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ON LOCAL COHOMOLOGY OF A TETRAHEDRAL CURVE

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ABSTRACT. It is shown that the diameter $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I))$ of the first local cohomology module of a tetrahedral curve $C = C(a_1, ..., a_6)$ can be explicitly expressed in terms of the a_i and is the smallest non-negative integer k such that $\mathfrak{m}^k H^1_{\mathfrak{m}}(R/I) = 0$. From that one can describe all arithmetically Cohen-Macaulay or Buchsbaum tetrahedral curves.

INTRODUCTION

A tetrahedral curve $C = C(a_1, ..., a_6)$ is a curve in $\mathbb{P}^3$ defined by the ideal

$$I = (x_1, x_2)^{a_1} \cap (x_1, x_3)^{a_2} \cap (x_1, x_4)^{a_3} \cap (x_2, x_3)^{a_4} \cap (x_2, x_4)^{a_5} \cap (x_3, x_4)^{a_6}$$

of the polynomial ring $R = K[x_1, x_2, x_3, x_4]$ over a field K, where $a_1, ..., a_6$ are non-negative integers and not all of them are zero. The case $a_2 = a_5 = 0$ was first considered by Schwartau [7]. He gave a characterization of the Cohen-Macaulay property of C in terms of a_1, a_3, a_4, a_6. The general case of tetrahedral curves, when a_2 and a_5 are not necessarily zero, was introduced in [6]. Using basic double linkage, Migliore and Nagel gave there an efficient numerical algorithm for determining when a particular tetrahedral curve is arithmetically Cohen-Macaulay and asked for an explicit characterization in terms of $a_1, ..., a_6$. This problem was solved later by Francisco in [3]. Moreover, it was shown in the papers [6, 4] that these curves have many nice properties.

In this paper we study the structure of the first local cohomology module $H^1_{\mathfrak{m}}(R/I)$ with the support in the maximal homogeneous ideal $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. This study is important because we can characterize many properties, such as the Cohen-Macaulayness or the Buchsbaumness, of C in terms of $H^1_{\mathfrak{m}}(R/I)$.

Recall that the diameter of a $\mathbb{Z}$-graded module M of finite length is the integer $\operatorname{diam}(M) = \max\{n|\ M_n \neq 0\} - \min\{n|\ M_n \neq 0\} + 1$ ($\operatorname{diam}(M) := 0$ if $M = 0$). Let J be the defining ideal of an arbitrary projective curve X in $\mathbb{P}^3$. Then the module $H^1_{\mathfrak{m}}(R/J)$ is of finite length and let $k(R/J)$ be the smallest non-negative integer k such that $\mathfrak{m}^k H^1_{\mathfrak{m}}(R/J) = 0$ (see [5, 1]). It is

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obvious that $k(R/J) \le \operatorname{diam}(H^1_{\mathfrak{m}}(R/J))$. The main result of this paper states that $k(R/I) = \operatorname{diam}(H^1_{\mathfrak{m}}(R/I))$ for an arbitrary tetrahedral curve (see Theorem 3.4). Thus our result implies that for all tetrahedral curves, $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I))$ has no gap and $k(R/I)$ is, in this sense, as large as possible. (Note that monomial curves in $\mathbb{P}^3$ also have this property, see [1].) Moreover, we can explicitly compute $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I))$ in terms of $a_1, ..., a_6$ (see Theorem 3.2 and Theorem 3.4). Since C is an arithmetically Cohen-Macaulay curve if and only if $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I)) = 0$, this result is much more general than the Francisco's one in [3]. In particular, it also enables us to determine all arithmetically Buchsbaum tetrahedral curves (Theorem 3.7), thus extending Corollary 5.4 in [6].

Our approach is to reduce the above question to a problem in integer programming. First, based on a description of local cohomology modules of monomial ideals given recently in [9], we reduce the problem to describing the set of integer solutions of a certain linear constraints. Then using the well-known Fourier-Motzkin elimination we can determine when the set of solutions is empty (Theorem 3.2). This is corresponding to the case of arithmetically Cohen-Macaulay curves. If this set is not empty, we can still use it to determine the module structure of the first local cohomology (Proposition 3.3). Thus our result is not only an interesting application of integer programming to Commutative Algebra, but it also shows the usefulness of Takayama's formula in [9]. We believe that Takayama's formula, which is a generalization of Hochster's formula, can be applied in many other situations.

