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ITERATED POLYHARMONIC GREEN FUNCTIONS FOR PLANE DOMAINS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday in respect of his work and fate

Abstract. Convoluting harmonic Green, Neumann, and Robin functions leads to certain hybrid polyharmonic Green functions and related boundary value problems for partial differential equations with some power of the Laplacian as leading operator. This paper is a survey about hybrid polyharmonic Green functions for plane domains and related results.

1. Introduction

Fundamental solutions to higher order partial differential operators can be found in a natural way through continued construction of primitives to fundamental solutions of lower order ones. In this manner, from the fundamental solution $-\frac{1}{\pi z}$ of the Cauchy-Riemann operator $\partial_{\overline{z}}$ a fundamental solution to the polyanalytic operator $\partial_{\overline{z}}^k$ is seen to be $-\frac{\overline{z}^{k-1}}{\pi(k-1)!z}$. Also, $-\frac{1}{\pi}\log|z|^2$ turns out as a fundamental solution to the Laplace operator $\partial_z\partial_{\overline{z}}$. From here $-\frac{z^{k-1}\overline{z}^{l-1}}{\pi(k-1)!(l-1)!}\log|z|^2$ appears to be a fundamental solution to the operator $\partial_z^k\partial_{\overline{z}}^l$ and, in particular, $-\frac{|z|^{2(k-1)}}{\pi(k-1)!^2}\log|z|^2$ a fundamental solution to the polyharmonic operator $(\partial_z\partial_{\overline{z}})^k$. By the way, as fundamental solutions are only defined modulo the kernel of the operator it sometimes is appropriate to use $-\frac{z^{k-1}\overline{z}^{l-1}}{\pi(k-1)!(l-1)!}[\log|z|^2-\sum\limits_{m=1}^{k-1}\frac{1}{m}-\sum\limits_{n=1}^{l-1}\frac{1}{n}]$ as a fundamental solution to $\partial_z^k\partial_{\overline{z}}^l$, see [23]. In context to boundary value problems fundamental solutions are adjusted to certain boundary conditions. For the well-known case of the Laplace operator such fundamental solutions are the Green, the Neumann, and the Robin functions. These harmonic kernel functions are known to exist for quite general domains. They are discussed in many textbooks for partial differential equations, e.g. [40, 42, 43]. For particular domains, they are expressed in explicit form [8, 10, 11, 16, 24, 26, 28, 30, 31, 43].

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The general complex model partial differential operator $\partial_z^k \partial_{\overline{z}}^l$ can be written as a product of a polyanalytic operator $\partial_{\overline{z}}^m$ or its complex conjugate ∂_z^m and an polyanalytic operator $(\partial_z \partial_{\overline{z}})^n$. Boundary value problems to such general model equations are investigated in [39, 44, 45] while boundary value problems for the polyanalytic operator are solved in [1, 2, 8, 20, 25, 38, 52–57]. Certain classes of general linear complex higher order equations are treated in [2–5, 22].

2. Iterated polyharmonic Green functions

Recently, some particular polyharmonic Green functions were constructed via an iteration procedure. In case of the polyharmonic operator $(\partial_z \partial_{\overline{z}})^n$ of order n, there are n boundary data to be described in order to get a well-posed boundary value problem for the n-th order Poisson equation $(\partial_z \partial_{\overline{z}})^n w = f$. Obviously, the theory of related polyharmonic Green functions is quite involved. The polyharmonic Dirichlet problem

$$\partial_\nu^\mu w = \gamma_\mu, 0 \leq \mu \leq n-1 \text{ on } \partial D, \tag{2.1}$$

the polyharmonic Neumann problem

$$\partial_\nu^\mu w = \gamma_\mu, 1 \leq \mu \leq n \text{ on } \partial D, \tag{2.2}$$

where ∂_ν denotes the outward normal derivative on the boundary ∂D of the regular domain D under consideration, the polyharmonic Riquier problem, [46],

