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STABILITY AND UNIFORM BOUNDEDNESS RESULTS FOR NON-AUTONOMOUS LIENARD-TYPE EQUATIONS WITH A VARIABLE DEVIATING ARGUMENT

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Abstract. In this paper, we establish two new results related to the stability and uniform boundedness of the following non-autonomous Liénard type equation with a variable deviating argument $r(t)$:

$$x''(t) + f(t, x(t), x(t-r(t)), x'(t), x'(t-r(t)))x'(t) + g_1(x(t))$$
$$+g_2(x(t-r(t))) = p(t, x(t), x(t-r(t)), x'(t), x'(t-r(t))),$$

when $p(.) \equiv 0$, and $p(.) \neq 0$, respectively. By the Lyapunov functional approach, we prove our results and give an example to illustrate the theoretical analysis in this work. By this work, we extend and improve an important result in the literature.

1. Introduction

In applied sciences, some pratical problems concerning mechanics, the engineering technique fields, economy, control theory, physics, biology, medicine, atomic energy, information theory, etc. are associated with certain second order linear or nonlinear differential equations with a deviating argument (see, for example, the book of Kolmanovskii and Myshkis [20]). Among these equations, especially, Liénard type equations with a deviating argument have a great important place. Because, in fact, many actual systems have the property of aftereffect, i.e. the future states depend not only on the present, but also on the past history. Therefore, the investigation of the qualitative properties of Liénard type equations with a deviating argument; in particular, stability and boundedness of solutions of these type equations, are very considerable. So far, the qualitative properties of Liénard type equations with and without a deviating argument have been intensively studied and are still being investigated in the literature. We refer the reader to the papers or books of Ahmad and Rama Mohana Rao [1], Barnett [2], Burton ([3], [4]), Burton and Zhang [5], Caldeira-Saraiva [6], Cantarelli [7], Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Gao and Zhao [10], Hale [11], Hara and Yoneyama ([12], [13]), Heidel ([14], [15]), Huang and Yu [16], Jitsuro and Yusuke [17], Kato ([18], [19]), Krasovskii [21], Li [22], Liu and Huang ([23], [24]), Liu and

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Xu [25], Liu [26], Long and Zhang [27], Luk [28], Malyseva [30], Muresan [31], Nápoles Valdés [32], Sugie [33], Sugie and Amano [34], Sugie et al. [35], Tunç [36-42], C. Tunç and E. Tunç [43], Yang [44], Ye et al. [45], Yoshizawa [46], Zhang ([47], [48]), Zhang and Yan [49], Zhou and Jiang [50], Zhou an Liu [51], Zhou and Xiang [52], Wei and Huang [53], Wiandt [54] and the references therein.

Besides, in 2008, Liu and Huang [24] discussed the uniform boundedness of solutions to the following Lienard type equation with a deviating argument $\tau(t)$:

$$x''(t) + f(x(t))x'(t) + g_1(x(t)) + g_2(x(t-\tau(t))) = e(t), \tag{1}$$

where f, g_1 and g_2 are continuous functions on $\Re = (-\infty, \infty)$, $\tau(t) \geq 0$ is a bounded continuous function on $\Re$, and $e(t)$ is a bounded continuous function on $\Re^+ = [0, \infty)$.

Define

$$\varphi(x) = \int_0^x [f(u) - 1]du$$

and

$$y = \frac{dx}{dt} + \varphi(x).$$

Then, Eq. (1) can be transformed into the following system:

$$\begin{aligned} \frac{dx(t)}{dt} &= y(t) - \varphi(x(t)), \\ \frac{dy(t)}{dt} &= -y(t) - \left\{g_1(x(t)) - \varphi(x(t))\right\} - g_2(x(t-\tau(t))) + e(t). \end{aligned} \tag{2}$$

Under the above acceptations, Liu and Huang [24] proved the following theorem, and introduced an example for illustration of the topic.

Theorem A. (Liu and Huang [24, Theorem 3.1]). *Assume that the following conditions hold:*

(C_1) *There exists a constant* $\underline{d} > 1$ *such that* $\underline{d}\,|u| \leq \operatorname{sign}(u)\varphi(u)$ *for all* $u \in \Re$.