The paper has four sections with the current one being an introduction. In Section 1 we recall the main result of Takayama in [9] and relate the problem of describing $H^1_{\mathfrak{m}}(R/I)$ to a problem in integer programming (Lemma 1.4). In Section 2 we apply the Fourier-Motzkin elimination to solve that integer programming problem. The structure of the first local cohomology module is given in the last Section 3, where the main Theorem 3.4 is proved and some of its consequences are derived.

1. Preliminaries

Let $I \subset R = K[x_1, ..., x_n]$ be a monomial ideal. Denote by $G(I)$ the minimal set of monomial generators of I. Let Δ be the simplicial complex corresponding to the radical ideal $\sqrt{I}$, i.e.

$$\Delta = \{\{i_1, ..., i_k\} \subseteq \{1, ..., n\} | \ x_{i_1} \cdots x_{i_k} \notin \sqrt{I}\}.$$

A simplicial complex is uniquely defined by the set $\operatorname{Max}(\Delta)$ of its facets. Following [9], for $\underline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$, we set

$$G_{\underline{\alpha}} = \{i | \ \alpha_i < 0\},$$

and

$$\Delta_{\underline{\alpha}} = \{F \subset \{1, ..., n\} \setminus G_{\underline{\alpha}} | \quad \text{for all } \underline{x}^{\underline{\beta}} = x_1^{\beta_1} \cdots x_n^{\beta_n} \in G(I) \text{ there exists } i \notin F \cup G_{\underline{\alpha}} \text{ such that } \beta_i > \alpha_i \ge 0\}.$$

Lemma 1.1. *Denote by $I_{(x_{i_1}\dots x_{i_k})}$ the monomial ideal generated by I in the localization $K[\underline{x}]_{(x_{i_1}\dots x_{i_k})}$ with respect to (w.r.t., for short) the set of all monomials in the variables $x_{i_1},\dots,x_{i_k}$. Then*

$$\Delta_{\underline{\alpha}} = \{F \subset \{1,\dots,n\} \setminus G_{\underline{\alpha}} |\ \prod_{i \notin F \cup G_{\underline{\alpha}}} x_i^{\alpha_i} \notin I_{(\prod_{j \in F \cup G_{\underline{\alpha}}} x_j)}\}.$$

Proof. For simplicity we may assume that $F \cup G_{\underline{\alpha}} = \{1,\dots,r\}$. For a monomial $m \in K[x_1,\dots,x_n]$ let $m' \in K[x_{r+1},\dots,x_n]$ be the monomial obtained from m by deleting all powers of x_i, $i \leq r$. Let $G' = \{m' |\ m \in G(I)\}$. Then G' is a generating set of $I' := I_{(x_1\dots x_r)}$. Note that the monomial $\prod_{i>r} x_i^{\alpha_i} \in I'$ if and only if there exists $m' = \prod_{i>r} x_i^{\beta_i} \in G'$ such that $\beta_i \leq \alpha_i$ for all $i > r$, or equivalently, there exists $m = \prod_{i=1}^{n} x_i^{\beta_i} \in G(I)$ such that $\beta_i \leq \alpha_i$ for all $i > r$. From that we immediately get the claim. $\square$

Note that all local cohomology modules $H^i_{\mathfrak{m}}(R/I)$, $i \geq 0$, inherit a natural $\mathbb{Z}^n$-grading. Theorem 1 in [9] can be reformulated as follows.

Lemma 1.2. *Let $\rho_i = \max\{\beta_i |\ \underline{x}^{\underline{\beta}} \in G(I)\}$. For all $i \geq 0$ and $\underline{\alpha} \in \mathbb{Z}^n$ we have*

$$\dim H^i_{\mathfrak{m}}(R/I)_{\underline{\alpha}} = \begin{cases} \dim \tilde{H}_{i-|G_{\underline{\alpha}}|-1}(\Delta_{\underline{\alpha}}, K) & \text{if } G_{\underline{\alpha}} \in \Delta \text{ and } \alpha_j \leq \rho_j - 1,\ j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

From now on we consider ideals of tetrahedral curves

$$I = (x_1,x_2)^{a_1} \cap (x_1,x_3)^{a_2} \cap (x_1,x_4)^{a_3} \cap (x_2,x_3)^{a_4} \cap (x_2,x_4)^{a_5} \cap (x_3,x_4)^{a_6}$$

of the polynomial ring $R = K[x_1,x_2,x_3,x_4]$.