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \leq \mu \leq n-1, \text{ on } \partial D, \tag{2.3}$$

the problem

$$\partial_\nu (\partial_z \partial_{\overline{z}})^\mu = \gamma_\mu, 0 \leq \mu \leq n-1, \text{ on } \partial D, \tag{2.4}$$

and the problem

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_{0\mu}, 0 \leq 2\mu \leq n-1, \partial_\nu (\partial_z \partial_{\overline{z}})^\mu w = \gamma_{1\mu}, 0 \leq 2\mu \leq n-2, \text{ on } \partial D, \tag{2.5}$$

are possible boundary value problems for the n-Poisson equation, see e.g. [9, 14, 15, 17, 26–28, 32, 35, 36, 41, 44, 46, 47]. Naturally, there are many other proper boundary conditions.

The polyharmonic Green-Almansi function, given in explicit form for the unit disc, see [7, 9, 49] and for the upper half plane, see [41], is adjusted to problem (2.5), see [9,14,35,41]. As the Riquier problem (2.4) can be viewed as the iterated form of the Dirichlet problem for the Poisson equation

$$\partial_z \partial_{\overline{z}} w_\mu = w_{\mu+1} \text{ in } D, \ w_\mu = \gamma_\mu \text{ on } \partial D, 0 \leq \mu \leq n-1,$$

with $w_0 = w$ and $w_n = f$ its solution is composed from

$$w_\mu(z) = -\frac{1}{4\pi} \int_{\partial D} \gamma_\mu(\zeta) \partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta - \frac{1}{\pi} \int_D w_{\mu+1}(\zeta) G_1(z,\zeta) d\xi d\eta$$

as

$$w(z) = -\frac{1}{4\pi}\sum_{\mu=0}^{n-1}\int_{\partial D}\gamma_\mu(\zeta)\partial_{\nu_\zeta}\widehat{G}_{\mu+1}(z,\zeta)ds_\zeta - \frac{1}{\pi}\int_D f(\zeta)\widehat{G}_n(z,\zeta)d\xi d\eta$$

where with $\widehat{G}_1(z,\zeta) = G_1(z,\zeta)$

$$\widehat{G}_\mu(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\widetilde{\zeta})\widehat{G}_{\mu-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}, 2 \le \mu \le n,$$

is the iterated polyharmonic Green function of order μ. From the properties of the harmonic Green function $G_1(z,\zeta)$, it is seen that $\widehat{G}_\mu(z,\zeta)$ is a solution for the Dirichlet problem for the Poisson equation

$$\partial_z\partial_{\overline{z}}\widehat{G}_\mu(z,\zeta) = \widehat{G}_{\mu-1}(z,\zeta) \text{ in } D,$$
$$\widehat{G}_\mu(z,\zeta) = 0 \text{ on } \partial D.$$

Hence $\widehat{G}_\mu(\cdot,\zeta)$ satisfies for any $\zeta \in D$

- $\widehat{G}_\mu(\cdot,\zeta)$ is a polyharmonic function in $D\backslash\{\zeta\}$,
- $\widehat{G}_\mu(z,\zeta) + \frac{|\zeta-z|^{2(\mu-1)}}{(\mu-1)!^2}\log|\zeta-z|^2$ is polyharmonic of order μ in D
- $(\partial_z\partial_{\overline{z}})^\nu\widehat{G}_\mu(z,\zeta) = 0, 0 \le \nu \le \mu-1,$ on ∂D.

