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LYAPUNOV EXPONENTS AND CENTRAL EXPONENTS OF LINEAR ITO STOCHATIC DIFFERENTIAL EQUATIONS

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Abstract. We study Lyapunov, central and auxiliary exponents of linear Ito stochastic equations. We show that the central exponents are nonrandom like Lyapunov exponents, the nonrandomness of which was proved in [8]. We prove that under a nondegeneracy condition the central exponents Θ_k of a linear Ito stochastic differential equation coincide with its auxiliary exponents γ_k, and, moreover, all the first exponents coincide: $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$.

1. Introduction

We consider a linear n-dimensional Ito stochastic differential equation

$$\begin{aligned} dX(t) &= F_0(t)X(t)dt + \sum_{k=1}^{m} F_k(t)X(t)dW_t^k, \\ X(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where $F_k(t) = \left(f_{ik}^j\right)_{n\times n}$ $(k \in \{0, 1, 2, ..., m\})$ are continuous matrix-valued functions bounded by a constant K, x_0 is a non-random initial value, W_t^j $(j \in \{1, 2, ..., m\})$ are independent 1-dimensional standard Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is known that, with the above assumption, the Cauchy problem of (1.1) has unique solution (see Khasminskii [11, Theorem 3.2 page 79]). The linear Ito stochastic differential equation (1.1) generates a two-parameter stochastic flow $\Phi_{t_0,t}(\omega)$ of linear operators of $\mathbb{R}^n$ (see Kunita [13, page 116 and Theorem 4.5.1 page 155]). The solution of (1.1) satisfying initial value condition $X(t_0) = x_0$, is a stochastic process given by the formula $X(t) = \Phi_{t_0,t}(\omega)x_0$. Note that fixing an $\omega \in \Omega$ the two-parameter flow $\Phi_{t_0,t}(\omega)$ is an analogue of the Cauchy operator of a linear system of differential equations.

We denote by G_r the Grassmannian manifold of all r-dimentional subspaces in $\mathbb{R}^n$. For a linear subspace U of $\mathbb{R}^n$, we denote by U_* the subset of all nonvanishing

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vectors of U. For any nondegenerate $n \times n$ matrix X, let us denote by X^* the transposed matrix of X and by $d_1(X) \geq d_2(X) \geq ... \geq d_n(X)$ the singular numbers of X, i.e. they are the positive square roots of the eigenvalues of the matrix X^*X. Clearly, for any $k \in \{1, 2, ..., n\}$ we have

$$d_k(X) = \inf_{U \in G_{n-k+1}} \sup_{x \in U_*} \frac{\|Xx\|}{\|x\|} = \sup_{V \in G_k} \inf_{x \in V_*} \frac{\|Xx\|}{\|x\|}.$$

Definition 1.1. The random variables $\lambda_k(\omega),\ \Omega_k(\omega),\ \Theta_k(\omega)\,(k \in \{1, 2, ..., n\})$ defined by

$$(1.2)\ \lambda_k(\omega) := \min_{U \in G_{n-k+1}} \max_{x \in U} \limsup_{t \to +\infty} \frac{1}{t} \ln \|\Phi_{0,t}(\omega)x\|,$$

$$(1.3)\ \Theta_k(\omega) := \sup_{V \in G_k} \sup_{T \in \mathbb{R}^+} \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1},$$

$$(1.4)\ \Omega_k(\omega) := \inf_{U \in G_{n-k+1}} \inf_{T \in \mathbb{R}^+} \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U} \right\|,$$

where $\Phi|_S$ denotes the restriction of the operator Φ on S, are respectively called *Lyapunov exponents* and *central exponents* of the equation (1.1).

It will be shown in the proof of Theorem 2.6 that for any $V \in G_k$ and $T \in \mathbb{R}^+$

$$\limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$$
$$= \limsup_{m \to +\infty} \frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$$

and

$$\frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$$
$$\leq \frac{1}{m(2T)} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)2T,i2T}(\omega)\big|_{\Phi_{0,(i+1)2T}(\omega)V} \right\|^{-1}.$$

Therefore, formula (1.3) is equivalent to the following formula which can serve as a definition of $\Theta_k(\omega)$ as well

$$(1.5)\qquad \Theta_k(\omega) = \sup_{V \in G_k} \sup_{T > 1} \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)\big|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}.$$

By the same argument, we have the following equivalent definition of $\Omega_k(\omega)$

$$(1.6)\qquad \Omega_k(\omega) = \inf_{U \in G_{n-k+1}} \inf_{T > 1} \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U} \right\|.$$

Definition 1.2. The random variables $\gamma_k(\omega)$ defined by

$$\gamma_k(\omega) := \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big], \quad k \in \{1,2,...,n\}\,, \tag{1.7}$$

are called *auxiliary exponents* of the equation (1.1).
The function $\gamma_k(T)$ defined by

$$\gamma_k(T) := \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E}\ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big], \quad k \in \{1,2,...,n\}\,, T \in \mathbb{R}^+, \tag{1.8}$$

where $\mathrm{E}\xi(\omega)$ denotes the expectation of a random variable $\xi(\omega)$, are called *auxiliary functions* of the equation (1.1).

The above definitions of Lyapunov exponents, central exponents and auxiliary exponents for the stochastic differential equations have been introduced by Millionshchikov (see [14, 15]). Millionshchikov considered equation $\dot{u} = [B(t) + C(t,\omega)]u$, where $B(t)$ is a bounded continuous matrix-valued function and $C(t,\omega)$ is a piecewise-constant random matrix-valued process with independent values. Using Kolmogorov one-zero law Millionshchikov proved that Lyapunov exponents of such an equation do not depend on ω. Note that the equations Millionshchikov considered can be solved pathwisely, without the need of the Ito calculus. For the linear Ito stochastic differential equation

$$dX(t) = F_0(t)X(t)dt + \sigma \sum_{k=1}^{m} F_k X(t) dW_t^k, \tag{1.9}$$

where F_k, $(k \in \{1,2,...,m\})$ are constant matrix and $F_0(t)$ is continuous matrix-valued function bounded by a constant K, N. D. Cong [7] noticed that Lyapunov exponents, central exponents, auxiliary exponents do not depend on ω. He gave in [8] a proof of independence of Lyapunov exponents of (1.1) (which is an equation of a more general type than (1.9)) on ω, i.e. the Lyapunov exponents are nonrandom.