Lemma 1.3. *If $H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} \neq 0$, then $\alpha_i \geq 0$ for all $i \geq 1$ and*

$$\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,i\},\{j,k\} |\ \{i,j,k\} = \{2,3,4\}\}.$$

Proof. Assume $H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} \neq 0$. By Lemma 1.2, either $G_{\underline{\alpha}} = \emptyset$ and $\Delta_{\underline{\alpha}}$ is disconnected, or $|G_{\underline{\alpha}}| = 1$ and $\Delta_{\underline{\alpha}} = \{\emptyset\}$.

If $|G_{\underline{\alpha}}| = 1$, without loss of generality (w.l.o.g., for short) we may assume that $G_{\underline{\alpha}} = \{1\}$, i. e. $\alpha_1 < 0$ and $\alpha_2,\alpha_3,\alpha_4 \geq 0$. By Lemma 1.1, $\Delta_{\underline{\alpha}} = \{\emptyset\}$ is equivalent to the following two conditions

(i) $x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4} \notin I_{(x_1)} = (x_2,x_3)^{a_4} \cap (x_2,x_4)^{a_5} \cap (x_3,x_4)^{a_6}$, and

(ii) $x_i^{\alpha_i}x_j^{\alpha_j} \in I_{(x_1,x_k)}$ for all $\{i,j,k\} = \{2,3,4\}$.

This is impossible, because

$$\text{(i)} \Leftrightarrow \left[\begin{array}{l} \alpha_2+\alpha_3 \leq a_4-1, \text{ or} \\ \alpha_2+\alpha_4 \leq a_5-1, \text{ or} \\ \alpha_3+\alpha_4 \leq a_6-1, \end{array}\right. \quad \text{and} \quad \text{(ii)} \Leftrightarrow \left\{\begin{array}{l} \alpha_2+\alpha_3 \geq a_4, \text{ and} \\ \alpha_2+\alpha_4 \geq a_5, \text{ and} \\ \alpha_3+\alpha_4 \geq a_6. \end{array}\right.$$

Hence we must have $G_{\underline{\alpha}} = \emptyset$ and $\Delta_{\underline{\alpha}}$ is disconnected. The first condition implies that $\alpha_i \geq 0$ for all $i \geq 1$. Since $\Delta_{\underline{\alpha}}$ is a disconnected simplicial complex on a subset of $\{1,2,3,4\}$, in order to show the second statement of the lemma it suffices to show that $\Delta_{\underline{\alpha}}$ does not contain a facet consisting of a single point. Assume, by contrary, that $\{1\}$ is a facet of $\Delta_{\underline{\alpha}}$. Then we again get (i) and (ii) (the only difference now is that all $\alpha_i \geq 0$ which, however, have no effect on (i) and (ii)). This is a contradiction. $\square$

As an example let us consider the well-known Buchsbaum curve defined by $I = (x_1, x_2) \cap (x_3, x_4)$. In this case $H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} \neq 0$ if and only if $\underline{\alpha} = (0,0,0,0)$. We have $\mathrm{Max}(\Delta_{(0,0,0,0)}) = \{\{1,2\},\{3,4\}\}$.

Lemma 1.4. *Fix an integer d. Assume that $\deg(\underline{\alpha}) := \alpha_1 + \cdots + \alpha_4 = d$. Then $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,2\},\{3,4\}\}$ if and only if $\underline{\alpha}$ satisfies the following system of inequalities*

$$(1)\qquad \begin{aligned} \alpha_1 + \alpha_3 &\geq a_2\\ \alpha_1 + \alpha_4 &\geq a_3\\ \alpha_2 + \alpha_3 &\geq a_4\\ \alpha_2 + \alpha_4 &\geq a_5\\ \alpha_1 + \alpha_2 &\leq a_1 - 1\\ \alpha_3 + \alpha_4 &\leq a_6 - 1\\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= d\\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 &\geq 0. \end{aligned}$$

In this case $\dim H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} = 1$.