Moreover, its symmetry

- $\widehat{G}_\mu(z,\zeta) = \widehat{G}_\mu(\zeta,z), z,\zeta \in D, z \ne \zeta,$

can be shown. The outward normal derivative of $\widehat{G}_\mu(z,\zeta)$ on ∂D with respect to the variable ζ is the polyharmonic Poisson kernel of order μ,

$$g_\mu(z,\zeta) = \partial_{\nu_\zeta}\widehat{G}_\mu(z,\zeta), z \in D, \zeta \in \partial D.$$

For the particular domain $D = \mathbb{D} = \{|z| < 1\}$, the first four iterated polyharmonic Green functions are

$$\widehat{G}_1(z,\zeta) = \log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2,$$

$$\widehat{G}_2(z,\zeta) = |\zeta-z|^2\widehat{G}_1(z,\zeta) + (1-|z|^2)(1-|\zeta|^2\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big],$$

$$\begin{aligned}\widehat{G}_3(z,\zeta) = {}& \frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2 + \frac{1}{4}(1-|z|^2)(1-|\zeta|^2)(z\overline{\zeta}+\overline{z}\zeta-4)\\ & -\frac{1}{4}(1-|z|^4)(1-|\zeta|^4)\left[\frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^2} + \frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^2} + \frac{1}{z\overline{\zeta}} + \frac{1}{\overline{z}\zeta}\right]\\ & +\frac{1}{2}(1-|z|^2)(1-|\zeta|^2)(|z|^2+|\zeta|^2)\left[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\right]\\ & +(1-|z|^2)(1-|\zeta|^2)\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(z\overline{\zeta})^l + (\overline{z}\zeta)^l\right],\end{aligned}$$

$$\begin{aligned}
\widehat{G}_4(z,\zeta) = {} & \frac{1}{36}\,|\zeta - z|^6 \log\left|\frac{1-z\overline{\zeta}}{\zeta - z}\right|^2 - \frac{1}{36}(1-|z|^2)(1-|\zeta|^2)(\overline{z}^2\zeta^2 + z^2\overline{\zeta}^2 - 30) \\
& + \frac{1}{12}(1-|z|^2)(1-|\zeta|^2)(|\zeta|^2+|z|^2)(\overline{z}\zeta + z\overline{\zeta} - 4) \\
& - \frac{1}{72}(1-|z|^4)(1-|\zeta|^4)(\overline{z}\zeta + z\overline{\zeta}) \\
& + \frac{1}{36}(1-|z|^6)(1-|\zeta|^6)\Big[\frac{1}{2}\left(\frac{1}{\overline{z}\zeta} + \frac{1}{z\overline{\zeta}}\right) + \frac{1}{(\overline{z}\zeta)^2} + \frac{1}{(z\overline{\zeta})^2} \\
& + \frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^3} + \frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^3}\Big] \\
& - \frac{1}{12}\Big[(1-|z|^6)(1-|\zeta|^4) + (1-|z|^4)(1-|\zeta|^6) + 3(1-|z|^2)(1-|\zeta|^4) \\
& + 3(1-|z|^4)(1-|\zeta|^2)\Big]\left[\frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^2} + \frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^2} + \frac{1}{\overline{z}\zeta} + \frac{1}{z\overline{\zeta}}\right] \\
& + \frac{1}{12}\Big[(1-|z|^6)(1-|\zeta|^2) + (1-|z|^2)(1-|\zeta|^6) + 3(1-|z|^4)(1-|\zeta|^4) \\
& + 3(1-|z|^2)(1-|\zeta|^2)\Big]\left[\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta} + \frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}\right] \\
& - (1-|z|^2)(1-|\zeta|^2)\sum_{l=0}^{\infty}\frac{(z\overline{\zeta})^l + (\overline{z}\zeta)^l}{(l+1)^3} \\
& + \frac{1}{2}(1-|z|^2)(1-|\zeta|^2)(1+|z|^2+|\zeta|^2)\sum_{l=0}^{\infty}\frac{(z\overline{\zeta})^l + (\overline{z}\zeta)^l}{(l+1)^2}, \quad z,\zeta\in\mathbb{D},
\end{aligned}$$

see [28, 37, 53]. For an induction proof of the respective formula for the general iterated polyharmonic Green function for the unit disc $\mathbb{D}$, this function has to be guessed from the first samples. It seems that more samples of lower order Green functions have to be calculated before such a conjecture becomes available. But the iteration procedure is involved. Many area integrals need to be evaluated for every step, see [37, 53].

