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ON THE TOPOLOGY OF RATIONAL FUNCTIONS IN TWO COMPLEX VARIABLES

NGUYEN TAT THANG

Dedicated to Professor Hà Huy Vui on the occasion of his sixtieth anniversary.

Abstract. We give some characterizations for the critical values at infinity of a rational function in two complex variables in terms of the Euler characteristic, the Malgrange condition and the M-tameness.

1. Introduction

Let F be a rational function in n complex variables. It is well-known that F is a locally trivial fibration outside some finite subsets of $\mathbb{C}$ (see [T1]). The smallest such subset is called the bifurcation value set of F and is denoted by $B(F)$. A natural question is how to compute this set $B(F)$.

We recall the definition of the so-called critical values at infinity (or atypical values) of a rational function.

Definition 1.1. A value $t_0 \in \mathbb{C}$ is called a *regular value at infinity* of F if there is a positive real number $\delta > 0$ and a compact subset $K \subset \mathbb{C}^n$ such that the restriction

$$F : F^{-1}(D_\delta(t_0)) \setminus K \to D_\delta(t_0)$$

is a C^∞-trivial fibration, where $D_\delta(t_0) := \{t \in \mathbb{C} : |t - t_0| < \delta\}$.

If $t_0 \in \mathbb{C}$ is not a regular value at infinity, we call it a *critical value at infinity* (or *atypical value*) of the rational function F. Denote the set of critical values at infinity of F by $B_\infty(F)$.

Obviously $B(F)$ contains the set $K_0(F)$ of the critical values and the set $B_\infty(F)$, $B(F) \supseteq B_\infty(F) \cup K_0(F)$.

The aim of this article is to study the sets $B(F)$ and $B_\infty(F)$ of a rational function F in two complex variables, that is, the case $n = 2$.

Let $f, g \in \mathbb{C}[x, y]$ be two non-zero polynomials without common factors and set $F = f/g$. If $\deg(g) = 0$, or equivalently F is a polynomial function, then one can prove that $B(F) = B_\infty(F) \cup K_0(F)$. Our first result is the following theorem.

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Theorem 1.2. *Assume that* $\deg(f) > \deg(g)$. *We have*

$$B(F) = B_\infty(F) \cup K_0(F) \cup K_1(F).$$

Here $K_1(F)$ is a subset of $\mathbb{C}$ which is defined in Section 2 by using the Milnor number at indeterminant points of F.

Our second result is about the set of critical values at infinity $B_\infty(F)$. In the case of polynomial functions, there have been several interesting characterizations of this set. One of those via Euler characteristic is due to Suzuki [S] and Hà Huy Vui-Lê Dũng Tráng [HL].

Theorem 1.3. ([S], [HL]) *Let F be a polynomial function in two complex variables and $t_0 \in \mathbb{C}$ be a regular value of F. Then $t_0 \in B_\infty(F)$ if and only if the Euler characteristic of the fiber $F^{-1}(t)$ is not a constant in every neighborhood of t_0.*

Another characterization of the set $B_\infty(F)$ is by the Fedoryuk condition, Malgrange condition and the M-tameness. Recall that for a rational function F, we denote by $\widetilde{K}_\infty(F)$ the set of $t \in \mathbb{C}$ such that there exists a sequence $\{x_k\}_k \subset \mathbb{C}^n$, $x_k \to \infty$, such that $F(x_k) \to t$ and $||\mathrm{grad} F(x_k)|| \to 0$. We say that F satisfies *Fedoryuk condition* at a value $t \in \mathbb{C}$ if $t \notin \widetilde{K}_\infty(F)$. If, in addition, we require that $||x_k||.||\mathrm{grad} F(x_k)|| \to 0$, then we get a subset $K_\infty(F) \subseteq \widetilde{K}_\infty(F)$. We say that F satisfies *Malgrange condition* at a value $t \in \mathbb{C}$ if $t \notin K_\infty(F)$.

Let $M_\infty(F)$ denote the set of values $t \in \mathbb{C}$ such that there are sequences $\{\lambda_k\}_k \subset \mathbb{C}$ and $\{x_k\}_k \subset \mathbb{C}^n$ with $x_k \to \infty$, $F(x_k) \to t$ and $\mathrm{grad} F(x_k) = \lambda_k x_k$ for all $k = 0, 1, 2, \ldots$. We say that the rational function F is *M-tame* at a value $t \in \mathbb{C}$ if $t \notin M_\infty(F)$. The notion of M-tame was introduced for polynomial functions by A. Némethi and A. Zaharia (see [NZ1] and [NZ2]). In a recent paper [BP], the authors showed that if f/g is a rational function in two variables then a non-zero value $t_0 \in \mathbb{C}$ belongs to $M_\infty(f/g)$ if and only if outside a large compact set of $\mathbb{C}^2$, the topological type of the curve $(f/g)^{-1}(t)$ is constant for all t near t_0.

For the case of polynomial functions in two variables, we have the following characterizations of the set $B_\infty(F)$ due to Hà Huy Vui and Ishikawa.

Theorem 1.4. ([H], [I]) *Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and $t \in \mathbb{C}$. The following are equivalent:*

(i) $t \in B_\infty(F)$;
(ii) $t \in \widetilde{K}_\infty(F)$;
(iii) $t \in K_\infty(F)$;
(iv) $t \in M_\infty(F)$.

Our second result in this article is a generalization of Theorems 1.3 and 1.4 to the case of rational functions. We will show that under some wild assumptions, the critical values at infinity of rational functions in two variables can be determined in terms of the Euler characteristic, the Malgrange condition and the M-tameness (Theorems 3.8, 3.9 and 3.16). Moreover, we give examples showing that the Fedoryuk condition can not characterize the critical values at infinity of those functions.

The article consists of three sections. Theorem 1.2 is proved in Section 2. Section 3 is devoted to a generalization of Theorems 1.3 and 1.4 for rational functions in two complex variables as mentioned above. The main results of this section are Theorems 3.8, 3.9 and 3.16.

2. The bifurcation set

In this section we give some descriptions for the set of bifurcation values of a rational function in two complex variables.