For deriving the main results of the paper presented in Section 3 we will need the following *nondegeneracy condition* of the random part of the equation (1.1): There are positive numbers μ_1, μ_2 such that for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$

$$\mu_1 \|x\|^2 \|y\|^2 \le \big\langle D(t,x)y, y\big\rangle \le \mu_2 \|x\|^2 \|y\|^2\,, \tag{1.10}$$

where $\big\langle y_1, y_2\big\rangle$ denotes the scalar product of two vectors $y_1, y_2 \in \mathbb{R}^n$,

$$D(t,x) = \big(d_{ij}(t,x)\big)_{n\times n} \text{ with } d_{ij}(t,x) = \sum_{k=1}^{m}\bigg(\sum_{r,l=1}^{n} f_{ik}^r(t) f_{jk}^l(t) x_r x_l\bigg).$$

The central exponents of deterministic linear differential equations were initially introduced to give lower and upper estimates for Lyapunov exponents and are different from Lyapunov exponents in general, as shown by Example 13.5.1

in Bylov et al. [3, page 187]. Beside giving estimates for Lyapunov exponents, the central exponents also serve as qualitative and quantitive characteristics of the equations under considerations. The auxiliary exponents γ_k are interesting from computational point of view: for their computation we do not have to follow trajectories of solutions on the whole time axis, but only compute the Cauchy matrix on each compact time interval.

In this paper, under the nondegeneracy condition specified above we will show that the central exponents Θ_k of the linear Ito stochastic differential equation (1.1) coincide with its auxiliary exponents γ_k, and, moreover, the first exponents coincide: $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$. The observation on equality of exponents was made by one of the authors in 1993 (see [7]) for the case of equations (1.9). However, the proof given there is incomplete since the technique of changing from series of random variables forming a Markov chain to a series of independent random variables is not completely verified. Here in this paper, we overcome this problem for the case of Θ_k and the first exponents by using another technique, namely, we use the law of large numbers and inequalities provided by Rosenblatt-Roth [17] for a series of random variables depending on a Markov chain. Moreover, we are able to prove the results for a more general equation (1.1).

The paper is organized as follows. In Section 2, we prove some properties of central and auxiliary exponents of (1.1). Namely, using standard techniques in the theory of Lyapunov exponents, we show that central exponents are nonrandom and make upper and lower estimates for Lyapunov exponents; the auxiliary exponents are also nonrandom, and the biggest auxiliary exponent γ_1 is larger than the biggest central exponent Ω_1, the least auxiliary exponent γ_n equals the least central exponent Θ_n. Section 3 presents the main result of the paper with assumption of the nondegeneracy condition, $\Theta_k = \gamma_k$ and $\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1$.

2. Properties of central and auxiliary exponents

Theorem 2.1. *For any $k \in \{1, 2, ..., n\}$ the exponent $\gamma_k(\omega)$ is nonrandom and $\gamma_k(\omega) = \limsup_{T\to+\infty} \gamma_k(T)$.*

Proof. First of all, we will prove the random variables $\frac{1}{T} \ln \left\| \Phi_{iT,(i+1)T}(\omega) \right\|$ have second moments, bounded by a constant independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$. For any $N \in \mathbb{N}$, put $\eta(\omega) = \frac{1}{N} \ln \|\Phi_{0,N}(\omega)\|$. Since

$$\|\Phi_{0,N}(\omega)\| \le \|\Phi_{N-1,N}(\omega)\| \dots \|\Phi_{1,2}(\omega)\| \, \|\Phi_{0,1}(\omega)\| ,$$

and $\|A\| \ge \frac{1}{\|A^{-1}\|}$ for any inverse matrix A then

$$\|\Phi_{0,N}(\omega)\| \ge \frac{1}{\|\Phi_{N,0}(\omega)\|} \ge \frac{1}{\|\Phi_{1,0}(\omega)\| \dots \|\Phi_{N,N-1}(\omega)\|} \ge \frac{1}{\left\|\Phi_{0,1}^{-1}(\omega)\right\| \dots \left\|\Phi_{N-1,N}^{-1}(\omega)\right\|}.$$

Therefore,

$$-\sum_{i=0}^{N-1} \ln \left\| \Phi_{i,i+1}^{-1}(\omega) \right\| \leq \ln \|\Phi_{0,N}(\omega)\| \leq \sum_{i=0}^{N-1} \ln \|\Phi_{i,i+1}(\omega)\| .$$

It follows that for any $\omega \in \Omega$, we have

$$\big|\frac{1}{N} \|\ln \Phi_{0,N}(\omega)\| \big| \leq \big|\frac{1}{N} \sum_{i=0}^{N-1} \ln \left\| \Phi_{i,i+1}^{-1}(\omega) \right\| \big| + \big|\frac{1}{N} \sum_{i=0}^{N-1} \ln \|\Phi_{i,i+1}(\omega)\| \big|.$$

Put $\eta_1(\omega) = \frac{1}{N} \sum\limits_{i=0}^{N-1} \ln \left\| \Phi_{i,i+1}^{-1}(\omega) \right\|$, $\eta_2 = \frac{1}{N} \sum\limits_{i=0}^{N-1} \ln \|\Phi_{i,i+1}(\omega)\|$ then

$$E\eta^2(\omega) \leq 2[E\eta_1^2(\omega) + E\eta_2^2(\omega)]. \tag{2.1}$$

Using Minkowski's Inequality for $p = 2$ (see Shiryaev [18, page 194]), we have

$$\big(E|\eta_1(\omega)|^2\big)^{\frac{1}{2}} \leq \frac{1}{N} \sum_{i=0}^{N-1} \big[E\big| \ln \|\Phi_{i,i+1}(\omega)\| \big|^2\big]^{\frac{1}{2}} \leq K_1, \tag{2.2}$$

where the constant K_1 is independent of N.
Similarly, by considering the backward Ito differential equation we get

$$\big(E|\eta_2(\omega)|^2\big)^{\frac{1}{2}} \leq K_2,$$

where the constant K_2 is independent of N.
It follows from (2.1) that

$$E\eta^2(\omega) \leq K_3 = 2(K_1^2 + K_2^2)$$

where the constant K_3 is independent of N. Now, it is easily seen that, for any $s \in \mathbb{R}^+, N \in \mathbb{N}$ the variable $\widetilde{\eta} = \frac{1}{N} \ln \|\Phi_{s,s+N}(\omega)\|$ has second moment bounded by a constant independent of s, N.
In case $\hat{\eta}(\omega) = \frac{1}{T} \ln \|\Phi_{s,s+T}(\omega)\|$, for arbitrary $s \in \mathbb{R}^+, T \in \mathbb{R}^+, T > 1$, we put $N = [T]$, the integer part of T, and get

$$\|\Phi_{s,s+T}(\omega)\| \leq \|\Phi_{s,s+1}(\omega)\| \dots \|\Phi_{s+N-1,s+N}(\omega)\| \, \|\Phi_{s+N,s+T}(\omega)\| .$$