By the way, W. Ying [53] did calculate $\widehat{G}_4(z,\zeta)$ for some arbitrary disc sector and then just has guessed the function for $\mathbb{D}$. Once having made some guess, the verification for being the proper - uniquely given - Green function is easily done by checking the homogeneous Dirichlet conditions for the respective Poisson equation.

Without knowing the polyharmonic Green function, the related polyharmonic Poisson kernel $g_n(z,\zeta) = \partial_{\nu_\zeta}\widehat{G}_n(z,\zeta), z\in\mathbb{D}, \zeta\in\partial\mathbb{D}$, iteratively defined by

$$g_n(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} \widehat{G}_1(z,\widetilde{\zeta})g_{n-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},$$

can be found from the system of conditions

- $\partial_z\partial_{\overline{z}}g_1(z,\zeta) = 0, \partial_z\partial_{\overline{z}}g_n(z,\zeta) = g_{n-1}(z,\zeta), 2 \le n,$
- $\lim\limits_{z\to t,|z|<1,|t|=1} \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)g_1(z,\zeta)\frac{d\zeta}{\zeta} = \gamma(t), \text{ for } \gamma \in C(\partial\mathbb{D};\mathbb{C}),$
- $\lim\limits_{z\to t,|z|<1,|t|=1} \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)g_2(z,\zeta)\frac{d\zeta}{\zeta} = 0 \text{ for } \gamma \in C(\partial\mathbb{D};\mathbb{C}),$
- $\lim\limits_{z\to t,|z|<1,|t|=1} g_n(z,\zeta) = 0 \text{ for } 2 < n \text{ and } |\zeta| = 1,$
- $g_n(\cdot,\zeta) \in C^{2n}(\mathbb{D};\mathbb{C})$ for any $\zeta \in \partial\mathbb{D}$, $g_n(z,\zeta), \partial_z g_n(z,\zeta), \partial_{\overline{z}}g_n(z,\zeta) \in C(\mathbb{D}\times\partial\mathbb{D};\mathbb{C}), n \in \mathbb{N}$,

see [17, 39]. The first ones are for $z \in \mathbb{D}, \zeta \in \partial\mathbb{D}$,

$$
\begin{aligned}
g_1(z,\zeta) &= \frac{1}{1-z\overline{\zeta}} + \frac{1}{1-\overline{z}\zeta} - 1,\\
g_2(z,\zeta) &= (1-|z|^2)\left[1 + \frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\right],\\
g_3(z,\zeta) &= (1-|z|^2)\left[1 + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2}\right]\\
&\quad - \frac{1-|z|^4}{2}\left[\frac{1}{2} + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k(k+1)^2}\right],\\
g_4(z,\zeta) &= -(1-|z|^2)\left[1 + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2}\right]\\
&\quad + \frac{1-|z|^4}{2!}\left[\frac{1}{2!} + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2(k+1)}\right]\\
&\quad + \frac{1-|z|^2}{2!}\left[\frac{1}{2!} + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2(k+1)}\right]\\
&\quad - \frac{1-|z|^6}{3!}\left[\frac{1}{3!} + \sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k(k+1)(k+2)}\right].
\end{aligned}
$$

In [39], an involved formula is given for general $g_n(z,\zeta)$ using so-called vertical sums, see also [18]. The formula obviously produces all polyharmonic Poisson kernels for the unit disc. But as well a concise form as an induction proof still need to be given.

3. Hybrid polyharmonic Green functions

The iteration process applied to the Green function can in the same way be used to produce higher order polyharmonic Neumann and Robin functions,

$$N_n(z,\zeta) = -\frac{1}{\pi}\int\limits_D N_1(z,\widetilde{\zeta})N_{n-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}, n\in\mathbb{N}, 2\le n,$$
$$R_n(z,\zeta) = -\frac{1}{\pi}\int\limits_D R_1(z,\widetilde{\zeta})R_{n-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}, n\in\mathbb{N}, 2\le n,$$

see [15, 21, 26, 28, 32].