Let $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ have no common factor. Let

$$A(F) := \{(x, y) \in \mathbb{C}^2 : f(x, y) = g(x, y) = 0\}.$$

For each $t \in \mathbb{C}$ set

$$d_t := \deg(f - tg),$$
$$V_t := \{(x, y) \in \mathbb{C}^2 : f(x, y) - tg(x, y) = 0\},$$
$$G(x, y, z, t) := z^{d_t} f(x/z, y/z) - tz^{d_t} g(x/z, y/z)$$

and

$$\overline{V_t} := \{[x : y : z] \in \mathbb{C}P^2 : G(x, y, z, t) = 0\}.$$

Let $V^t_\infty = \overline{V_t} \cap H_\infty$ be the set of points at infinity of $\overline{V_t}$.

Remark 2.1. (i) $A(F)$ contains finitely many points.

(ii) For all $t \in \mathbb{C}$ we have $A(F) \subset \{f - tg = 0\}$. Moreover, if t_0 is a regular value of F and t is near t_0 enough then every point $p \in A(F)$ is either a regular point or an isolated singular point of the curve V_t.

Definition 2.2. We denote by $K_1(F)$ the set of $t_0 \in \mathbb{C} \setminus K_0(F)$ such that there exists $p \in A(F)$ with $\mu_p(f - t_0g) \neq \mu_p(f - tg)$ for all $t \neq t_0$ near t_0 enough, where $\mu_p(f - tg)$ is the Milnor number of $f - tg$ at p.

Remark 2.3. For each curve $V \subset \mathbb{C}^2$ we denote by Sing$V$ the set of singular points of V. Then $t_0 \notin K_1(F)$ if $\mathrm{Sing}\{f - t_0g = 0\} \cap A(F) = \emptyset$.

Lemma 2.4. *Let* $F := \frac{f}{g} \colon \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ *be a rational function, where* $f, g \in \mathbb{C}[x, y]$ *have no common factor. Then*

$$B(F) \subset K_0(F) \cup B_\infty(F) \cup K_1(F).$$

To prove the lemma, we need the following results.

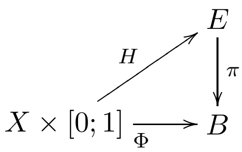
Theorem 2.5. ([T2, LR]) *Let* $g_s \colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), n \neq 3, s \in \mathbb{R}^m$ *be a differential family of holomorphic germs such that for all* s*, the origin* $0 \in \mathbb{C}^n$ *is an isolated singular point of* g_s*. Assume that* $\mu_0(g_s) = \mu_0(g_0)$ *for all* s *near* $0 \in \mathbb{R}^m$ *enough. Then, there exist a neighborhood* D *of* $0 \in \mathbb{R}^m$*, a small ball* B *centered at the origin and a continuous family of homeomorphisms*

$$\Phi_s \colon g_s^{-1}(0) \cap B \to g_0^{-1}(0) \cap B, s \in D$$

such that $\Phi_s(0) = 0$.

Let $h : X \to Y$ be a continuous map. A *homotopy* of h is a continuous map $H : X \times [0;1] \to Y$ such that $H(x;0) = h(x)$ for all $x \in X$.

Definition 2.6. A continuous map $\pi : E \to B$ is called a *fibration*, or equivalently, *has homotopy lifting property*, if for all polytopes X and for any continuous map $h : X \to E$, every homotopy Φ of $\pi \circ h$ can be lifted to a homotopy of h, i.e. there exists a homotopy H of h such that the diagram



commutes.

Definition 2.7. ([M]) Let X, Y be topological spaces. Two homotopies

$$H, H^{'} : X \times [0,1] \to Y$$

are said to *have the same germ* if they coincide in a neighborhood of $X \times \{0\}$.

Definition 2.8. ([M]) A continuous map $\pi : E \to B$ is called a *homotopic submersion*, or equivalently say that it has the *germ-of-homotopy lifting property*, if for every polytope X and every continuous map $h : X \to E$ every germ-of-homotopy of $\pi \circ h$ lifts to a germ-of-homotopy for h.

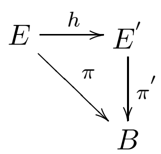
Definition 2.9. ([M]) A continuous map $\pi : E \to B$ is called a *local homotopic submersion* if for every $x \in E$ there is a neighborhood $U(x) \subset E$ such that the restriction $\pi_{|U(x)}$ is a homotopic submersion from $U(x)$ onto $\pi(U(x))$.

Lemma 2.10. ([M], Lemma 6) *Let $\pi : E \to B$ be an open continuous map. Assume that π is a local homotopic submersion. Then π is a homotopic submersion.*

It deduces from Lemma 2.10 that

Lemma 2.11. *Let $f : V_1 \to V_2$ be a differential map. Assume that f is a submersion. Then f is a homotopic submersion.*

Lemma 2.12. ([M]) *In the following commutative diagram of continuous maps*



assume that π and $\pi^{'}$ are surjective homotopic submersions. If $\pi^{'}$ is a fibration and for every $b \in B$, the restriction $h_b := h_{|\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{'-1}(b)$ is a weak homotopy equivalence, then π is a fibration.

Lemma 2.13. ([M], Corollary 32) *Let $\pi : E \to B$ be a differential map such that $\dim_{\mathbb{R}} E = \dim_{\mathbb{R}} B + 2$. Assume that the following are satisfied:*

(i) *π is surjective;*

(ii) *π is a fibration;*
(iii) *π is a submersion.*

Then π is a locally $C^\infty-$trivial fibration.

Proof of Lemma 2.4. Let $t_0 \notin K_0(F) \cup B_\infty(F) \cup K_1(F)$. Without loss of generality, we may assume that $t_0 = 0$.