By the same arguments as for deriving (2.2), we get

$$\begin{aligned}\big(E|\hat{\eta}_1(\omega)|^2\big)^{\frac{1}{2}} &\leq \frac{1}{T}\Big(\sum_{i=0}^{N-1} [E\big| \ln \|\Phi_{i,i+1}(\omega)\| \big|^2]^{\frac{1}{2}} + [E\big| \|\ln |\Phi_{N,T}(\omega)\| \big|^2]^{\frac{1}{2}} \Big) \\ &\leq \frac{(N+1)\hat{K}_1}{T} \leq 2\hat{K}_1.\end{aligned}$$

and

$$\big(E|\hat{\eta}_2(\omega)|^2\big)^{\frac{1}{2}} \leq 2\hat{K}_2,$$

where the constant $\hat{K}_1$ and $\hat{K}_2$ are independent of s, T. So

$$E\hat{\eta}(\omega)^2 \leq \hat{K}_3 = 4(\hat{K}_1^2 + \hat{K}_2^2).$$

To summarize, we have shown that there exists a positive constant M_1 independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$ such that

$$E\Big(\frac{1}{T} \ln \big\|\Phi_{iT,(i+1)T}(\omega)\big\|\Big)^2 \le M_1.$$

Similarly, we can find a constant $M_2 > 0$ independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$ such that

$$E\big(\frac{1}{T} \ln \big\|\Phi_{(i+1)T,iT}(\omega)\big\|\big)^2 \le M_2.$$

Fix a $k \in \{1, 2, ..., n\}$. Since

$$0 < d_n[\Phi_{iT,(i+1)T}(\omega)] \le d_k[\Phi_{iT,(i+1)T}(\omega)] \le d_1[\Phi_{iT,(i+1)T}(\omega)],$$

we have

$$\big(\frac{1}{T} \ln d_k[\Phi_{iT,(i+1)T}(\omega)]\big)^2 \le \big(\frac{1}{T} \ln \big\|\Phi_{iT,(i+1)T}(\omega)\big\|\big)^2 + \big(\frac{1}{T} \ln \big\|\Phi_{(i+1)T,iT}(\omega)\big\|\big)^2.$$

Consequently,

$$E\big(\frac{1}{T} \ln d_k[\Phi_{iT,(i+1)T}(\omega)]\big)^2 \le M_1 + M_2.$$

Hence, $\frac{1}{T} \ln d_k[\Phi_{iT,(i+1)T}(\omega)]$ $(i \in \{0, 1, 2, ...\})$ is a sequence of independent random variables having second moments bounded by $M_1 + M_2$. By virtue of the Kolmogorov strong law of large numbers (see Shiryaev [18, Theorem 2, page 389]), the following equality holds with probability 1

$$\lim_{m\to+\infty} \Big(\frac{1}{mT} \sum_{i=0}^{m-1} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] - \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E}\ \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\Big) = 0.$$

Consequently,

$$\limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] = \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E}\ \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big],$$

hence,

$$\gamma_k(\omega) = \limsup_{T\to+\infty} \gamma_k(T).$$

The theorem is proved. □

Theorem 2.2. *For any $k \in \{1, 2, ..., n\}$ the central exponent $\Theta_k(\omega)$ of* (1.1) *does not depend on $\omega \in \Omega$.*

Proof. Denote by $\{\mathcal{F}_s^t\}_{t\ge s\ge 0}$ the filtration of σ-algebras generated by the Wiener processes $(W_u^1, W_u^2, ..., W_u^m)_{t\ge u\ge s}$ (see e.g. Arnold [1, pages 91-92]). Clearly, the $\Phi_{0,t}(\omega)$ is adapted to the filtration $\{\mathcal{F}_0^t\}_{t\ge 0}$. From the formula (1.5), it follows that the random variable $\Theta_k(\omega)$ is measurable with respect to the limit σ-algebra

$\mathcal{F}_0^{+\infty} := \lim_{t\to+\infty} \mathcal{F}_0^t = \bigvee_{t\geq 0} \mathcal{F}_0^t$. We note that for any fixed $k \in \{1, 2, ..., n\}$ and $N \in \mathbb{N}$,

$$\begin{aligned}\Theta_k(\omega) &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\ &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \Bigg(\sum_{i=0}^{N} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\ &\qquad + \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \Bigg).\end{aligned}$$

Since N is a fixed number, $\sum_{i=0}^{N} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$ is a random variable with finite second moment, hence, the limit

$$\lim_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{N} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} = 0$$

exists and the equality holds with probability 1. Therefore,

$$\begin{aligned}\Theta_k(\omega) &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\ &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\left(\Phi_{0,NT}(\omega)V\right)} \right\|^{-1} \\ &\leq \sup_{\widetilde{V}\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\widetilde{V}} \right\|^{-1} =: r(\omega).\end{aligned}$$

On the other hand, by the definition of $r(\omega)$ just given above, for any $\epsilon > 0$ and $\omega \in \Omega$, there exists $\widetilde{V}_1 \in G_k$ such that

$$r(\omega) - \epsilon < \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\widetilde{V}_1} \right\|^{-1}.$$