For the unit disc $\mathbb{D}$, the first polyharmonic Neumann functions are

$$\begin{aligned}
N_1(z,\zeta) &= -\log|(\zeta-z)(1-z\overline{\zeta})|^2,\\
N_2(z,\zeta) &= |\zeta-z|^2\,[4+N_1(z,\zeta)] - 4\sum_{k=2}^{\infty}\frac{(z\overline{\zeta})^k+(\overline{z}\zeta)^k}{k^2}\\
&\quad - 2(z\overline{\zeta}+\overline{z}\zeta)\log|1-z\overline{\zeta}|^2\\
&\quad - (1+|z|^2)(1+|\zeta|^2)\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}+\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big],\\
N_3(z,\zeta) &= \frac{3}{2}(|\zeta|^4+|z|^4)+5(1+|\zeta|^2)(1+|z|^2)+2(6+|\zeta|^2+|z|^2)\\
&\quad + \frac{1}{4}(1-|z|^2)(1-|\zeta|^2)(\overline{z}\zeta+z\overline{\zeta})+\frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2\\
&\quad - \left[2(2+|z|^2)(2+|\zeta|^2)+\frac{1}{2}(|\zeta|^4+|z|^4)\right]\log|1-z\overline{\zeta}|^2\\
&\quad + \left[\frac{1}{2}(|\zeta|^2+|z|^2)(1+|z|^2)(1+|\zeta|^2)+4(2+|z|^2+|\zeta|^2)\right]\\
&\quad \times\left[\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}+\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}\right]\\
&\quad - \frac{1}{4}(1+|z|^4)(1+|\zeta|^4)\left[\frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^2}+\frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^2}+\frac{1}{\overline{z}\zeta}+\frac{1}{z\overline{\zeta}}\right]\\
&\quad + \sum_{l=0}^{\infty}\Big(\frac{8}{(l+1)^3}-\frac{(|\zeta|^2+1)(|z|^2+1)}{(l+2)^2}-\frac{4|z|^2+4|\zeta|^2+6}{(l+1)^2}\Big)\\
&\quad\quad \times\left[(\overline{z}\zeta)^{l+1}+(z\overline{\zeta})^{l+1}\right],\quad z,\zeta\in\mathbb{D},
\end{aligned}$$

see [15, 37, 53].

The harmonic Robin function $R_{1;\alpha,\beta}(z,\zeta)$ satisfying the boundary condition

$$\alpha R_{1;\alpha,\beta}(z,\zeta)+\beta\partial_{\nu_z}R_{1;\alpha,\beta}(z,\zeta)=0 \text{ for } z\in\partial D, \zeta\in D$$

is for the unit disc $\mathbb{D}$ and $\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0$,

$$R_{1;\alpha,\beta}(z,\zeta) = G_1(z,\zeta) + 2\beta \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{\alpha + \beta k} + \frac{2\beta}{\alpha}, \quad \text{if} \quad -\frac{\alpha}{\beta} \notin \mathbb{N}_0,$$

$$R_{1;\alpha,\beta}(z,\zeta) = G_1(z,\zeta) + 2\beta \sum_{k=1,\alpha+\beta k\neq 0}^{\infty} \frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{\alpha + \beta k} + \frac{2\beta}{\alpha} - \frac{\beta}{\alpha}[(z\overline{\zeta})^{-\frac{\alpha}{\beta}} + (\overline{z}\zeta)^{-\frac{\alpha}{\beta}}]$$
$$+ 2\beta[(z\overline{\zeta})^{-\frac{\alpha}{\beta}} \log(z\overline{\zeta}) + (\overline{z}\zeta)^{-\frac{\alpha}{\beta}} \log(\overline{z}\zeta)], \quad \text{if} \quad -\frac{\alpha}{\beta} \in \mathbb{N}_0,$$

see [31]. For $\beta = 0$, this is just the harmonic Green function. The iteration process has not yet been undertaken. However, for $\alpha = \beta = 1$ denoting $R_{1;1,1}(z,\zeta)$ just by $R_1(z,\zeta)$ for $\mathbb{D}$

$$R_1(z,\zeta) = G_1(z,\zeta) - 2\Big[1 + \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \overline{z}\zeta)}{\overline{z}\zeta}\Big],$$

$$R_2(z,\zeta) = G_2(z,\zeta) - 2[2 - |z|^2 - |\zeta|^2] \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{(k+1)^2}$$
$$- 4\Big[1 + \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{(k+1)^3}\Big],$$

see [21, 28].