Since $0 \notin K_1(F)$ there is a neighborhood D_1 of 0 such that

$$\mu_p(f - tg) = \mu_p(f), \forall t \in D_1, p \in A(F).$$

Hence, according to Theorem 2.5, there exists a small neighborhood D_2 of 0 such that for every $p \in A(F)$, there is a ball $B(p)$ centered at p and there is a continuous family of homeomorphisms

$$\Phi_p(s) \colon \{f - sg = 0\} \cap B(p) \to \{f = 0\} \cap B(p), s \in D_2$$

such that $\Phi_p(s)(p) = p$. The family $\Phi_p(s), s \in D_2$, generates a homeomorphism as follows:

$$\Phi_p \colon F^{-1}(D_2) \cap B(p) \to (F^{-1}(0) \cap B(p)) \times D_2$$
$$\Phi_p(x) = (\Phi_p(F(x)), F(x)).$$

Thus, we have the following commutative diagram of continuous maps

$$\begin{array}{ccc} F^{-1}(D_2) \cap B(p) & \xrightarrow{\Phi_p} & (F^{-1}(0) \cap B(p)) \times D_2. \\ & \searrow^{F} & \downarrow{\scriptstyle pr_2} \\ & & D_2 \end{array}$$

Since $0 \notin K_0(F)$, we can choose D_2 small enough such that the restriction

$$F_{|F^{-1}(D_2) \cap B(p)} : F^{-1}(D_2) \cap B(p) \to D_2$$

is a submersion, hence, is a homotopic submersion (according to Lemma 2.11). Moreover, since $\Phi_p(s)$ is a homeomorphism, the restriction

$$(\Phi_p)_{|F^{-1}(s)} : F^{-1}(s) \to pr_2^{-1}(s) = (F^{-1}(0) \cap B(p)) \times \{s\}$$

is also a homeomorphism. Therefore, it is a weak homotopy equivalence. Thus, by Lemma 2.12, the map

$$F_{|F^{-1}(D_2) \cap B(p)} : F^{-1}(D_2) \cap B(p) \to D_2$$

is a fibration. It is easy to check that fibers of the map are two dimensional. It follows from Lemma 2.13 that it is a $C^\infty-$trivial fibration. So there is a diffeomorphism

$$\Psi_p^1 : F^{-1}(D_2) \cap B(p) \to (F^{-1}(0) \cap B(p)) \times D_2$$

such that the following diagram

$$\begin{array}{ccc} F^{-1}(D_2) \cap B(p) & \xrightarrow{\Psi_p^1} & (F^{-1}(0) \cap B(p)) \times D_2 \\ & \searrow^{F} & \downarrow pr_2 \\ & & D_2 \end{array}$$

commutes.

On the other hand, since $0 \notin B_\infty(F)$, there exist a neighborhood D_3 of 0, a compact set $B \subset \mathbb{C}^2$ and a homeomorphism

$$\Psi_2 : F^{-1}(D_3) \setminus B \to (F^{-1}(0) \setminus B) \times D_3$$

such that the diagram

$$\begin{array}{ccc} F^{-1}(D_3) \setminus B & \xrightarrow{\Psi^2} & (F^{-1}(0) \setminus B) \times D_3 \\ & \searrow^{F} & \downarrow pr_2 \\ & & D_2 \end{array}$$

commutes.

Without loss of generality, we may assume that $D_2 = D_3 = D$. Since 0 is a regular value of F, we can choose D small enough such that every $t \in D$ is a regular value of F.

Now, we construct a convenient vector field $v(x)$ on $F^{-1}(D)$ trivializing the restriction $F_{|F^{-1}(D)}$. Let $x^\alpha \in F^{-1}(D)$ arbitrary, we consider the following cases.

a) *Case* 1: $x^\alpha \in B(p) \cap F^{-1}(D), p \in A(F)$. Let $U^\alpha := B(p) \cap F^{-1}(D)$ and $v^\alpha(x) := \frac{\partial (\Psi_p^1)^{-1}}{\partial s}(\Psi_p^1(x))$, where s is the coordinate on D.

It is easy to verify that

$$\langle v^\alpha(x), \operatorname{grad} F(x) \rangle = 1, x \in U^\alpha.$$

b) *Case* 2: $x^\alpha \in F^{-1}(D) \setminus B$. Let $U^\alpha = F^{-1}(D) \setminus B$ and $v^\alpha(x) = \frac{\partial (\Psi^2)^{-1}}{\partial s}(\Psi^2(x))$, $x \in U^\alpha$. Similarly, we have

$$\langle v^\alpha(x), \operatorname{grad} F(x) \rangle = 1, x \in U^\alpha.$$

c) *Case* 3: $x^\alpha \in (F^{-1}(D) \cap \operatorname{int} B) \setminus (\cup_{p \in A(F)} \overline{B(p)})$. Let

$$U^\alpha := (F^{-1}(D) \cap \operatorname{int} B) \setminus (\cup_{p \in A(F)} \overline{B(p)}).$$

Since $t \in D$ is a regular value of F then $\operatorname{grad} F(x) \neq 0$ for all $x \in F^{-1}(D)$. Let

$$v^\alpha(x) = \frac{\operatorname{grad} F(x)}{\| \operatorname{grad} F(x) \|}, x \in U^\alpha.$$

Then

$$\langle v^\alpha(x), \operatorname{grad} F(x) \rangle = 1, x \in U^\alpha.$$

d) *Case* 4: $x^\alpha \in \partial B \cup (\cup_{p\in A(F)} \partial B(p) \cap F^{-1}(D))$. Since $t \in D$ is a regular value of F, then x^α is a regular point of F. Hence there is a small neighborhood U^α of x^α such that $\operatorname{grad} F(x) \neq 0$ for all $x \in U^\alpha$. Similarly, let

$$v^\alpha(x) = \frac{\operatorname{grad} F(x)}{\| \operatorname{grad} F(x) \|}, x \in U^\alpha.$$

Let λ^α be a smooth unit partition on $F^{-1}(D)$ such that $\operatorname{supp}\lambda^\alpha \subset U^\alpha$ for all α. The vector field $v(x)$ on $F^{-1}(D)$ is defined by $v(x) := \sum \lambda^\alpha(x) v^\alpha(x)$. It is clear that $v(x)$ is smooth and satisfies

$$\langle v(x), \operatorname{grad} F(x) \rangle = 1, x \in F^{-1}(D).$$

By integrating the vector field $v(x)$, we get the diffeomorphism trivializing the map

$$F_{|F^{-1}(D)} : F^{-1}(D) \to D.$$

Thus $0 \notin B(F)$. □

Now, we assume that $\deg f > \deg g$. Then d_t and V^t_∞ do not depend on t, set $d := d_t$. The following is deduced from Corollary 4.4 in [D].

Proposition 2.14. *For all* $t \in \mathbb{C}$, *we have*

$$\chi(\overline{V_t}) = \chi(V) + \Sigma_{p\in \, Sing(V_t)} \mu_p(G(x, y, z, t)),$$

where V *is a smooth projective curve of degree* d *and* $\overline{V_t}$ *is the projective closure of the curve* $V_t \subset \mathbb{C}^2$.