Let $\widetilde{V}_2 \in G_k$ denote the subspace $\Phi_{0,NT}^{-1}(\omega)\widetilde{V}_1$. We have $\Phi_{0,NT}(\omega)\widetilde{V}_2 = \widetilde{V}_1$, hence,

$$\begin{aligned}r(\omega) - \epsilon &< \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{NT,(i+1)T}(\omega)\Phi_{0,NT}\widetilde{V}_2} \right\|^{-1} \\ &= \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2} \right\|^{-1} \\ &= \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \Bigg(\sum_{i=0}^{N} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2} \right\|^{-1}\end{aligned}$$

$$+\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}_2}\right\|^{-1}\Bigg)$$
$$\le \sup_{V\in G_k}\sup_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V}\right\|^{-1}$$
$$=\Theta_k(\omega).$$

Since $\epsilon>0$ is arbitrary so we have $r(\omega)\le\Theta_k(\omega)$. Thus, for any fixed N we have $\Theta_k(\omega)=r(\omega)\in\mathcal{F}^{+\infty}_{(N+1)T}$ so $\Theta_k(\omega)$ is measurable with respect to the $\mathcal{F}^{+\infty}_{(N+1)T}=\lim_{t\to+\infty}\mathcal{F}^{t}_{(N+1)T}=\bigvee_{t\ge(N+1)T}\mathcal{F}^{t}_{(N+1)T}$. Hence, $\Theta_k(\omega)$ is measurable with respect to the tail σ-algebra $\bigcap_{N=1}^{+\infty}\mathcal{F}^{+\infty}_{(N+1)T}$. By the zero-or-one law (see Shiryaev [18, page 381]) the random variable $\Theta_k(\omega)$ is degenerate, i.e. nonrandom. □

Theorem 2.3. *For any $k\in\{1,2,...,n\}$ the central exponents $\Omega_k(\omega)$ of* (1.1) *does not depend on $\omega\in\Omega$.*

Proof. We note that for any fixed $k\in\{1,2,...,n\}$ and $N\in\mathbb{N}$,

$$\Omega_k(\omega)=\inf_{U\in G_{n-k+1}}\inf_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U}\right\|$$
$$=\inf_{U\in G_{n-k+1}}\inf_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\Bigg(\sum_{i=0}^{N}\ln\left\|\Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U}\right\|$$
$$+\sum_{i=N+1}^{m-1}\ln\left\|\Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U}\right\|\Bigg).$$

Using an argument similar to that of the proof of Theorem 2.2, we can show that $\Omega_k(\omega)$ is measurable with respect to the tail σ-algebra $\bigcap_{N=1}^{+\infty}\mathcal{F}^{+\infty}_{(N+1)T}$, hence, is degenerate, i.e. nonrandom. □

Now, since the Lyapunov, central and auxiliary exponents are independent of ω, we will drop ω in their notations.

Theorem 2.4. *For any $k\in\{1,2,...,n\}$ the central exponent Ω_k of* (1.1) *is larger or equal to the Lyapunov exponent λ_k.*

Proof. Fixing an $\omega\in\Omega$, for any $\epsilon>0$, by the definition of Ω_k, there exists an $U\in G_{n-k+1}$ such that

$$\inf_{T>1}\limsup_{m\to+\infty}\frac{1}{mT}\sum_{i=0}^{m-1}\ln\left\|\Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U}\right\|<\Omega_k+\epsilon.$$

For any vector $x\in U$ and $T>1$, we have

$$\|\Phi_{0,mT}(\omega)x\|=\left\|\Phi_{(m-1)T,mT}(\omega)\circ\Phi_{0,(m-1)T}(\omega)x\right\|$$

$$\le \left\| \Phi_{(m-1)T,mT}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| \cdot \left\| \Phi_{0,(m-1)T}(\omega)x \right\| \le ...$$
$$\le \left\| \Phi_{(m-1)T,mT}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| ... \left\| \Phi_{0,T}(\omega)|_U \right\| \|x\| .$$

Therefore,

$$\limsup_{m\to+\infty} \frac{1}{mT} \ln \|\Phi_{0,mT}(\omega)x\| \le \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| .$$

By Theorem 3.4 of N. D. Cong [8], for any $h \in \mathbb{R}^+$, we have

$$\lambda_k(\omega) = \min_{\tilde{U}\in G_{n-k+1}} \max_{x\in\tilde{U}} \limsup_{\substack{m\to+\infty \\ m\in\mathbb{N}}} \frac{1}{mh} \ln \|\Phi_{0,mh}(\omega)x\| .$$

Therefore,

$$\lambda_k \le \inf_{T>1} \max_{x\in U} \limsup_{m\to+\infty} \frac{1}{mT} \ln \|\Phi_{0,mT}(\omega)x\|$$
$$\le \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{iT,(i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)U} \right\| < \Omega_k + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we derive $\lambda_k \le \Omega_k$. □

Theorem 2.5. *For any $k \in \{1, 2, ..., n\}$ the central exponent Θ_k of (1.1) is smaller or equal to the Lyapunov exponent λ_k.*

Proof. Fixing an $\omega \in \Omega$, for any $\epsilon > 0$, by the definition of Θ_k, there exists an $V \in G_k$ such that

$$\sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} > \Theta_k - \epsilon.$$

By the definition of λ_k and Theorem 3.4 of N. D. Cong [8], there exists an $U \in G_{n-k+1}$ such that for any $h \in \mathbb{R}^+$

$$\lambda_k = \max_{x\in U} \limsup_{m\to+\infty} \frac{1}{mh} \ln \|\Phi_{0,mh}(\omega)x\| .$$

Since U, V are linear subspaces in $\mathbb{R}^n$, the dimension of U is $n - k + 1$ and the dimension of V is k, the dimension of the subspace $U \cap V$ is larger or equal to 1. Take an $x_0 \in U \cap V \setminus \{0\}$ we have

$$\sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}$$
$$\le \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \ln \frac{\|\Phi_{0,mT}(\omega)x_0\|}{\|x_0\|}$$
$$\le \sup_{T>1} \max_{x\in U_*} \limsup_{m\to+\infty} \frac{1}{mT} \ln \frac{\|\Phi_{0,mT}(\omega)x\|}{\|x\|}$$
$$= \lambda_k.$$

Therefore, $\Theta_k - \epsilon < \lambda_k$ for every $\epsilon > 0$, hence, $\Theta_k \le \lambda_k$. □

Theorem 2.6. *For any $k \in \{1, 2, ..., n\}$ the central exponent Θ_k of (1.1) is smaller or equal to the auxiliary exponent γ_k.*