Rather than iterating Green, Neumann, and Robin functions, also called Green functions of first, second, and third kind, respectively, convolutions of them with one another can be used to determine a variety of polyharmonic Green functions. They are given as

$$K_m\widehat{K}_n(z,\zeta) = -\frac{1}{\pi}\int_D K_m(z,\widetilde{\zeta})\widehat{K}_n(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}$$

with

$$K_m(z,\zeta) \in \{G_m(z,\zeta), \widehat{G}_m(z,\zeta), N_m(z,\zeta), R_{m;\alpha,\beta}(z,\zeta)\}$$

or any other polyharmonic Green function of order m and similarly with $\widehat{K_n}(z,\zeta)$ any polyharmonic Green function of order n. $K_m\widehat{K}_n(z,\zeta)$ is a polyharmonic Green function of order $m+n$. In general, it is not symmetric with respect to its both variables. While its boundary behavior as a function of z is determined from that of $K_m(z,\zeta)$; the boundary properties as a function of ζ is copying the ones from $\widehat{K}_n(z,\zeta)$. In detail, this can be formulated as a result.

Theorem. *Let the m boundary conditions and additional side conditions, if needed, for $K_m(z,\zeta)$ for $z \in \partial D, \zeta \in D$ be denoted by B_m such that $K_m(z,\zeta)$ satisfies*

- $K_m(\cdot,\zeta)$ *is polyharmonic of order m in $D\backslash\{\zeta\}$,*
- $K_m(z,\zeta) + \frac{|\zeta-z|^{2(m-1)}}{(m-1)!^2}\log|\zeta - z|^2$ *is polyharmonic of order m in D for any $\zeta \in D$,*

- *$K_m(z,\zeta)$ satisfies the conditions B_m for $z \in \partial D, \zeta \in D$.*

As a function of z, $K_m\widehat{K}_n(z,\zeta)$ satisfies for any $\zeta \in D$ the boundary value problem

$$(\partial_z\partial_{\overline{z}})^m K_m\widehat{K}_n(z,\zeta) = \widehat{K}_n(z,\zeta) \ \text{ in } \ D, \tag{3.1}$$

$$B_m(K_m\widehat{K}_n(z,\zeta)) = -\frac{1}{\pi}\int\limits_D (B_m(K_m(z,\widetilde{\zeta})))\widehat{K}_n(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta} \ \text{ on } \ \partial D, \tag{3.2}$$

and as a function of ζ for any fixed $z \in D$

$$(\partial_\zeta\partial_{\overline{\zeta}})^n(K_m\widehat{K}_n(z,\zeta)) = K_m(z,\zeta) \ \text{ in } \ D, \tag{3.3}$$

$$\widehat{B}_n(K_m\widehat{K}_n(z,\zeta)) = -\frac{1}{\pi}\int\limits_D K_m(z,\widetilde{\zeta})\widehat{B}_n\widehat{K}_n(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta} \ \text{ on } \ \partial D. \tag{3.4}$$

In cases where $K_m(z,\zeta)$ and $\widehat{K}_n(z,\zeta)$ are both symmetric, then

$$K_m\widehat{K}_n(z,\zeta) = \widehat{K}_n K_m(\zeta,z)$$

follows.