Lemma 2.15. *Let* $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ *be a rational function and* t_0 *be a regular value of* F. *Assume that* $\deg f > \deg g$ *and* $\chi(F^{-1}(t)) = \chi(F^{-1}(t_0))$ *for all* t *near* t_0 *enough. Then* $t_0 \notin K_1(F)$ *and there exists a neighborhood* D *of* t_0 *such that*

$$\mu_p(G(x, y, z, t)) = \mu_p(G(x, y, z, t_0)), p \in V_\infty, t \in D.$$

Proof. Let D be a neighborhood of t_0 such that $\chi(V_t) = \chi(V_{t_0})$ for all $t \in D$. Since $t_0 \notin K_0(F)$, we can choose D small enough such that $F^{-1}(t)$ is smooth for all $t \in D$.

By using the Mayer-Vietoris exact sequence, we obtain

$$\chi(F^{-1}(t)) = \chi(\bar{V_t}) - \#V_\infty - \#A(F).$$

Therefore, according to Proposition 2.14, we have

$$\begin{aligned}
\chi(F^{-1}(t)) - \chi(F^{-1}(t_0)) &= \Sigma\mu_p(G(x, y, z, t)) - \Sigma\mu_q(G(x, y, z, t_0)) \\
&= \Sigma_{p\in A(F)}(\mu_p(f - tg) - \Sigma\mu_p(f - t_0 g)) + \Sigma_{p\in V_\infty}(\mu_p(G(x, y, z, t)) - \\
&- \mu_p(G(x, y, z, t_0))).
\end{aligned}$$

Since the Milnor number is a semi-continuous function in t, then $\chi(F^{-1}(t)) - \chi(F^{-1}(t_0)) \leq 0$, the equality occurs if and only if

$$\mu_p(G(x, y, z, t)) = \mu_p(G(x, y, z, t_0)),$$

for all $p \in V_\infty, p \in A(F)$. □

Proof of Theorem 1.2. According to Lemma 2.4, it is enough to prove that

$$K_0(F) \cup B_\infty(F) \cup K_1(F) \subset B(F).$$

Let $t_0 \notin B(F)$ arbitrary. Then F defines a locally $C^\infty-$trivial fibration at t_0. Let D be the neighborhood of t_0 such that the restriction

$$F_{|F^{-1}(D)} \colon F^{-1}(D) \to D$$

is a $C^\infty-$trivial fibration. That implies $t_0 \notin B_\infty(F)$.

According to the Sard's Theorem, we can take a regular value t_1 of $F_{|F^{-1}(D)}$. Therefore the fiber $F^{-1}(t_1)$ is smooth. Since $F_{|F^{-1}(D)}$ is trivial, it is also smooth. Thus $t_0 \notin K_0(F)$.

On the other hand, for all $t \in D$ the fiber $F^{-1}(t_0)$ is homeomorphic to $F^{-1}(t)$. Therefore their Euler characteristics are equal, by Lemma 2.15, we get $t_0 \notin K_1(F)$. The proof is complete. □

3. Critical values at infinity

Let $F = \frac{f}{g} : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ have no common factor. This section is to characterize the critical values at infinity of F.

Let $t_0 \in \mathbb{C} \setminus (K_0(F) \cup K_1(F))$ such that

$$d := \deg(f - t_0 g) = \max\{\deg f, \deg g\}. \tag{1}$$

Without loss of generality, we may assume that

$$d = \deg_x(f - t_0 g).$$

Remark 3.1. The assumption (1) holds in the following situations:

1) $\deg f > \deg g$;

2) $\deg g > \deg f$ and $t_0 \neq 0$;

3) $\deg f = \deg g = d$ and $t_0 \neq \frac{f_d}{g_d}$, where f_d, g_d are respectively the highest-degree homogeneous components of f, g.

3.1. **Geometrical and topological characterizations.** We denote by L the following linear function

$$\mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto y.$$

For each $t \in \mathbb{C}$ let

$$L_t := L_{|V_t} : V_t \to \mathbb{C}$$

and

$$l_t := L_{|F^{-1}(t)} : F^{-1}(t) \to \mathbb{C},$$

where $V_t = \{(x, y) \in \mathbb{C}^2 : f(x, y) - tg(x, y) = 0\}$. It is easy to prove the following.

Lemma 3.2. *For all $\delta > 0$ small enough and $t \in D_\delta(t_0)$, the map*

$$L_t : V_t \to \mathbb{C}$$

is proper and $\#L_t^{-1}(c) = d$, where c is a generic constant and $D_\delta(t_0) = \{t \in \mathbb{C} : |t - t_0| < \delta\}$.

The following lemma follows from Lemma 3.2 and the argument in the proof of Lemma 3.2 in [HT].

Lemma 3.3. *Under the hypothesis in Lemma 3.2, for all $\delta > 0$ small enough, the restriction*

$$L_\delta := L_{|\cup_{t\in\overline{D_\delta(t_0)}} V_t} : \cup_{t\in\overline{D_\delta(t_0)}} V_t \to \mathbb{C}$$

is proper.

Remark 3.4. 1) The critical points of $l_t : F^{-1}(t) \to \mathbb{C}$ are exactly the critical points of $L_t : V_t \to \mathbb{C}$ not belonging to the set V_t.

2) The critical points of $L_t : V_t \to \mathbb{C}$ are algebraic functions in t, we can divide them into two types:

(i) The points which tend to critical points of L_{t_0} as $t \to t_0$. The number of points in this type, counting with multiplicity, is equal to the number of critical points, counting with multiplicity, of L_{t_0}.

(ii) The points that tend to infinity as $t \to t_0$ (the points in this type are also critical points of L_t).

Lemma 3.5. *For each $a > 0$ and $\delta > 0$ let*

$$U(a,\delta) := \{|L| \leqslant a\} \cap \overline{F^{-1}(D_\delta(t_0))}.$$

For a large enough and δ small enough we have

$$\begin{aligned}\chi(V_t) - \chi(V_{t_0}) &= \chi(F^{-1}(t_0)) - \chi(F^{-1}(t)) \\ &= \chi(F^{-1}(t_0) \setminus U(a,\delta)) - \chi(F^{-1}(t) \setminus U(a,\delta)),\end{aligned}$$

where $V_t = \{(x,y) \in \mathbb{C}^2 : f(x,y) - tg(x,y) = 0\}$.