Proof. First of all, as claimed in the Introduction, we will show that for definition of $\Theta_k(\omega)$ the formula (1.3) is equivalent to the formula (1.5). For $V_1 \in G_k, T \in \mathbb{R}^+, \omega \in \Omega, m \in \mathbb{N}$ we set

$$g(T, m, V_1) := \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V_1} \right\|^{-1}.$$

Then

$$\begin{aligned}
g(T, 2m, V_1) &= \frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V_1} \right\|^{-1} \\
&= \frac{1}{2mT} \sum_{i=0}^{2m-1} \ln \inf_{z \in V_1*} \frac{\left\|\Phi_{0,(i+1)T}(\omega)z\right\|}{\|\Phi_{0,iT}(\omega)z\|} \\
&= \frac{1}{2mT} \sum_{i=0}^{m-1} \left(\ln \inf_{z \in V_1*} \frac{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} + \ln \inf_{z \in V_1*} \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|} \right) \\
&\le \frac{1}{2mT} \sum_{i=0}^{m-1} \ln \inf_{z \in V_1*} \frac{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} \times \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\left\|\Phi_{0,(2i+1)T}(\omega)z\right\|} \\
&= \frac{1}{2mT} \sum_{i=0}^{m-1} \ln \inf_{z \in V_1*} \frac{\left\|\Phi_{0,(2i+2)T}(\omega)z\right\|}{\|\Phi_{0,2iT}(\omega)z\|} \\
&= \frac{1}{2mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)2T,i2T}(\omega)|_{\Phi_{0,(i+1)2T}(\omega)V_1} \right\|^{-1} = g(2T, m, V_1).
\end{aligned}$$

Thus, for all $V_1 \in G_k, T \in \mathbb{R}^+, \omega \in \Omega, m \in \mathbb{N}$, we have

$$g(T, 2m, V_1) \le g(2T, m, V_1). \tag{2.3}$$

Now we will prove that for any fixed $V_1 \in G_k, T \in \mathbb{R}^+$ the following equality

$$\limsup_{m \to +\infty} g(T, 2m, V_1) = \limsup_{m \to +\infty} g(T, m, V_1) \tag{2.4}$$

holds with probability 1. Note that

$$\frac{m+1}{m} g(T, m+1, V_1) - g(T, m, V_1) = \frac{1}{mT} \ln \left\| \Phi_{(m+1)T,mT}(\omega)|_{\Phi_{0,(m+1)T}(\omega)V_1} \right\|^{-1},$$

and

$$-\frac{1}{mT} \ln \left\| \Phi^{-1}_{mT,(m+1)T}(\omega) \right\| \le \frac{1}{mT} \ln \left\| \Phi_{(m+1)T,mT}(\omega)|_{\Phi_{0,(m+1)T}(\omega)V_1} \right\|^{-1}$$

$$\le \frac{1}{mT} \ln \left\| \Phi_{mT,(m+1)T}(\omega) \right\|. \tag{2.5}$$

Put

$$\begin{aligned} B_j &:= \left\{\omega \in \Omega |\ \left\|\Phi_{jT,(j+1)T}(\omega)\right\| \geq jT + n^2 e^{KT}\right\}, \quad j \in \{0,1,2...\}, \\ \tilde{B}_j &:= \left\{\omega \in \Omega |\ \left\|\Phi_{(j+1)T,jT}(\omega)\right\| \geq jT + n^2 e^{KT}\right\}, \quad j \in \{0,1,2...\}, \\ B &:= \bigcup_{i=1}^{+\infty} \bigcap_{j=i}^{+\infty} (\Omega \setminus (B_j \cup \tilde{B}_j)). \end{aligned}$$

By Lemma 3.3 of N. D. Cong [8] we have $\mathbb{P}(B) = 1$. Let $\omega \in B$ be arbitrary, then there exists $M(\omega) > 0$ such that for all $m > M(\omega)$ the following inequalities hold

$$\frac{1}{mT} \ln \left\|\Phi_{mT,(m+1)T}(\omega)\right\| \leq \frac{\ln(mT + n^2 e^{KT})}{mT}, \tag{2.6}$$

$$-\frac{1}{mT} \ln \left\|\Phi^{-1}_{mT,(m+1)T}(\omega)\right\| \geq -\frac{\ln(mT + n^2 e^{KT})}{mT}. \tag{2.7}$$

From (2.5), (2.6) and (2.7), it follows that

$$\lim_{m\to+\infty} \Big(\frac{m+1}{m} g(T, m+1, V_1) - g(T, m, V_1)\Big) = 0$$

with probability 1, in particular,

$$\lim_{l\to+\infty} \Big(\frac{2l+1}{2l} g(T, 2l+1, V_1) - g(T, 2l, V_1)\Big) = 0.$$

Therefore, since $\lim_{l\to+\infty} \frac{2l+1}{2l} = 1$, with probability 1, we have

$$\limsup_{l\to+\infty} g(T, 2l+1, V_1) = \limsup_{l\to+\infty} g(T, 2l, V_1),$$

from which (2.4) follows.
By (2.3) and (2.4), for all $V_1 \in G_k, T \in \mathbb{R}^+$ with probability 1, we have

$$\limsup_{m\to+\infty} g(T, m, V_1) \leq \limsup_{m\to+\infty} g(2T, m, V_1).$$

From this, it follows that the formula (1.3) is equivalent to the formula (1.5). Moreover, this inequality also implies that

$$\sup_{T>1} \limsup_{m\to+\infty} g(T, m, V_1) = \limsup_{T\to+\infty} \limsup_{m\to+\infty} g(T, m, V_1).$$

Therefore, taking into account Theorem 2.2, we have

$$\begin{aligned} \Theta_k &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\ &= \sup_{V\in G_k} \sup_{T>1} \limsup_{m\to+\infty} g(T, m, V) \\ &= \sup_{V\in G_k} \limsup_{T\to+\infty} \limsup_{m\to+\infty} g(T, m, V) \\ &= \sup_{V\in G_k} \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T,iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{V\in G_k} \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \inf_{z\in\Phi_{0,iT}(\omega)V_*} \frac{\left\|\Phi_{iT,(i+1)T}(\omega)z\right\|}{\|z\|}\\
&\le \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln\, d_k\left[\Phi_{iT,(i+1)T}(\omega)\right]\\
&= \limsup_{T\to+\infty} \gamma_k(T) = \gamma_k.
\end{aligned}
$$

