k x = 0$.

In particular a nilpotent module is 0-connected. A suspension is nilpotent. In fact one has the following:

Proposition 5.2. *An unstable module M is nilpotent if and only if it is the colimit of unstable modules which have a finite filtration whose quotients are suspensions.*

This allows to extend the definition for $p > 2$.

More generally one can define a filtration on $\mathcal{U}$. It is filtered by subcategories $\mathcal{N}il_s$, $s \geq 0$, $\mathcal{N}il_s$ is the smallest thick subcategory stable under colimits and containing all s-suspensions.

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \ldots \supset \mathcal{N}il_s \supset \ldots$$

By the very definition *any $M \in \mathcal{N}il_s$ is $(s-1)$-connected.*.

Proposition 5.3. *Any M has a convergent decreasing filtration $\{M_s\}_{s\geq 0}$ with $M_s/M_{s+1} \cong \Sigma^s R_s(M)$ where $R_s(M)$ is a reduced unstable module, i.e. does not contain a non trivial suspension.*

Only the second part of the proposition needs a small argument see [24]. The following results are easy consequences of the commutation of T with suspension, the definition, and of 4.3. Just the last part needs a small amount of additional care because T does not commute with limits.

Proposition 5.4. *One has*

- *if $M \in \mathcal{N}il_m$, $N \in \mathcal{N}il_n$ then $M \otimes N \in \mathcal{N}il_{m+n}$;*
- *if $M \in \mathcal{N}il_m$ then $T(M) \in \mathcal{N}il_m$,*
- *$M \in \mathcal{U}_n$ if and only if for any s $f(R_s(M)) \in \mathcal{F}_n$.*

Proposition 5.5. *The indecomposable elements of an augmented unstable algebra are in* $\mathcal{N}il_1$.

This is a suspension if $p = 2$.

Following Kuhn, for an unstable module M, one defines a function $w_M : \mathbb{N} \to \mathbb{Z} \cup \infty$ by:

$$w_M(i) = \deg f(R_i(M)).$$

The following lemma is a consequence of Corollary 4.5 and Proposition 5.4:

Lemma 5.6. $M \in \mathcal{N}il_s \Rightarrow T(M) \in \mathcal{N}il_s$.

Let M be such that $w_M(i) \leq i$, $w_{\mathbb{T}(M)}$ the tensor algebra on M. Then the function $w_{\mathbb{T}(M)}$ has the same property.

Below are two statements that imply Theorems 3.4 and 4.7

Let X be a space, define $w_X = w_{H^*X}$ and $q_X = w_{QH^*X}$.

Theorem 5.7 (Gaudens, Nguyen T. Cuong, Schwartz)**.** *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function q_X either is equal to 0 or $q_X - Id$ takes at least one positive (non zero) value.*

Theorem 5.8 (Gaudens, Schwartz)**.** *Let X be such that $\tilde{H}^*X \in \mathcal{N}il_1$. The function w_X either is equal to 0 or $w_X - Id$ takes arbitrary large values.*

One would like to reformulate Theorem 5.7 like Theorem 5.8 but there are some (presumably small) difficulties.

6. Lannes' theorem and Kuhn's reduction, proof of Theorems 5.7 and 5.8

Given X p-complete, 1-connected, assume that TH^*X is finite dimensional in each degree. The following theorem of Lannes is the major geometrical application of 4.3. The evaluation map:

$$B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, X) \to X$$

induces a map in cohomology:

$$H^*X \to H^*B\mathbb{Z}/p \otimes H^*map(B\mathbb{Z}/p, X)$$

and by adjunction

$$TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X).$$

Theorem 6.1 (J. Lannes)**.** *Under the hypothesis mentioned above the natural map $TH^*X \to H^*\mathrm{map}(B\mathbb{Z}/p, X)$ is an isomorphism of unstable algebras.*

Following François Xavier Dehon and Gaudens these conditions could be relaxed using Morel's machinery of pro-p spaces [23].

This theorem extends replacing $B\mathbb{Z}/p$ by BV, V an elementary abelian p-group. It is linked to the Sullivan conjecture and has a lot of applications, in particular in the theory of p-compact groups (William Dwyer and Clarence Wilkerson [8]) and the one of p-local groups or Bob Oliver [5]. In particular it leads to a complete solution of Steenrod's problem. The question is to determine which polynomial algebras are the cohomology of a space. One famous example is the algebra of modular invariants. Let V be an elementary abelian 2-group and $d = dim(V)$. They are the elements invariant under the action of $\mathbb{GL}(V)$ in

$$S^*(V^*) \cong H^*BV \cong \mathbb{F}_2[x_1, \dots, x_d].$$

It is known (Dickson) to be polynomial in generators of degree $2^d - 2^{d-1}, \dots, 2^d - 1$. This is the cohomology of a space if and only if $d \leq 4$, the case $d = 4$ is "exotic" [9], all this would deserve other reports and one will not proceed further.