Proof. As $K_m(z,\zeta)$ is a fundamental solution of $(\partial_z\partial_{\overline{z}})^m$, differential equation (3.1) holds. Applying the conditions B_m shows (3.2). Similarly, because $\widehat{K}_n(z,\zeta)$ as a function of ζ is a fundamental solution to $(\partial_\zeta\partial_{\overline{\zeta}})^n$, equation (3.3) follows. And (3.4) again is seen by applying the boundary operator $\widehat{B}_n$. From the symmetry conditions

$$K_m\widehat{K}_n(z,\zeta) = -\frac{1}{\pi}\int\limits_D \widehat{K}_n(\zeta,\widetilde{\zeta})K_m(\widetilde{\zeta},z)d\widetilde{\xi}d\widetilde{\eta} = \widehat{K}_n K_m(\zeta,z)$$

follows. □

Such hybrid Green functions are used in [13] to solve the Dirichlet-Neumann biharmonic problem which is differently treated in [50] via the Goursat representation of biharmonic functions. In [29], as an example of a higher order hybrid Green function, the convolution of the polyharmonic iterated Green function $\widehat{G}_m(z,\zeta)$ of order m with the polyharmonic Green-Almansi function $G_n(z,\zeta)$ of order n, see [7, 9, 49], is investigated and used to solve the related polyharmonic Dirichlet problem. Independently, other higher order hybrid polyharmonic Green functions are studied and applied for certain boundary value problems in [5], see also [6]. For some simple cases, see [12, 13, 15, 24, 26]. The case when explicit formulas are aimed for the unit disc is mostly considered in the mentioned literature. There are some other simple domains treated also as e.g. half, quarter, octo planes, half discs, half rings, disc sectors, triangles, rectangles, concentric and arbitrary rings, see [16, 19, 28–31, 33, 34, 41, 47, 48, 51, 53].

As an example for the situation of the theorem, the mentioned polyharmonic Green function $G_m\widehat{G}_n(z,\zeta)$ of order $m+n$ is discussed. The properties of the iterated polyharmonic Green function are listed in Section 2. The polyharmonic Green-Almansi function of order n has the properties, see e.g. [9, 49]

- $G_n(\cdot,\zeta)$ is polyharmonic of order n in $D\backslash\{\zeta\}$,
- $G_n(z,\zeta)+\dfrac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\log|\zeta-z|^2$ is polyharmonic of order n in D for any $\zeta\in D$,
- $\partial^{\mu}_{\nu_z}G_n(z,\zeta)=0$ for $z\in\partial D,\zeta\in D,0\le\mu\le n-1$,
- $G_n(z,\zeta)=G_n(\zeta,z)$ for $z,\zeta\in D,z\neq\zeta$.

Thus, $G_m\widehat{G}_n(z,\zeta)$ has the following properties.

For any $\zeta\in D$

- $G_m\widehat{G}_n(\cdot,\zeta)$ is polyharmonic of order $m+n$ in $D\backslash\{\zeta\}$,
- $G_m\widehat{G}_n(z,\zeta)+\dfrac{|\zeta-z|^{2(m+n-1)}}{(m+n-1)!^2}\log|\zeta-z|^2$ is polyharmonic of order $m+n$ in D,
- $\partial^{\mu}_{\nu_z}G_m\widehat{G}_n(z,\zeta)=0$ for $z\in\partial D,0\le\mu\le m-1$,
- $(\partial_z\partial_{\overline{z}})^{(m+\lambda)}G_m\widehat{G}_n(z,\zeta)=0$ for $z\in\partial D,0\le\lambda\le n-1$.

For any $z\in D$

- $G_m\widehat{G}_n(z,\cdot)$ is polyharmonic of order $m+n$ in $D\backslash\{z\}$,
- $G_m\widehat{G}_n(z,\zeta)+\dfrac{|\zeta-z|^{2(m+n-1)}}{(m+n-1)!^2}\log|\zeta-z|^2$ is polyharmonic of order $m+n$ in D,
- $(\partial_\zeta\partial_{\overline{\zeta}})^{\mu}G_m\widehat{G}_n(z,\zeta)=0$ for $\zeta\in\partial D,0\le\mu\le n-1$,
- $\partial^{\mu}_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^{n}G_m\widehat{G}_n(z,\zeta)=0$ for $\zeta\in\partial D,0\le\mu\le m-1$.