The theorem is proved. □

Theorem 2.7. *For the equation* (1.1) *we always have*

$$\gamma_1 \ge \Omega_1 \quad \textit{and} \quad \gamma_n = \Theta_n.$$

Proof. Note that

$$
\begin{aligned}
d_1[\Phi_{iT,(i+1)T}(\omega)] &= \left\|\Phi_{iT,(i+1)T}(\omega)\right\|,\\
d_n[\Phi_{iT,(i+1)T}(\omega)] &= \left\|\Phi_{iT,(i+1)T}(\omega)^{-1}\right\|^{-1}.
\end{aligned}
$$

Since the space G_n has only one point $\mathbb{R}^n$, using (1.3) and the argument in the proof of Theorem 2.6, we get

$$
\begin{aligned}
\Theta_n &= \sup_{T\in\mathbb{R}^+} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln d_n[\Phi_{iT,(i+1)T}(\omega)]\\
&= \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln d_n[\Phi_{iT,(i+1)T}(\omega)] = \gamma_n.
\end{aligned}
$$

Using an argument similar to that of the proof of Θ_k in Theorem 2.6, we can prove for any subspace $U \in G_{n-k+1}$,

$$
\begin{aligned}
\Omega_k &= \inf_{U\in G_{n-k+1}} \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \left\|\Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U}\right\|\\
&= \inf_{U\in G_{n-k+1}} \liminf_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \left\|\Phi_{iT,(i+1)T}(\omega)\big|_{\Phi_{0,iT}(\omega)U}\right\|.
\end{aligned}
$$

So

$$
\begin{aligned}
\Omega_1 &= \inf_{T>1} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \left\|\Phi_{iT,(i+1)T}(\omega)\right\|\\
&= \liminf_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \left\|\Phi_{iT,(i+1)T}(\omega)\right\|\\
&\le \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln \left\|\Phi_{iT,(i+1)T}(\omega)\right\|
\end{aligned}
$$

$$
\begin{aligned}
&= \limsup_{T\to+\infty} \limsup_{m\to+\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln d_1[\Phi_{iT,(i+1)T}(\omega)] \\
&= \gamma_1.
\end{aligned}
$$

The theorem is proved. □

3. Lyapunov exponents of nondegenerate stochastic differential equations coincide with central exponents

In the whole of this section, we will assume the *nondegeneracy condition* (1.10) for the equation (1.1).

Proposition 3.1. *For any $\epsilon > 0$ one can find $0 < \delta = \delta(\epsilon) < 1$ such that for any $V \in G_k$ and $U \in G_{n-k}$ $(k \in \{1, 2, ..., n-1\})$ and any $\tau \in \mathbb{R}^+$ the set of $\omega \in \Omega$, for which*

$$
\left[\Phi_{\tau,\tau+1}(\omega)V\right] \cap \hat{U}(\delta(\epsilon)) \not\equiv \{0\}
$$

has $\mathbb{P}$-measure $\le \epsilon$, where $\hat{U}(\varrho)$ denotes the cone consisting of vetors in $\mathbb{R}^n$ which make an angle $\le \varrho$ with the subspace U.

Proof. The proof of this proposition is completely similar to the proof of the Lemma 2 in N. D. Cong [5] and Theorem in N. D. Cong [7]. □

Theorem 3.2. *There exists a positive constant c_1 such that for any $\epsilon \in (0, 1)$, $T > 1$ and $k \in \{1, 2, \cdots, n\}$ the following inequality holds*

$$
\Theta_k \ge \gamma_k(T) + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}, \tag{3.1}
$$

where $\delta = \delta(\epsilon)$ is determined according to Proposition 3.1.

Proof. Let $\epsilon \in (0, 1)$ and fix $k \in \{1, 2, ..., n\}$. Determine $\delta = \delta(\epsilon)$ from ϵ according to Proposition 3.1. Fix an arbitrary $T > 1$ and an arbitrary k-dimentional linear subspace V of $\mathbb{R}^n$. Let $i \in \{0, 1, 2, ...\}$, for brevity in expression, let Φ_i denotes the matrix $\Phi_{iT,(i+1)T}(\omega)$. Denote by $\{f_1, ..., f_k, f_{k+1}, ..., f_n\}$ such the eigenvectors, corresponding to the eigenvalues

$$
d_1^2(\Phi_i) \ge \cdots \ge d_k^2(\Phi_i) \ge d_{k+1}^2(\Phi_i) \ge \cdots \ge d_n^2(\Phi_i)
$$

of the matrix $\Phi_i^*\Phi_i$, that they depend measurably on ω and form an orthonormal basis of $\mathbb{R}^n$ (for the existence of such a measurable orthonormal basis see, e.g., Arnold [1, pages 196-197]). Furthermore, we denote by $U_{i,\omega}^{n-k}$ the linear subspace spanned by the last $n-k$ eigenvectors $\{f_{k+1}, ..., f_n\}$ of $\Phi_i^*\Phi_i$. We introduce some notations

$$
\begin{aligned}
C_i &:= \left\{\omega \in \Omega : \left[\Phi_{0,iT}(\omega)V\right] \cap \hat{U}_{i,\omega}^{n-k}\left[\delta(\epsilon)\right] \not\equiv \{0\} \right\}, \\
\eta_i(\omega) &:= \frac{1}{T} \ln \left\|\Phi_{(i+1)T,iT}(\omega)\right\|, \\
\zeta_i(\omega) &:= \inf_{y\in V_*} \frac{1}{T} \ln \frac{\left\|\Phi_{0,(i+1)T}(\omega)y\right\|}{\|\Phi_{0,iT}(\omega)y\|}.
\end{aligned}
$$