(C_2) *There exist non-negative constants* L_1, L_2, q_1 *and* q_2 *such that*

$$|(g_1(u) - \varphi(u))| \leq L_1\,|u| + q_1,$$

$$|g_2(u)| \leq L_2\,|u| + q_2, \;\; \textit{for all } u \in \Re,$$

$$L_1 + L_2 < 1,$$

$$e(t) \textit{ is a bounded continuous function on } \Re^+ = [0, \infty).$$

Then the solutions of (2) are uniformly bounded.

It should be noted that by the above work Liu and Huang [24] obtained some new sufficient conditions for all solutions of (1) and their derivatives to be uniformly bounded, which substantially extend and improve some important results in the literature (see [24]). The authors also showed that some results obtained in the literature cannot be applicable to the systems of the form (2) (see [24]). That is to say that by the mentioned work, Liu and Huang [24] achieved an important contribution to the topic of boundedness of solutions of Liénard equations with a variable delay in the literature.

In this paper, we consider the non-autonomous Liénard type equation with a variable deviating argument $r(t)$:

$$\begin{aligned} x''(t) + f(t, x(t), x(t-r(t)), x'(t), x'(t-r(t)))x'(t) + g_1(x(t)) \\ + g_2(x(t-r(t))) = p(t, x(t), x(t-r(t)), x'(t), x'(t-r(t))), \end{aligned} \tag{3}$$

where $r(t)$ is a continuous, differentiable and bounded variable deviating argument, $0 \le r(t) \le \gamma$, $r'(t) \le \beta$, $0 < \beta < 1$, γ and β are some positive constants and γ will be determined later; the primes in Eq. (3) denote differentiation with respect to $t \in \Re^+$, $\Re^+ = [0, \infty)$; f, g_1, g_2 and p are continuous functions in their respective arguments on $\Re^+ \times \Re^4, \Re, \Re$ and $\Re^+ \times \Re^4$ and depend only on the arguments displayed explicitly with $g_1(0) = g_2(0) = 0$. The continuity of the functions f, g_1, g_2 and p is a sufficient condition for the existence of the solutions of Eq. (3). It is also assumed as basic that the functions f, g_1, g_2 and p satisfy a Lipschitz condition in $x', \ldots, x'(t-r(t))$. In this way, the uniqueness of the solutions of (3) is guaranteed. Besides, it is assumed that the derivative $\frac{dg_2}{dx} \equiv g_2'(x)$ exists and is continuous. We examine here the stability and the uniform boundedness of the solutions of Eq. (3), when $p \equiv 0$ and $p \neq 0$, respectively.

In addition to the above information, it is worth mentioning that, in 2003, Liu and Huang [23] also discussed the boundedness of solutions to the following equation with the constant delay h:

$$x'' + f_1(x)x' + f_2(x)(x')^2 + g(x(t-h)) = e(t).$$

After that, the author in [36] improved the result of Liu and Huang [23] for the stability and boundedness of solutions to the following Liénard type equation with a variable deviating argument $r(t)$:

$$\begin{aligned} & x''(t) + f_1(x(t), x(t-r(t)), x'(t), x'(t-r(t)))x'(t) \\ & + f_2(x(t), x(t-r(t)), x'(t), x'(t-r(t)))\left(x'(t)\right)^2 + g_1(x(t)) \\ & + g_2(x(t-r(t))) = p(t, x(t), x(t-r(t)), x'(t), x'(t-r(t))). \end{aligned}$$

In addition, in 2009, Ye et al. [45] studied the uniform boundedness of all solutions of the following Liénard equation with a variable deviating argument $\tau(t)$:

$$x''(t) + f_1(x(t))x'(t) + f_2(x(t))x'(t) + g_1(x(t)) + g_2(x(t-\tau(t))) = e(t).$$

At the same line, in 2010, Long and Zhang [27] considered the following Liénard equation with a variable deviating argument $\tau(t)$:

$$x''(t) + f_1(x(t))\left(x'(t)\right)^2 + f_2(x(t))x'(t) + g_1(x(t)) + g_2(x(t-\tau(t))) = e(t).$$

The authors proved a theorem which ensures that all solutions of the above Liénard equation satisfying given initial conditions are bounded.

Instead of Eq. (3), we consider the following system:

$$
\begin{aligned}
x'(t) =& y(t) \\
y'(t) =& -f(t,x(t),x(t-r(t)),y(t),y(t-r(t)))y(t) - g_1(x(t)) - g_2(x(t)) \\
&+ \int_{t-r(t)}^{t} g_2'(x(s))y(s)ds + p(t,x(t),x(t-r(t)),y(t),y(t-r(t))),
\end{aligned}
\tag{4}
$$

which was obtained from Eq. (3). Throughout the paper $x(t), y(t)$ are also abbreviated as x and y, respectively.