Proof. According to Remark 3.4, for a sufficiently small δ and a sufficiently large a, all critical points of L_t in $U(a,\delta)$ are in the first type. Let $Q_i, i = 1,\dots,s$ be the critical points of L_{t_0} in $U(a,\delta)$. Let D_{β_i} be the disc centered at $L(Q_i)$, with the radius sufficiently small β_i, then for sufficiently small δ there is no critical point of L_t in $U(a,\delta) \setminus \cup_{i=1}^s L^{-1}(D_{\beta_i})$.

For each $i = 1,\dots,s$ and $t \in \mathbb{C}$ we denote

$$M_t^i := (V_t \cap U(a,\delta)) \setminus L_\delta^{-1}(D_{\beta_i}),\ N_t^i := (V_t \cap U(a,\delta)) \cap L_\delta^{-1}(D_{\beta_i})$$

and

$$C^i := \{z \in \mathbb{C} : |z| \leqslant a\} \setminus D_{\beta_i}.$$

According to Lemma 3.2, for all $t \in D_\delta(t_0)$ the restriction $L_t = L_{|V_t}$ is proper, then $L(V_t)$ is closed and constructible. Hence $L(V_t) = \mathbb{C}$ and the restriction map $L_{|M_t^i} \colon M_t^i \to C^i$ is surjective. Moreover, it is easy to check that $L_{|M_t^i}$ does not have critical points. Thus, the map $L_{|M_t^i}$ is a d-sheeted unbranched covering. Then

$$\chi(M_t^i) = d \cdot \chi(C^i), \forall t \in D_\delta(t_0). \tag{2}$$

On the other hand, the restriction map $L_{|N_t^i} : N_t^i \to D_{\beta_i}$ is a d-sheeted covering branching over the critical points. By the same argument as in the proof of Theorem 3.1 in [HT], we have

(3) $$\chi(N_t^i) = d - \rho_i(t), \forall t \in D_\delta(t_0),$$

where $\rho_i(t)$ is the number of critical points in D_{β_i}, counting with multiplicity, of L_t. By using the Mayer-Vietoris exact sequence, we get

$$\chi(V_t) - \chi(V_{t_0}) = (\chi(V_t \setminus U(a,\delta)) - \chi(V_{t_0} \setminus U(a,\delta))) + \sum_{i=1}^{s}(\rho_i(t_0) - \rho_i(t)).$$

Moreover, by Remark 3.4, the second term is equal to 0 and $V_t \setminus U(a,\delta) = F^{-1}(t) \setminus U(a,\delta)$. Then

$$\chi(V_t) - \chi(V_{t_0}) = \chi(F^{-1}(t) \setminus U(a,\delta)) - \chi(F^{-1}(t_0) \setminus U(a,\delta)).$$

Similarly, since $t_0 \notin K_1(F)$, by using the Mayer-Vietoris exact sequence again, we can prove that

$$\chi(V_t) - \chi(V_{t_0}) = \chi(F^{-1}(t)) - \chi(F^{-1}(t_0)).$$

From the last two equalities, we get the conclusion of the lemma. □

Theorem 3.6. *Let $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x,y]$ have no common factor, and let $t_0 \notin K_0(F) \cup K_1(F)$ such that*

$$\deg(f - t_0 g) = \deg_x(f - t_0 g) = \max\{\deg f, \deg g\}.$$

Then the following are equivalent:

(i) $t_0 \notin B_\infty(F)$.

(ii) *There are no critical points of $l_t = L_{|F^{-1}(t)}$ which tend to infinity as $t \to t_0$.*

Proof. ii) $\Longrightarrow$ i): Assume that there is no critical point of l_t going to infinity as $t \to t_0$. It follows that if a number a is large enough, then the set

$$U(a) = \{|L| \leqslant a\} \cap (\overline{F^{-1}(D_\delta(t_0))})$$

contains all the critical points of the maps $l_t, t \in D_\delta(t_0)$. It follows from Lemma 3.3 that $U(a)$ is bounded, hence, is a compact set. By the same argument as in the proof of Theorem 3.1 in [HT], the restriction

$$F_{|F^{-1}(D_\delta(t_0)) \setminus U(a)} : F^{-1}(D_\delta(t_0)) \setminus U(a) \to D_\delta(t_0)$$

is a trivial fibration. Hence $t_0 \notin B_\infty(F)$.

i) $\Longrightarrow$ ii): By contradiction, assume that there exist critical points of l_t going to infinity as $t \to t_0$.

Let

$$K := U(a) = \{|L| \leqslant a\} \cap \overline{F^{-1}(D_\delta(t_0))},$$

where $|a| \gg 1$ such that all critical points of l_{t_0}, all critical points in the first type of $L_t, t \in D_\delta(t_0)$ and the points of the set $A(F)$ are contained in K. It follows from the assumption that for arbitrarily small δ, there exists $t \in D_\delta(t_0)$ such that l_t has critical points $P_1(t), \dots, P_m(t)$ that do not belong to K.

Let $D_{\epsilon_i}, i = 1, \dots, m$, be the discs centered at $\alpha_i := L(P_i(t))$ with radii ϵ_i small enough. We consider the following restrictions

$$L : (F^{-1}(t_0) \setminus K) \setminus \cup_{i=1}^m l_\delta^{-1}(D_{\epsilon_i}) \to (\mathbb{C} \setminus l_\delta(K)) \setminus \cup_{i=1}^m D_{\epsilon_i}$$

and

$$L : (F^{-1}(t) \setminus K) \setminus \cup_{i=1}^m l_\delta^{-1}(D_{\epsilon_i}) \to (\mathbb{C} \setminus l_\delta(K)) \setminus \cup_{i=1}^m D_{\epsilon_i}.$$

These maps are well-defined. By the similar arguments as in the proof of Lemma 3.5, we can prove that these maps are $d-$sheeted unbranched coverings. As a consequence, we have the following

$$\begin{aligned}\chi((F^{-1}(t_0) \setminus K) \setminus \cup_{i=1}^m l_\delta^{-1}(D_{\epsilon_i})) &= \chi((F^{-1}(t) \setminus K) \setminus \cup_{i=1}^m l_\delta^{-1}(D_{\epsilon_i})) \\ &= d\chi((\mathbb{C} \setminus l_\delta(K)) \setminus \cup_{i=1}^m D_{\epsilon_i}).\end{aligned}$$