We have shown in the proof of Theorem 2.1 that for the equation (1.1) the random variables $\eta_i(\omega)$ have second moments, bounded by a constant independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$. Let $\chi_i(\omega)$ denote the indicator function of the set C_i. By Proposition 3.1 and the Markov property of the solutions of the systems (1.1) we have

$$\mathbb{P}(C_i) \leq \epsilon, \qquad E(\chi_i(\omega)) \leq \epsilon.$$

By the definitions of C_i and χ_i, if $\chi_i(\omega) = 0$, then any vector of $\Phi_{0,iT}(\omega)V$ is separated from $U^{n-k}_{i,\omega}$ by an angle bigger than $\delta(\epsilon)$, hence,

$$\begin{aligned}
\zeta_i(\omega) &= \inf_{z \in \Phi_{0,iT}(\omega)V_*} \frac{1}{T} \ln \frac{\|\Phi_i z\|}{\|z\|} \\
&\geq \inf_{z \in \Phi_{0,iT}(\omega)V_*} \frac{1}{T} \ln \Big(d_k(\Phi_i) \sin \angle(z, U^{n-k}_{i,\omega}) \Big) \\
&\geq \frac{1}{T} \ln \Big(d_k(\Phi_i) \sin \big[\delta(\epsilon)\big] \Big) \\
&\geq \frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2}.
\end{aligned}$$

If $\chi_i(\omega) = 1$ then

$$\zeta_i(\omega) \geq \inf_{z \in V_*} \frac{1}{T} \ln \frac{\big\|\Phi_{iT,(i+1)T}(\omega) z\big\|}{\|z\|} \geq -\frac{1}{T} \ln \Big\|\Phi^{-1}_{iT,(i+1)T}(\omega)\Big\| = -\eta_i(\omega).$$

Consequently,

$$\begin{aligned}
\zeta_i(\omega) &\geq \big[1 - \chi_i(\omega)\big] \left(\frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} \right) - \chi_i(\omega)\eta_i(\omega) \\
&\geq \frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} \\
&\quad - \chi_i(\omega)\frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] - \chi_i(\omega)\eta_i(\omega).
\end{aligned}$$

It is easily seen that, the random variables $\frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]$, $\zeta_i(\omega)$ have second moments bounded by a constant independent of $T > 1$, $i \in \{0, 1, 2, ...\}$ and $\epsilon \in (0, 1)$. Thererefore, there exists a positive constant $c_1 > 0$ which is independent of $T > 1$, $i \in \{0, 1, 2, ...\}$ and $\epsilon \in (0, 1)$ such that

$$\begin{aligned}
E\left|\chi_i(\omega)\frac{1}{T} \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\right| &\leq \frac{1}{T}\Big(E\chi_i^2(\omega)\Big)^{\frac{1}{2}} \Big(\mathrm{E} \ln^2 d_k\big[\Phi_{iT,(i+1)T}(\omega)\big]\Big)^{\frac{1}{2}} \\
&\leq c_1 \Big(\int_\Omega \chi_i^2(\omega) d\mathbb{P}\Big)^{\frac{1}{2}} = c_1 \mathbb{P}\Big(\big\{\omega \,\big|\, \chi_i(\omega) = 1\big\}\Big)^{\frac{1}{2}} \\
&\leq c_1 \sqrt{\epsilon},
\end{aligned}$$

and

$$E\big|\chi_i(\omega)\eta_i(\omega)\big| \leq \big[E\chi_i^2(\omega)\big]^{\frac{1}{2}} \big[E\eta_i^2(\omega)\big]^{\frac{1}{2}} \leq c_1 \sqrt{\epsilon}.$$

Consequently,

$$E\zeta_i(\omega) \ge \frac{1}{T} E \ln d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}. \tag{3.2}$$

Now we use results by Rosenblatt-Roth [17] to prove that the sequence of random variables $\zeta_i(\omega)$, $i = 1, 2, \ldots$, satisfies the strong law of large numbers. This is a crucial argument in the proof of this theorem. Note that the random variables $\zeta_i(\omega)$, $i = 1, 2, \ldots$, are not independent.

To this end, we define a Markov chain in the state space $G_k \times Gl(n, \mathbb{R})$ with the Borel σ-algebra using the fundamental matrix of the equation (1.1) as follows:

Our Markov chain starts (at time $\tau = 0$) from the state $(V, I) \in G_k \times Gl(n, \mathbb{R})$. From the state $(V_1, Y_1) \in G_k \times Gl(n, \mathbb{R})$ at time $\tau = iT$ it goes to the state $(V_2, Y_2) \in G_k \times Gl(n, \mathbb{R})$ next time $\tau = (i + 1)T$ by the rule $V_2 = \Phi_{iT,(i+1)T}(\omega)V_1$, $Y_2 = \Phi_{(i+1)T,(i+2)T}(\omega)$.

Note that the second coordinate of our chain is a sequence of independent random variables, and the first coordinate is a Markov chain on the compact state space G_k generated by the solutions of the equation (1.1). The transition probability of our Markov chain is the product of transition probabilities on two coordinates because the second coordinate is independent on the present and past of the first coordinate. Denote by μ the Riemannian volume on the compact space G_k, and $P_i(V_1, B_1)$, where $V_1 \in G_k$ is a point and $B_1 \subset G_k$ is a measurable subset of G_k, the transition probability of the Markov chain of the first coordinate of our chain at the time moment $\tau = iT$. This Markov chain on G_k has density satisfying a parabolic partial equation which is determined by the equation (1.1) (see Khasminskii [11, page 96]). Since G_k is a compact manifold and our nondegeneracy condition (1.10) is uniform with respect to time, we can find positive constants $K_3, K_4 > 0$ (see Aronson [2, page 891]) such that for any $i = 1, 2, \ldots$, any $V_1 \in G_k$ and any measurable subset $B \subset G_k$ we have

$$K_3\mu(B) \le P_i(V_1, B) \le K_4\mu(B), \tag{3.3}$$

where the constants K_3, K_4 depend only on n, T, μ_1, μ_2 and the Lipschitz constant K of the equation (1.1). From this, it follows that for any $i = 1, 2, \ldots$, any pair of points $V_1, V_2 \in G_k$ and any measurable subset $B \subset G_k$ we have

$$|P_i(V_1, B) - P_i(V_2, B)| \le \frac{K_4}{K_3 + K_4}.$$

Therefore,

$$\sup |P_i(V_1, B) - P_i(V_2, B)| \le \frac{K_4}{K_3 + K_4}, \tag{3.4}$$

where the sup is taken over the collections of all points $V_1, V_2 \in G_k$ and all measurable set $B \subset G_k$. Since the transition probability of our Markov chain on the product space $G_k \times Gl(n, \mathbb{R})$ is the product of two transition probabilities on its coordinates, it is easily seen that, the ergodic coefficient α_i of the transition function P_i of our Markov chain (see Dobrushin [9] and Rosenblatt-Roth [17] for definition of ergodic coefficient and its properties) satisfies for any $i = 1, 2, \dots$ the inequality

$$\alpha_i = \alpha(P_i) \geq \frac{K_3}{K_3 + K_4}. \tag{3.5}$$

Thus, for any $m \in \{1, 2, 3, ...\}$ we have

$$\alpha^{(m)} := \min_{0 \leq i \leq m-1} \alpha_i \geq \frac{K_3}{K_3 + K_4} > 0.$$