The motivation for the present paper has been inspired basically by the paper Liu and Huang [24]. It is worth mentioning that, to the best of our knowledge, it is not found any research on the stability and boundedness of solutions of Eq. (3) in the literature. It is also clear that the equation discussed in Liu and Huang [24], Eq. (1), is a special case of our equation, Eq. (3). Liu and Huang [24] studied the boundedness of solutions of Eq. (1). In addition to the boundedness of solutions, we also discuss the stability of solutions for Eq. (3), and introduce an example on the topic. By defining a new Lyapunov functional (see, also, Krasovskii [21] and Lyapunov [29]) some sufficient conditions for the stability and boundedness of solutions of Eq. (3) are obtained, when $p \equiv 0$ and $p \neq 0$, respectively. Thus, in view of the above information, it is worthwhile to continue the investigation of the stability and boundedness of solutions of Eq. (3). When we compare the results established in the above papers, books and that to be established here, it can be seen that our results are different from that obtained in the foregoing sources and that in the literature.

Consider the general non-autonomous delay differential system:

$$\dot{x} = F(t, x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0, \tag{5}$$

where F: $\Re^+ \times C_H \to \Re^n$ is a continuous mapping, $F(t,0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous functions ϕ: $[-r,\ 0] \to \Re^n$ with supremum norm; $r > 0$, C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r,0],\ \ \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [4], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0 + \alpha)$ satisfying (5) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote a convenient norm in $\Re^n$ with $|x| = \max\limits_{1\leq i\leq n} |x_i|$. Let us assume that $C(t) = \{\phi : [t-\alpha, t] \to \ \Re^n) \mid \phi$ is continuous$\}$ and ϕ_t denotes the ϕ in the particular $C(t)$, and that $\|x\| = \max\limits_{t-\alpha\leq s\leq t} |\phi(t)|$. Evidently, Eq. (3) is a particular case of (5).

Definition 1. (Burton [4]). A continuous positive definite function W: $\Re^n \to [0,\infty)$ is called a wedge.

Definition 2. (Burton [4]). A continuous strictly increasing function $W\colon [0,\infty) \to [0,\infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$ is a wedge. (We denote wedges by W or W_i , where i is an integer.)

Definition 3. (Burton [4]). Let D be an open set in $\Re^n$ with $0 \in D$. A function $V\colon [0,\infty) \times D \to [0,\infty)$ is called positive definite if $V(t,0) = 0$ and if there is a wedge W_1 with $V(t,x) \geq W_1(|x|)$, and is called a decrescent function if there is a wedge W_2 with $V(t,x) \leq W_2(|x|)$.

Definition 4. (Burton [4]). Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (5) will be denoted by $\dot{V}$ and is defined by the relation:

$$\dot{V}(t,\phi) = \limsup_{h\to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution with $x_{t_0}(t_0,\phi) = \phi$.

Theorem 1. (Burton [4]). *Let $V(t,x_t)$ be a differentiable scalar functional defined when $x : [\alpha,t] \to \Re^n$ is continuous and bounded by some $D \leq \infty$. If*

$$V(t,0) = 0, W_1(|x|) \leq V(t,x_t), \quad \textit{(where } W_1(r) \textit{ is a wedge)},$$

and

$$\dot{V}(t,x_t) \leq 0,$$

then the null solution of system (5) *is stable.*

Theorem 2. (Yoshizawa [46]). *Suppose that there exists a continuous Lyapunov functional $V(t,\phi)$ defined for all $0 \leq t < \infty$, $\phi \in C_H$, which satisfies the following conditions:*

(i) *$a(\|\phi\|) \leq V(t,\phi) \leq b(\|\phi\|)$, where $a(r) \in CI$, positive for $r > H$, $a(r) \to \infty$ as $r \to \infty$ and $b(r) \in CI$ (CI denotes the family of continuous increasing functions),*

(ii) $\dot{V}(t,\phi) \leq 0$.

Then the solutions of system (5) *are uniformly bounded.*

2. Main results

For the case $p(.) = 0$, the first main result of this paper is the following theorem.