Kuhn considers the homotopy cofiber $\Delta(X)$ of the natural map$X \to \mathrm{map}(B\mathbb{Z}/p, X)$. Reduction is to consider the cofiber $\Delta(X)$ of $X \to \mathrm{map}(B\mathbb{Z}/p, X)$. Then Theorem 6.1 immediately yields:

Proposition 6.2.

$$H^*(\Delta(X)) \cong \bar{T}H^*X,$$
$$w_{\Delta(X)} = w_X - 1.$$

As a consequence, if $H^*X \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, then $H^*\Delta(X) \in \mathcal{U}_{n-1} \setminus \mathcal{U}_{n-2}$.

Given an augmented unstable algebra K the indecomposable functor Q does commute with T:

$$T(Q(K)) \cong Q(TK)$$

but this is not true with $\bar{T}$. However if Z is an H-space, then [6].

Proposition 6.3. $QH^*\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z) = \bar{T}^n QH^*Z$.

This follows from the homotopy equivalence:

$$\mathrm{map}(B\mathbb{Z}/p, Z) \cong Z \times \mathrm{map}_*(B\mathbb{Z}/p, Z).$$

On the way one notes the beautiful result of these authors.

Theorem 6.4 (N. Castellana, C. Crespo, J. Scherer)**.** *Let X be an H-space such that $QH^*X \in \mathcal{U}_n$, then $QH^*\Omega X \in \mathcal{U}_{n-1}$.*

In order to prove 5.7 or 5.8 one shows a space cannot be such that $\tilde{H}^*X \in \mathcal{N}il_1$ and such that q_X is not 0 and less or equal to Id. Kuhn's reduction allows us to suppose that the reduced mod-p cohomology is exactly s-nilpotent, $s > 0$ and that $R_s(H^*X) \in \mathcal{U}_1 \setminus \mathcal{U}_0$.

Let Z be $\Omega\Sigma X$, then $H^*Z \cong \mathbb{T}(\tilde{H}^*X)$, recall that $\mathbb{T}()$ denotes the tensor algebra.

The first part of the proof consists of the following chain of implications:

- $q_X \leq Id \Rightarrow w_X \leq Id$, in fact this holds for any unstable algebra K so that $K^{>0}$ is nilpotent;
- Kuhn's reduction plus the collapse of a low dimensional skeleton allows to suppose that the reduced mod-p cohomology is exactly s-nilpotent, $s > 0$ and that $R_s(H^*X) \in \mathcal{U}_1 \setminus \mathcal{U}_0$, and that $w_X(s+i) \leq i+1$.
- Let Z be $\Omega\Sigma X$, then $H^*Z \cong \mathbb{T}(\tilde{H}^*X)$, the condition above implies for algebraic reasons that $\bar{T}^n H^*Z$ is $(n+s-2)$-connected, it follows from Proposition 6.3 that $map_*(B^{\wedge n}, Z)$ is $(n+s-2)$-connected.

Proposition 6.5. *$\tilde{H}^*Z \in \mathcal{N}il_s$, $\bar{T}^n(H^*Z)$ is $(n+s-2)$-connected, thus $\mathrm{map}_*(B^{\wedge n}, Z)$ is $(n+s-2)$-connected.*

Then, one gets a non trivial algebraic map of unstable algebras

$$\varphi_s^* : H^*Z \to \Sigma^s R_s H^*Z \to \Sigma^s F(1) \subset \Sigma^s \tilde{H}^* B\mathbb{Z}/p \ .$$

It cannot factor through $H^*\Sigma^{s-1}K(\mathbb{Z}/p,2)$, because there are no non trivial map from an s-suspension (and thus from an unstable module in $\mathcal{N}il_s$) to an $(s-1)$-suspension of a reduced module, and

Proposition 6.6. *$H^*K(\mathbb{Z}/p,2)$ is reduced.*

Then comes the second part of the proof, the contradiction comes from the fact that using obstruction theory one can construct a factorisation.

The existence of a map realising φ_s^* is a consequence (using Lannes' theorem) and of the Hurewicz theorem because $\mathrm{map}_*(B\mathbb{Z}/p, Z)$ is $(s-1)$-connected.

$K(\mathbb{Z}/p,2)$ is built up, starting with $\Sigma B\mathbb{Z}/p$, as follows (Milnor's construction). There are a filtration $* = C_0 \subset C_1 = \Sigma B\mathbb{Z}/p \subset C_2 \subset \ldots \subset \cup_n C_n = K(\mathbb{Z}/p,2)$, a diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & B^{*n+1} & \longrightarrow & B^{*n+2} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \cdots
\end{array}$$

and cofibrations, up to homotopy

$$\Sigma^{n-1}B^{\wedge n} \to C_{n-1} \to C_n$$

$$\Sigma^{n-2+s}B^{\wedge n} \to \Sigma^{s-1}C_{n-1} \to \Sigma^{s-1}C_n.$$

The obstructions to extend $\varphi_s \colon \Sigma^s B\mathbb{Z}/p \to Z$ to $\Sigma^{s-1}K(\mathbb{Z}/p,2)$ are in the groups

$$[\Sigma^{n+s-2}(B\mathbb{Z}/p)^{\wedge n}, Z] = \pi_{n+s-2}\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$$

but they are trivial. It follows that one can do the extension, this is a contradiction.