Now, for each $i = 1, \dots, m$ and $t \in D_\delta(t_0)$ let us consider the restricted map

$$L_t^i := L_{|F^{-1}(t) \cap L^{-1}(D_{\epsilon_i})} : F^{-1}(t) \cap L^{-1}(D_{\epsilon_i}) \to D_{\epsilon_i}.$$

Since $V_t = F^{-1}(t) \cup A(F)$, we have $\mathbb{C} \setminus L(K) \subset L(F^{-1}(t))$. Hence, we can choose ϵ_i small enough such that L_t^i is surjective. Moreover, for all t the map L_t^i is proper, for $t \neq t_0$ the map L_t^i has critical points $P_i(t)$, and $L_{t_0}^i$ has no critical points, then by using the same argument as in the proof of Theorem 3.1 in [HT], we have

$$\chi(F^{-1}(t_0) \cap L^{-1}(D_{\epsilon_i})) = d,$$

and

$$\chi(F^{-1}(t) \cap L^{-1}(D_{\epsilon_i})) = d - r_i,$$

where r_i is the multiplicity of the critical point $P_i(t)$ of the map l_t.

It follows from the Mayer-Vietoris sequence that

$$\chi(F^{-1}(t_0) \setminus K) - \chi(F^{-1}(t) \setminus K) = \sum_{i=1}^m r_i \neq 0,$$

since there are critical points of l_t tending to infinity when t tends to t_0. By applying Lemma 3.5, we get $\chi(F^{-1}(t)) \neq \chi(F^{-1}(t_0))$ for all t near t_0. Thus $t_0 \in B_\infty(F)$. □

Let $\delta(y, t) = \mathrm{disc}_x(f - tg)$ be the discriminant of $f - tg$ with respect to x. Then the critical points of l_t are $(x(t), y(t))$ such that $y(t)$ is a root of $\delta(y, t) = 0$. Those points go to infinity as $t \to t_0$ if and only if $y(t) \to \infty$ when $t \to t_0$. We can write

$$\delta(y, t) = q_k(t)y^k + q_{k-1}y^{k-1} + \cdots .$$

Then $\delta(y, t)$ has a root tending to infinity when $t \to t_0$ if and only if $q_k(t_0) = 0$. The following is an immediate corollary of Theorem 3.6.

Corollary 3.7. *Let $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ have no common factor and $t_0 \notin K_0(F) \cup K_1(F)$ such that*

$$\deg(f - t_0 g) = \deg_x(f - t_0 g) = \max\{\deg f, \deg g\}.$$

Then, $t_0 \in B_\infty(F)$ if and only if $q_k(t_0) = 0$.

Theorem 3.8. *Let $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ have no common factor, and let $t_0 \notin K_0(F) \cup K_1(F)$ such that*

$$\deg(f - t_0 g) = \deg_x(f - t_0 g) = \max\{\deg f, \deg g\}.$$

Then the following are equivalent:
(i) $t_0 \notin B_\infty(F)$.
(ii) *There exists a compact subset K of $\mathbb{C}^2$ such that*

$$\chi(F^{-1}(t_0) \setminus K) > \chi(F^{-1}(t) \setminus K),$$

for all t generic.

Proof. (i) $\Longrightarrow$ (ii): Let $K := U(a, \delta) = \{|L| \leqslant a\} \cap \overline{F^{-1}(D_\delta(t_0))}$. According to the proof of Theorem 3.6, if a is large enough and δ is small enough then

$$\chi(F^{-1}(t_0) \setminus K) - \chi(F^{-1}(t) \setminus K) = \rho, \forall t \in D_\delta(t_0),$$

where ρ is the number of critical points $P(t)$, counting with multiplicity, of the map l_t such that $\|P(t)\| \to \infty$ as $t \to t_0$.

Since $t_0 \in B_\infty(F)$ then according to Theorem 3.6, we have $\rho \neq 0$. Thus

$$\chi(F^{-1}(t_0) \setminus K) > \chi(F^{-1}(t) \setminus K),$$

for all t different and near t_0.

(ii) $\Longrightarrow$ (i): Let K be the compact set such that

$$\chi(F^{-1}(t_0) \setminus K) > \chi(F^{-1}(t) \setminus K).$$

By contradiction, assume that $t_0 \notin B_\infty(F)$. Since $t_0 \notin K_0(F) \cup K_1(F)$ then F defines a locally $C^\infty-$trivial fibration at t_0. Let D be a neighborhood of t_0 and $\Phi : F^{-1}(D) \to F^{-1}(t_0) \times D$ be the diffeomorphism trivializing $F_{|F^{-1}(D)}$.

Hence, the restriction $\Phi_{|F^{-1}(D)\setminus K}$ induces a diffeomorphism trivializing the map

$$F_{|F^{-1}(D)\setminus K} : F^{-1}(D) \setminus K \to D.$$

This implies that

$$\chi(F^{-1}(t) \setminus K) = \chi(F^{-1}(t_0) \setminus K)$$

which contradicts the assumption. Thus $t_0 \in B_\infty(F)$. □

Theorem 3.9. *Let $F = f/g : \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ have no common factor, and let $t_0 \notin K_0(F) \cup K_1(F)$ such that*

$$\deg(f - t_0 g) = \deg_x(f - t_0 g) = \max\{\deg f, \deg g\}.$$

Then the following are equivalent:
(i) $t_0 \in B_\infty(F)$;
(ii) $\chi(\{f - t_0 g = 0\}) > \chi(\{f - tg = 0\})$, *for all t generic;*
(iii) $\chi(F^{-1}(t_0)) > \chi(F^{-1}(t))$, *for all t generic.*

Proof. The proof is straightforward from Lemma 3.5 and the proof of Theorem 3.6. □

3.2. **Analytic characterization.** In this section, we will determine the critical values at infinity of rational functions in two variables in terms of Malgrange condition and M-tameness. Assume that $d := \deg f > \deg g$.