Theorem 3. *In addition to the basic assumptions imposed on the functions f, g_1 and g_2, we suppose that there exist positive constants α_1 and L such that the following conditions hold:*

$$f\left(t,x,x(t-r(t)),y,y(t-r(t))\right) \geq \alpha_1,$$

$$xg_1(x) > 0, xg_2(x) > 0, (x \neq 0), \left|g_2'(x)\right| \leq L.$$

If

$$\gamma < \frac{2\alpha_1(1-\beta)}{L(2-\beta)},$$

then the null solution of Eq. (3) *is stable.*

Proof. Define the Lyapunov functional

$$V(t,x_t,y_t) = \exp\Big(-2\int_0^t |e(s)|\,ds\Big)\Big\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2$$

$$+1+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\Big\}, \tag{6}$$

where λ is a positive constant which will be determined later and $e(t)$ is a continuous function on $\Re^+ = [0,\infty)$ and $\int_0^t |e(s)|\,ds < \infty$.

Then we have

$$\exp\Big(-2\int_0^\infty |e(s)|\,ds\Big)\Big\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1$$

$$+\lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\Big\} \le V(t,x_t,y_t).$$

The time derivative of the Lyapunov functional $V = V(t,x_t,y_t)$ along the system (4) is

$$\frac{dV}{dt} = -2\,|e(t)|\exp\Big(-2\int_0^t |e(s)|\,ds\Big)$$

$$\times\Big\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\Big\}$$

$$-\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\Big\{f\,(t,x,x(t-r(t)),y,y(t-r(t))) - \lambda r(t)\Big\}y^2$$

$$+\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\Big\{y\int_{t-r(t)}^{t} g_2'(x(s))y(s)ds - \lambda\{1-r'(t)\}\int_{t-r(t)}^{t} y^2(s)ds\Big\}.$$

By using the assumptions of Theorem 3 and the inequality $|mn| \le \frac{m^2}{2} + \frac{n^2}{2}$, we obtain

$$-\Big\{f\,(t,x,x(t-r(t)),y,y(t-r(t))) - \lambda r(t)\Big\}y^2 \le -(\alpha_1 - \lambda\gamma)y^2,$$

$$y\int_{t-r(t)}^{t} g_2'(x(s))y(s)ds \le |y|\int_{t-r(t)}^{t} \left|g_2'(x(s))\right| |y(s)|\,ds \le |y|\int_{t-r(t)}^{t} L\,|y(s)|\,ds$$

$$\leq \frac{1}{2}r(t)Ly^2 + \frac{1}{2}L\int_{t-r(t)}^{t} y^2(s)ds \leq \frac{1}{2}\gamma Ly^2 + \frac{1}{2}L\int_{t-r(t)}^{t} y^2(s)ds$$

and

$$-\lambda\{1-r'(t)\}\int_{t-r(t)}^{t} y^2(s)ds \leq -\lambda(1-\beta)\int_{t-r(t)}^{t} y^2(s)ds$$

so that

$$\begin{aligned}
\frac{dV}{dt} \leq & -2\,|e(t)|\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\\
& \times\Big\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\Big\}\\
& -\alpha_1\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2 + \lambda\gamma\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
& +\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\Big\{2^{-1}L\int_{t-r(t)}^{t} y^2(s)ds\Big\}\\
& +2^{-1}L\gamma\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
& -\lambda(1-\beta)\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\int_{t-r(t)}^{t} y^2(s)ds\\
\leq & -\big\{\alpha_1 - 2^{-1}(L+2\lambda)\gamma\big\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
& -\big\{\lambda(1-\beta) - 2^{-1}L\big\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\int_{t-r(t)}^{t} y^2(s)ds.
\end{aligned}$$

Let $\lambda = \frac{1}{2}L(1-\beta)^{-1}$. Hence

$$\frac{dV}{dt} \leq -\Big\{\alpha_1 - 2^{-1}L[1+(1-\beta)^{-1}]\gamma\Big\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2.$$

If

$$\gamma < \frac{2\alpha_1(1-\beta)}{L(2-\beta)},$$

then the last estimate is
$$\frac{dV}{dt} = \frac{d}{dt}V(t, x_t, y_t) \le -ky^2 \le 0$$
for a positive constant k.
Hence we conclude that the null solution of Eq. (3) is stable (see also Theorem 1). $\square$

Remark 1. Theorem 3 arises a new result in the literature on the stability of solutions of a certain non-autonomous Liénard type equation with a variable delay. This case is a clear improvement on the topic of the paper.