To prove 5.8 it is now enough to use again $w_{\Delta(X)} = w_X - 1$.

References

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, *Ann. of Math.* **72**(1) (1960), 20-104.

[2] J. F. Adams and M. Atiyah, K-theory and the Hopf invariant, *Quart. J. Math. Oxford Ser.* **17**(1) (1966), 31-38.

[3] W. Browder, The Kervaire invariant of framed manifolds and its generalization, *Ann. of Math.* **90** (1969), 157-186.

[4] S. Büscher, F. Hebestreit, O. Röndigs and M. Stelzer, *The Arone-Goodwillie spectral sequence for $\Sigma^\infty\Omega^n$ and topological realization at odd primes*, arXiv:1110.2370 [math.AT].

[5] C. Broto, R. Levi and B. Oliver, The homotopy theory of fusion systems, *J. Amer. Math. Soc.* **16**(4) (2003), 779-856.

[6] N. Castellana, J. A. Crespo, and J. Scherer, Deconstructing Hopf spaces, *Invent. Math.* **167**(1), (2007), 1-18.

[7] W. Dwyer and C. Wilkerson, Spaces of null homotopic maps, *Proc. Luminy*, 1988, Astérisque 191 (1990), 97-108.
[8] W. Dwyer and C. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, *Ann. of Math.* (2) **139** (1994), 395-442.
[9] W. Dwyer and C. Wilkerson, A new finite loop space at the prime two, *J. Amer. Math. Soc.* **6** (1993), 37-64.
[10] Y. Félix, S. Halperin, J. M. Lemaire and J. C. Thomas, Mod p loop space homology, *Invent. Math.* **95** (1989), 247-262.
[11] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323-448.
[12] G. Gaudens and L. Schwartz, Applications depuis $K(\mathbb{Z}/p, 2)$ et une conjecture de N. Kuhn, *Annales Institut Fourier* 2012.
[13] P. Hoffman, On the realizibility of Steenrod modules, *Quart. J. Math. Oxford Ser.* (2) **20** (1969), 403-407.
[14] J. D. Hua, Un 3-polyGEM de cohomologie modulo 2 nilpotente, Tome 54, Annales Institut Fourier, 2004.
[15] M. A. Hill, M. J. Hopkins and D. C. Ravenel, On the non-existence of elements of Kervaire invariant one http://arxiv.org/abs/0908.3724.
[16] H. W. Henn, J. Lannes, and L. Schwartz, The categories of unstable modules and unstable algebra over the Steenrod algebra modulo its nilpotent objects, *Amer. J. Math.* **115**(5) (1993), 1053-1106.
[17] N. Kuhn, On topologically realizing modules over the Steenrod algebra, *Ann. Math.* **141** (1995), 321-347.
[18] N. Kuhn, Topological non-realisation results via the Goodwillie tower approach to iterated loopspace homology, *Algebr. Geom. Topol.* **8** (2008), 2109-2129.
[19] J. Lannes, Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire. *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 135-244.
[20] J. Lannes and L. Schwartz, Sur les groupes d'homotopie des espaces dont la cohomologie modulo 2 est nilpotente, *Israel J. Math.* **66** (1989), 260-273.
[21] J. Lannes and S. Zarati, Sur les $\mathcal{U}$-injectifs, *Ann. Sci. École. Norm. Sup.* **19** (1986), 303-333.
[22] H. Miller, The Sullivan conjecture on maps from classifying spaces, *Ann. of Math.* (2) **120**(1) (1984), 39–87.
[23] F. Morel, Quelques remarques sur la cohomologie modulo p continue des pro-p espaces et les résultats de Jean Lannes sur les espaces fonctionnels $\mathrm{map}(B\mathbb{Z}/p, X)$, *Ann. Sci. Ecole Norm. Sup.* (4) **26**(3) (1993), 309-360.
[24] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lectures in Mathematics, University of Chicago Press, (1994).
[25] L. Schwartz, A propos de la conjecture de non-réalisation due à N. Kuhn, *Invent. Math.* **134** (1998), 211-227.
[26] L. Schwartz, La filtration de Krull de la catégorie $\mathcal{U}$ et la cohomologie des espaces, *Algebr. Geom. Topol.* **1** (2001), 519-548.
[27] L. Schwartz, Sur l'anneau de grothendieck de la catégorie des modules instables, *Comm. Algebra* **34**(5) (2006) 1825-1845.
[28] L. Schwartz, Erratum to: La conjecture de non réalisation due à N. Kuhn, *Invent. Math.* **182**(2) (2010), 449-450.

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