Definition 3.10. ([LS], [P]) Let $H(t,x) : \mathbb{C}^{n+1} \to \mathbb{C}$ be an analytic function such that for every t the point $0 \in \mathbb{C}^n$ is an isolated singular point of $H(t,x)$. Then the following set

$$\Gamma_H = \overline{\{(t,x) \in \mathbb{C}^{n+1} : \partial H/\partial t \neq 0, \partial H/\partial x_1 = \ldots = \partial G/\partial x_n = 0\}}$$

is called the *relative polar curve* of the family of hypersurfaces $\{x \in \mathbb{C}^n : H(t,x) = 0\}$.

Theorem 3.11. ([LS], [P]) *Let $H(t,x) : \mathbb{C}^{n+1} \to \mathbb{C}$ be an analytic function such that for every t near t_0 enough the origin is an isolated singular point of $H_t(x) := H(t,x)$. Then, the following are equivalent:*

(i) $|\partial H/\partial t(t,x)| \ll \|(\partial H/\partial x_1, \ldots, \partial G/\partial x_n)(t,x)\|$ *for all* (t,x) *near* $(t_0, 0)$ *enough.*

(ii) $\mu_0(H(t,x)) = \mu_0(H(t_0,x))$ *for all t near t_0 enough.*

(iii) *There exists a neighborhood B of $(t_0,0)$ such that $\Gamma_H \cap B = \emptyset$.*

Since $\deg f > \deg g$ then $\deg(f - tg) = \deg f$ for all t and the set V_∞^t of points at infinity of V_t does not depend on t. Denote $V_\infty := V_\infty^t$.

Consider a point $p_0 \in V_\infty$. Without loss of generality, we may assume that $p_0 = [1 : 0 : 0] \in \mathbb{C}P^2$. Then (y,z) forms a local system of coordinates near p_0. Let $G(y,z,t) := G(1,y,z,t)$. Then $(0,0)$ is either a regular point or an isolated singular point of $G(y,z,t)$.

The following is a version of Lemma 3.1 in [P] for rational functions.

Lemma 3.12. *Let $F = f/g \colon \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x,y]$ and $\deg f > \deg g$. Let $t_0 \in \mathbb{C} \setminus (K_0(F) \cup K_1(F))$ and $p_0 = [1 : 0 : 0] \in V_\infty$. Assume that, either p_0 is a regular point of $\overline{V_{t_0}}$ or p_0 is a singular point of $\overline{V_{t_0}}$ and $\mu_{(0,0)}(G(y,z,t)) = \mu_{(0,0)}(G(y,z,t_0))$ for all t near t_0 enough.*

Then, for every positive integer N, the following holds

$$|\partial G/\partial t| \ll \|(\partial G/\partial y, z^{(N-1)/N} \partial G/\partial z)(y,z,t)\| \tag{4}$$

as $(y,z,t) \to (0,t_0)$.

Proof. The case that p_0 is nonsingular is easy. Now, we assume that for each t, p_0 is a singular point of $\bar{V}_t$ and $\mu_{(0,0)}(G(y,z,t)) = \mu_{(0,0)}(G(y,z,t_0))$ for all t near t_0 enough. For each $N > 1$, we consider the function

$$G_N(y,z,t) = G(y,z^N,t).$$

Then $\frac{\partial G_N}{\partial y} = \frac{\partial G}{\partial y}, \frac{\partial G_N}{\partial z} = N z^{N-1} \frac{\partial G}{\partial z}$ and $\frac{\partial G_N}{\partial t} = \frac{\partial G}{\partial t}$.

Since $(0,0)$ is a singular point of $\{G(y,z,t_0) = 0\}$, it is easy to prove that $(0,0)$ is also an isolated singular point of $\{G_N(y,z,t) = 0\}$ for all t.

According to Theorem 3.11, it suffices to show that the relative polar curve:

$$\Gamma_{G_N} = \overline{\{(y,z,t) | \partial G_N/\partial t \neq 0, \partial G_N/\partial y = 0, \partial G_N/\partial z = 0\}}$$

of the family $\{G_N(y,z,t)=0\}$ is empty in some small neighborhood of $(0,0,t_0)$.

By contradiction, we assume that there exists a sequence $(y_k,z_k,t_k)\to(0,0,t_0)$ such that

$$\frac{\partial G_N}{\partial t}(y_k,z_k,t_k)\neq 0, \frac{\partial G_N}{\partial y}(y_k,z_k,t_k)=\frac{\partial G_N}{\partial z}(y_k,z_k,t_k)=0, \forall k. \tag{5}$$

We have

$$\frac{\partial G_N}{\partial t}(y_k,z_k,t_k)=\frac{\partial G}{\partial t}(y_k,z_k,t_k)=-z_k^d g(1/z_k,y_k/z_k)\neq 0.$$

Since $d>\deg g$ then $z_k\neq 0$. Hence, it follows from (5) that $\frac{\partial G}{\partial z}(y_k,z_k,t_k)=0$.

That means the relative polar curve Γ_G of the family $\{G(y,z,t)=0\}$ is not empty in some neighborhood of $(0,0,t_0)$. This contradicts the assumption. The proof is complete. □

By the Curve Selection Lemma and by using the inequality in Lemma 3.12 for all N, we obtain

$$|\partial G/\partial t|\leqslant C\|(\partial G/\partial y, z\partial G/\partial z)\|. \tag{6}$$

By applying the same argument as in the proof of Lemma 3.2 in [P], we receive the following.

Lemma 3.13. *Under the hypothesis in Lemma 3.12, for all $(y,z,t)\in\{G(y,z,t)=0\}$ and $(y,z,t)\to(0,t_0)$, we have*

$$|z\partial G/\partial z|\ll|\partial G/\partial y|. \tag{7}$$

Theorem 3.14. *Let $F=f/g\colon \mathbb{C}^2\setminus\{g=0\}\to\mathbb{C}$ be a rational function, where $f,g\in\mathbb{C}[x,y]$ and $\deg f>\deg g$. Let $t_0\in\mathbb{C}\setminus(K_0(F)\cup K_1(F)\cup B_\infty(F))$. Then F satisfies the Malgrange condition at t_0.*

Proof. Let D be the neighborhood of t_0 such that

$$\chi(F^{-1}(t_0))=\chi(F^{-1}(t)), \forall t\in D.$$

According to Lemma 2.15, for all $p\in V_\infty$ and $t\in D$, either p is a regular point of $\bar{V_{t_0}}$ or is a singular point and $\mu_p(G(x,y,z,t))=\mu_p(G(x,y,z,t_0))$.