Let us come back to the random variables $\zeta_i(\omega)$ introduced above. We can consider them as random variables defined on our Markov chain as follows:

$$\zeta_i(\omega_i) = \zeta_i(V_i, \Phi_i) = \inf_{z \in V_i^*} \frac{1}{T} \ln \frac{\|\Phi_i(z)\|}{\|z\|}.$$

We know that ζ_i has second moments bounded by a constant c_2 independent of $T > 1$ and $i \in \{0, 1, 2, ...\}$, hence,

$$0 \leq D\zeta_i \leq E|\zeta_i|^2 \leq c_2,$$

where $D\xi(\omega)$ denotes the variance of the random variable $\xi(\omega)$. This implies that

$$\sum_{n=1}^{+\infty} n^{-2} D\zeta_n < c_2 \sum_{n=1}^{+\infty} n^{-2} < +\infty.$$

Therefore, according to Rosenblatt-Roth [17, Theorem 2, page 567] the sequence $\zeta_0, \zeta_1, \zeta_2, ...$ satisfies the strong law of large numbers, so we have with probability 1 the equalities

$$\begin{aligned}
&\limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T, iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1} \\
=& \limsup_{m \to +\infty} \frac{1}{m} \sum_{i=0}^{m-1} \zeta_i(\omega) \\
=& \limsup_{m \to +\infty} \frac{1}{m} \sum_{i=0}^{m-1} E\zeta_i(\omega) \\
=& \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \mathrm{E} \ln \left\| \Phi_{(i+1)T, iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}.
\end{aligned}$$

Using the definition of the central exponent Θ_k, we get

$$\begin{aligned}
\Theta_k =& \sup_{\widetilde{V} \in G_k} \sup_{T>1} \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T, iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)\widetilde{V}} \right\|^{-1} \\
\geq& \limsup_{m \to +\infty} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \left\| \Phi_{(i+1)T, iT}(\omega)|_{\Phi_{0,(i+1)T}(\omega)V} \right\|^{-1}
\end{aligned}$$

$$= \limsup_{m\to+\infty} \frac{1}{m}\sum_{i=0}^{m-1} \zeta_i(\omega) = \limsup_{m\to+\infty} \frac{1}{m}\sum_{i=0}^{m-1} E(\zeta_i(\omega))$$
$$\ge \limsup_{m\to+\infty} \frac{1}{m}\sum_{i=0}^{m-1} \left(\frac{1}{T} E\, \ln\, d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}\right)$$
$$= \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} E\, \ln\, d_k\big[\Phi_{iT,(i+1)T}(\omega)\big] + \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}.$$

Consequently, with probability 1, we have

$$\Theta_k \ge \gamma_k(T) + \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon}. \tag{3.6}$$

The theorem is proved. □

Theorem 3.3. *Assume the condition* (1.10)*, then for any* $k \in \{1, 2, ..., n\}$*, we have*

$$\gamma_k = \Theta_k.$$

Proof. Fix $k \in \{1, 2, ..., n\}$. Taking into account Theorem 2.6, it suffices to prove that $\Theta_k \ge \gamma_k$. Due to Theorem 2.1, we only need to prove $\Theta_k \ge \limsup_{T\to+\infty} \gamma_k(T)$. To do this, we will show that for any $\varrho > 0$, there exists $T_\varrho^{(1)} > 1$ such that, for any $T > T_\varrho^{(1)}$,

$$\gamma_k(T) < \Theta_k + \varrho.$$

By Theorem 3.2, for any $\epsilon \in (0, 1)$ and $T > 1$, we have

$$\gamma_k(T) + \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon} \le \Theta_k,$$

where $\delta = \delta(\epsilon)$ is specified as in Proposition 3.1. Fix an arbitrary $0 < \epsilon < \frac{\varrho^2}{16c_1^2}$. Since $\lim_{T\to+\infty} \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} = 0$, for any $\varrho > 0$ there exists $T_\varrho^{(2)} > 1$ such that for any $T > T_\varrho^{(2)}$ we have

$$-\frac{\varrho}{2} < \frac{1}{T} \ln \frac{\delta(\epsilon)}{2} < \frac{\varrho}{2},$$

which implies, for any $T > T_\varrho^{(2)}$,

$$\gamma_k(T) - \frac{\varrho}{2} - 2c_1\sqrt{\epsilon} < \gamma_k(T) + \frac{1}{T}\, \ln\, \frac{\delta(\epsilon)}{2} - 2c_1\sqrt{\epsilon} \le \Theta_k.$$

Thus, taking into account the choice $0 < \epsilon < \min\left\{1, \frac{\varrho^2}{16c_1^2}\right\}$, we have that for any $\varrho > 0$, there exists $T_\varrho = \max\left\{T_\varrho^{(1)}, T_\varrho^{(2)}\right\}$ such that for any $T > T_\varrho$

$$\gamma_k(T) < \Theta_k + \frac{\varrho}{2} + 2c_1 \frac{\varrho}{4c_1} < \Theta_k + \varrho.$$

Since $\varrho > 0$ is arbitrary, we obtain $\limsup_{T\to+\infty} \gamma_k(T) \le \Theta_k$. □

Theorem 3.4. *Assume the condition* (1.10) *holds. Then the following equalitities hold*

$$\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1.$$

Proof. By Theorem 2.7, Theorem 2.4 and Theorem 2.5, we have

$$\gamma_1 \geq \Omega_1 \geq \lambda_1 \geq \Theta_1.$$

Due to assumption of the condition (1.10), Theorem 3.3 implies that

$$\Theta_1 = \lambda_1 = \Omega_1 = \gamma_1.$$

The theorem is proved. □

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