As both $G_m(z,\zeta)$ and $\widehat{G}_n(z,\zeta)$ are symmetric functions, $G_m\widehat{G}_n(z,\zeta)$ satisfies the relation

- $G_m\widehat{G}_n(z,\zeta)=\widehat{G}_nG_m(\zeta,z)$.

Certain simple (biharmonic) hybrid Green functions for some particular domains are calculated, see e.g. [29]. For the unit disc $\mathbb{D}$, they are the Green-Neumann function

$$\begin{aligned}G_1N_1(z,\zeta)=&-|\zeta-z|^2\log|\zeta-z|^2\\&-(1-|z|^2)\Big[4+\frac{1-z\overline{\zeta}}{z\overline{\zeta}}\log(1-z\overline{\zeta})+\frac{1-\overline{z}\zeta}{\overline{z}\zeta}\log(1-\overline{z}\zeta)\Big]\\&-\frac{(\zeta-z)(1-z\overline{\zeta})}{z}\log(1-z\overline{\zeta})-\frac{(\overline{\zeta-z})(1-\overline{z}\zeta)}{\overline{z}}\log(1-\overline{z}\zeta),\end{aligned}$$

Green-Robin function for the particular case $\alpha = \beta = 1$

$$G_1R_1(z,\zeta) = G_2(z,\zeta) - 2(1-|z|^2)\Big[\sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2} - 1\Big],$$

Neumann-Robin function for the particular case $\alpha = \beta = 1$

$$\begin{aligned} N_1R_1(t,\zeta) = & - |\zeta - z|^2 \log|\zeta - z|^2 \\ & - (1-|\zeta|^2)\Big[4 + \frac{1-z\overline{\zeta}}{z\overline{\zeta}} \log(1-z\overline{\zeta}) + \frac{1-\overline{z}\zeta}{\overline{z}\zeta}\log(1-\overline{z}\zeta)\Big] \\ & + \frac{(\zeta - z)(1-\overline{z}\zeta)}{\zeta}\log(1-\overline{z}\zeta) + \frac{(\overline{\zeta - z})(1-z\overline{\zeta})}{\overline{\zeta}}\log(1-z\overline{\zeta}) \\ & - 4\log|1-z\overline{\zeta}|^2 + 4\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big] \\ & - 2(1+|z|^2)\sum_{k=1}^{\infty}\frac{(z\overline{\zeta})^{k-1} + (\overline{z}\zeta)^{k-1}}{k^2} + 16 - 2(1-|z|^2). \end{aligned}$$

As is mentioned before, the general Robin function $R_{1;\alpha,\beta}(z,\zeta)$ has not yet been involved in this convolution process.

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Added in Proof. The harmonic Robin function from Section 3 can be modified so that it becomes intermediate between the Green and the Neumann functions, see H. Begehr and T. Vaitekhovich, Modified harmonic Robin function, Preprint, FU Berlin, 2011. This is done by altering the above homogeneous boundary condition by introducing the inhomogeneity from the Neumann function. In case of the unit disc $\mathbb{D}$ the condition reads

$$\alpha R_{1;\alpha,\beta}(z,\zeta)+\beta\partial_{\nu_z}R_{1;\alpha,\beta}(z,\zeta)=-2\beta \text{ for } z\in\partial\mathbb{D}.$$

The Robin function then is in the case $-\frac{\alpha}{\beta}\notin\mathbb{N}$

$$R_{1;\alpha,\beta}(z,\zeta) = \log\Big|\frac{1-z\overline{\zeta}}{\zeta - z}\Big|^2 + 2\beta\sum_{k=1}^{\infty}\frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k}{\alpha + k\beta}.$$

If $\beta = 0$ this obviously is the Green function for $\mathbb{D}$, for $\alpha = 0$ it is the Neumann function.

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