For the case $p(.) \neq 0$, the second main result of this paper is the following theorem.

Theorem 4. *Let us assume that all the assumptions of Theorem 3 hold. In addition, we assume that*
$$\int_0^x g_1(s)ds \to +\infty \text{ and } \int_0^x g_2(s)ds \to +\infty \text{ as } |x| \to \infty,$$
and
$$|p\,(t, x, x(t - r(t)), y, y(t - r(t)))| \le |e(s)|\,, \qquad \int_0^t |e(s)|\, ds < \infty.$$
If
$$\gamma < \frac{2\alpha_1(1-\beta)}{L(2-\beta)},$$
then all solutions of Eq. (3) *are uniformly bounded.*

Proof. The essential implement to prove Theorem 4 is the Lyapunov functional $V(t, x_t, y_t)$, which was used in the proof of the preceding theorem. Evidently, from (6) we have
$$\exp\left(-2\int_0^\infty |e(s)|\, ds\right)\left\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds\right\}$$
$$\le V(t, x_t, y_t) \le \left\{\int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds\right\}.$$
The above inequality shows that the condition (i) of Theorem 2 holds.

Since $p(.) \neq 0$, then the time derivative of the Lyapunov functional $V = V(t, x_t, y_t)$ along the system (4) is
$$\frac{dV}{dt} = -2\,|e(t)| \exp\left(-2\int_0^t |e(s)|\, ds\right)$$

$$
\begin{aligned}
&\times \Big\{ \int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \Big\} \\
&- \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) \big\{ f\,(t, x, x(t-r(t)), y, y(t-r(t))) - \lambda r(t) \big\} y^2 \\
&+ \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) p\,(t, x, x(t-r(t)), y, y(t-r(t)))\, y \\
&+ \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) \Big\{ y \int_{t-r(t)}^{t} g_2'(x(s))y(s)ds - \lambda\{1 - r'(t)\} \int_{t-r(t)}^{t} y^2(s)ds \Big\}.
\end{aligned}
$$

Employing the assumption of Theorem 4 and the estimate $|mn| \leq \dfrac{m^2}{2} + \dfrac{n^2}{2}$, we have

$$
\begin{aligned}
\frac{dV}{dt} \leq &- 2\, |e(t)| \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) \\
&\times \Big\{ \int_0^x g_1(s)ds + \int_0^x g_2(s)ds + \frac{1}{2}y^2 + 1 + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \Big\} \\
&- \alpha_1 \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) y^2 + \lambda\gamma \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) y^2 \\
&+ \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) |e(t)|\, |y| \\
&+ \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) \Big\{ 2^{-1} L \int_{t-r(t)}^{t} y^2(s)ds \Big\} \\
&+ 2^{-1} L\gamma \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) y^2 \\
&- \lambda(1-\beta) \exp\Big(-2 \int_0^t |e(s)|\, ds \Big) \int_{t-r(t)}^{t} y^2(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\le -2\,|e(t)|\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\\
&\quad\times\Big\{\int_0^x g_1(s)ds+\int_0^x g_2(s)ds+\frac{1}{2}y^2+1+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds\Big\}\\
&\quad-\alpha_1\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2+\lambda\gamma\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
&\quad+\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\,|e(t)|+\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\,|e(t)|\,y^2\\
&\quad+\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\{2^{-1}L\int_{t-r(t)}^t y^2(s)ds\}+2^{-1}L\gamma\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
&\quad-\lambda(1-\beta)\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\int_{t-r(t)}^t y^2(s)ds\\
&\le -\alpha_1\exp(-2\int_0^t |e(s)|\,ds)y^2+\lambda\gamma\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
&\quad+\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\Big\{2^{-1}L\int_{t-r(t)}^t y^2(s)ds\Big\}+2^{-1}L\gamma\exp(-2\int_0^t |e(s)|\,ds)y^2\\
&\quad-\lambda(1-\beta)\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\int_{t-r(t)}^t y^2(s)ds\\
&\le -\{\alpha_1-2^{-1}(L+2\lambda)\gamma\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2\\
&\quad-\{\lambda(1-\beta)-2^{-1}L\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)\int_{t-r(t)}^t y^2(s)ds.
\end{aligned}
$$