Let $p\in V_\infty$ arbitrary. Without loss of generality, we may assume that $p=[1:0:0]$. It follows from inequalities (6) and (7) that

$$|\partial G/\partial t|\leqslant C|\partial G/\partial y(y,z,t)|$$

for all $(y,z,t)\in\{G(y,z,t)=0\}$ near $(0,0,t_0)$ enough, where $G(y,z,t)=z^d(f(1/z,y/z)-tg(1/z,y/z))$. We have

$$|z\cdot g(1/z,y/z)|\leqslant\left|\frac{\partial}{\partial Y}(f-tg)(1/z,y/z)\right|$$

for all (y,z,t) near $(0,0,t_0)$ enough such that $(f-tg)(1/z,y/z)=0$.

Now, set $z=\frac{1}{X}$ and $y=\frac{Y}{X}$, we obtain

$$0<1/C\leqslant\|(X,Y)\|\cdot\|\operatorname{grad}F(X,Y)\|$$

for all $(X,Y)\to\infty$ and $F(X,Y)\to t_0$. Thus F satisfies the Malgrange condition at t_0. □

Now we consider the M-tameness of F. Firstly, we prove the following.

Theorem 3.15. *If F satisfies the Malgrange condition at a value t_0, then F is M-tame at t_0.*

Proof. Assume that F is not M-tame at t_0, i.e. there are sequences $\{p_k\}_k$ and $\{\lambda_k\}_k$ such that

$$p_k \to \infty, F(p_k) \to t_0 \text{ and } \operatorname{grad} F(p_k) = \lambda_k p_k.$$

We will show that F does not satisfy the Malgrange condition at t_0.

Indeed, by the Curve Selection Lemma, there exist some real analytic curves $(x(\tau), y(\tau)) \to \infty$ and $\lambda(\tau), \tau \in (0, \epsilon)$ such that $\operatorname{grad}F(x(\tau), y(\tau)) = \lambda(\tau)(x(\tau), y(\tau))$ and $F(x(\tau), y(\tau)) \to t_0$ when $\tau \to 0$.

For each analytic curve $\phi(\tau) = c\tau^m +$ higher powers ($c \neq 0$), we denote $\deg(\phi(\tau)) = m$. If $\phi(\tau)$ and $\rho(\tau)$ are two analytic curves then we define $\deg(\frac{\phi(\tau)}{\rho(\tau)}) = \deg(\phi(\tau)) - \deg(\rho(\tau))$.

Since $F(x(\tau), y(\tau)) \to t_0$, then $\deg(f(x(\tau), y(\tau))) = \deg(g(x(\tau), y(\tau)))$. Hence $\deg F'(x(\tau), y(\tau)) > -1$. Thus

$$\deg(\langle (x(\tau), y(\tau)), \operatorname{grad}F\rangle) > 0 \quad \text{and} \quad \|(x(\tau), y(\tau))\| \cdot \|\operatorname{grad}F\| \to 0.$$

Therefore F does not satisfy the Malgrange condition at t_0. □

The main theorem in this section is the following.

Theorem 3.16. *Let $F = f/g\colon \mathbb{C}^2 \setminus \{g = 0\} \to \mathbb{C}$ be a rational function, where $f, g \in \mathbb{C}[x, y]$ and $\deg f > \deg g$. Let $t_0 \in \mathbb{C}\setminus(K_0(F) \cup K_1(F))$. Then the following are equivalent:*
(i) $t_0 \in B_\infty(F)$;
(ii) $t_0 \in K_\infty(F)$;
(iii) $t_0 \in M_\infty(F)$.

Proof. (i) $\Longrightarrow$ (iii): Assume that F is M-tame at t_0. Then for $\delta > 0$ small enough and $R > 0$ large enough, we can construct in $F^{-1}(D_\delta(t_0)) \setminus B_R$ a smooth vector field $v(x)$ such that

$$\langle v, x\rangle = 0;$$
$$\langle v, \operatorname{grad}F\rangle = 1.$$

Here B_R is the closed ball in $\mathbb{C}^2$ with radius R centered at the origin.

Now, by integrating the vector field we get a diffeomorphism trivializing the map

$$F : F^{-1}(D_\delta(t_0)) \setminus B_R \to D_\delta(t_0).$$

(ii) $\Longrightarrow$ (i): By Theorem 3.14.
(iii) $\Longrightarrow$ (ii): By Theorem 3.15. □

Remark 3.17. Theorem 3.16 remains valid if $\deg f < \deg g$ and $t_0 \neq 0$.

3.3. **Examples.** To conclude we give some examples showing that the Fedoryuk condition is not necessary for a value to be regular at infinity.

Example 3.18. Let $F(x,y) = \frac{xy+1}{x^2+1}$ and $L : \mathbb{C}^2 \to \mathbb{C}, (x,y) \mapsto y$.

The critical points of l_t are (x,y), where $x = \sqrt{(1-1/t)}, y = 2x/(1-x^2)$. It is easy to check that these points do not go to infinity when $t \to i$. According to Theorem 3.6 we have $i \notin B_\infty(F)$.

Now let $(x_k, y_k) = (k, ik)$. We see that $\|(x_k, y_k)\| \to \infty, F((x_k, y_k)) \to i$ and $\|\text{grad}F(x_k, y_k)\| \to 0$ as $k \to \infty$. That means $i \in \widetilde{K}_\infty(F)$. Thus $\widetilde{K}_\infty(F) \not\subset B_\infty(F)$.

Example 3.19. Let $F(x,y) = \frac{x^3+1}{xy+1}$. It is easy to check that $B_\infty(F) = \{0\}$. Let $p_k := (k, \frac{1}{c}k^2) \in \mathbb{C}^2, c \neq 0, k \geq 1$, we see that $F(p_k) \to c$ as $k \to \infty$, and $\text{grad}F(p_k) \to 0$. Therefore $\widetilde{K}_\infty(F) = \mathbb{C}$. In particular, $B_\infty(F) \neq \widetilde{K}_\infty(F)$.

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Institute of Mathematics
18 Hoang Quoc Viet road, 10307 Hanoi, Vietnam
E-mail address: ntthang@math.ac.vn