Let $\lambda=2^{-1}L(1-\beta)^{-1}$. Then

$$
\frac{d}{dt}V(t,x_t,y_t)\le -\{\alpha_1-2^{-1}(L+2\lambda)\gamma\}\exp\Big(-2\int_0^t |e(s)|\,ds\Big)y^2.
$$

In view of the last estimate, if $\gamma < \dfrac{2\alpha_1(1-\beta)}{L(2-\beta)}$, then a straightforward calculation leads to

$$\frac{dV}{dt} = \frac{d}{dt}V(t, x_t, y_t) \leq 0.$$

Hence one can conclude that all solutions of Eq. (3) are uniformly bounded (see also Theorem 2).

Remark 2. It is evident that the assumptions of Theorem 3 are completely different from that established by Liu and Huang [24, Theorem 3.1]. Next, the method and procedure used in the proof of Theorems 3 are different from that used in [24, Theorem 3.1], and the assumptions of Theorem 3 and 4 are very clear, elegant and comprehensible, and our assumptions can also be easily applied to the very general non-autonomous Liénard type equations (3). That is to say the following:

(r_1) Our equation, Eq. (3), includes and improves the equation discussed by Liu and Huang [24], Eq. (1).

(r_2) Evidently, the assumptions of Theorems 3 and 4 have very simple forms and the applicability of our assumptions can be easily verified.

Example. As a special case of Eq. (3), consider the nonlinear second order differential equation with a deviating argument $r(t)$:

$$\begin{aligned} &x'' + \left(4 + \tfrac{1}{1+t^2+x^2+x^2(t-r(t))+x'^2+x'^2(t-r(t))}\right)x' + x(x^2+2) \\ &+4x(t-r(t)) = \tfrac{1}{1+t^2+x^2+x^2(t-r(t))+x'^2+x'^2(t-r(t))}, \end{aligned} \tag{7}$$

whose associated system is

$$\begin{aligned} x' &= y \\ y' &= -\left(4 + \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))}\right)y - x(x^2+2) \\ &\quad - 4x + 4\int_{t-r(t)}^{t} y(s)ds + \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))}. \end{aligned}$$

Hence we see the following:

$$\begin{aligned} &f\left(t, x, x(t-r(t)), y, y(t-r(t))\right) \\ &= 4 + \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))} \\ &\geq 4 = \alpha_1, \\ g_1(x) &= x(x^2+2), \\ g_1(0) &= 0, \\ xg_1(x) &= x^2(x^2+2) > 0, \quad (x \neq 0), \end{aligned}$$

$$\int_0^x (s^3+2s)ds = \frac{1}{4}x^4 + x^2 \to +\infty \quad \text{as} \quad |x| \to \infty,$$

that is

$$\int_0^x g_1(s)ds \to +\infty \quad \text{as} \quad |x| \to \infty;$$

$$g_2(x) = 4x, g_2(0) = 0,$$

$$xg_2(x) = 4x^2 > 0, \quad (x \neq 0),$$

$$\int_0^x 4sds = 2x^2 \to +\infty \quad \text{as} \quad |x| \to \infty,$$

that is

$$\int_0^x g_2(s)ds \to +\infty \quad \text{as} \quad |x| \to \infty;$$

$$g_2'(x) = 4,$$

$$\left|g_2'(x)\right| = 4 = L,$$

$$p\left(t, x, x(t-r(t)), y, y(t-r(t))\right) = \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))}$$

$$\leq \frac{1}{1+t^2} = e(t),$$

$$\int_0^\infty e(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$

that is,

$$\int_0^t |e(s)|\, ds < \infty$$

and

$$\gamma < \frac{2\alpha_1(1-\beta)}{L(2-\beta)} = \frac{2(1-\beta)}{(2-\beta)}, \quad 0 < \beta < 1.$$

Thus, all the assumptions of Theorems 3 and 4 hold, when $p \equiv 0$ and $p \neq 0$, respectively. The above discussion implies that all solutions of Eq. (7) are stable and uniformly bounded, when $p \equiv 0$ and $p \neq 0$, respectively.

3. Conclusion

A non-autonomous Liénard type equation with a variable deviating argument is considered. The stability and uniform boundedness of solutions of this equation are discussed. In proving our results, we employ the Lyapunov functional approach by defining a Lyapunov functional. An example is also given to illustrate our theoretical findings.

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