Proof. The condition $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,2\},\{3,4\}\}$ implies $G_{\underline{\alpha}} = \emptyset$, i.e. α_1, α_2, α_3, $\alpha_4 \geq 0$. By Lemma 1.1, $\{1,2\} \in \Delta_{\underline{\alpha}}$ if and only if $x_3^{\alpha_3} x_4^{\alpha_4} \notin (x_3, x_4)^{a_6}$, or equivalently, $\alpha_3 + \alpha_4 \leq a_6 - 1$. Similarly, $\{3,4\} \in \Delta_{\underline{\alpha}}$ if and only if $\alpha_1 + \alpha_2 \leq a_1 - 1$. On the other hand, $\{1,3\},\{1,4\},\{2,3\},\{2,4\} \notin \Delta_{\underline{\alpha}}$ are equivalent to the first four inequalities given above. Thus, $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,2\},\{3,4\}\}$ implies (1). The converse is also clear from these arguments.

When $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,2\},\{3,4\}\}$ we have $\tilde{H}_0(\Delta_{\underline{\alpha}}, K) \cong K$ and $|G_{\underline{\alpha}}| = 0$. Hence, by Lemma 1.2, $\dim H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} = 1$, as required. $\square$

2. Fourier-Motzkin elimination

By Lemma 1.4 we are interested in finding an integer solution of the following system of inequalities

$$(2)\qquad \begin{aligned} y_1 + y_3 &\geq a_2\\ y_1 + y_4 &\geq a_3\\ y_2 + y_3 &\geq a_4\\ y_2 + y_4 &\geq a_5\\ y_1 + y_2 &\leq a_1 - 1\\ y_3 + y_4 &\leq a_6 - 1\\ y_1 + y_2 + y_3 + y_4 &= d\\ y_1, y_2, y_3, y_4 &\geq 0. \end{aligned}$$

For this purpose we apply the Fourier-Motzkin elimination which at first enables to find a real solution of a system of linear equalities and inequalities, see, e.g. [2], Section 2.3. We sketch here the algorithm by considering a concrete example.

Example. Consider the system

$$\begin{aligned} y_1+2y_2-y_3+4 &\geq 0\\ -2y_1+y_2+3y_3-2 &\geq 0\\ 2y_2-y_3 &\geq 0\\ y_1 &= y_2+y_3. \end{aligned} \tag{3}$$

First, replace the equality $y_1 = y_2 + y_3$ by two inequalities $y_1 \geq y_2 + y_3$ and $y_1 \leq y_2+y_3$. The obtained system is not reduced w.r.t. y_1, i.e. y_1 appears with a non-zero coefficient in at least one inequality. After dividing by the absolute value of the coefficient of y_1 when nonzero and rearranging the terms and the order of the constraints, we can then partition them in 3 groups, depending on whether in a particular constraint y_1 is on the right or the left hand, or its y_1-coefficient is zero.

$$\begin{aligned} \tfrac{1}{2}y_2+\tfrac{3}{2}y_3-1 &\geq y_1 && (E1)\\ y_2+y_3 &\geq y_1 && (E2)\\ y_1 &\geq -2y_2+y_3-4 && (E3)\\ y_1 &\geq y_2+y_3 && (E4)\\ 2y_2-y_3 &\geq 0. && (E5) \end{aligned}$$

Combining each inequality in the first group $\{(E1),\ (E2)\}$ with another one in the second group $\{(E3),\ (E4)\}$ and keep all inequalities in the third group ($\{(E5)\}$ in this example), we obtain a new system of inequalities

$$\begin{aligned} \tfrac{1}{2}y_2+\tfrac{3}{2}y_3-1 &\geq -2y_2+y_3-4 && (E1,E3)\\ \tfrac{1}{2}y_2+\tfrac{3}{2}y_3-1 &\geq y_2+y_3 && (E1,E4)\\ y_2+y_3 &\geq -2y_2+y_3-4 && (E2,E3)\\ y_2+y_3 &\geq y_2+y_3 && (E2,E4)\\ 2y_2-y_3 &\geq 0. && (E5) \end{aligned}$$

The temporary label $(E1, E3)$ means that this inequality appears by combining $(E1)$ and $(E3)$. Note that in the last system, $(E1, E3)$ follows from $(E1, E4)$ and $(E2, E3)$. For short, we will write this reduction as $(E1, E4) + (E2, E3) \Rightarrow (E1, E3)$. The constraint $(E2, E4)$ trivially holds. We say that $(E1, E3)$ and $(E2, E4)$ are redundant. Deleting the redundant inequalities, we finally get the system

$$\begin{aligned} \tfrac{1}{2}y_2+\tfrac{3}{2}y_3-1 &\geq y_2+y_3\\ y_2+y_3 &\geq -2y_2+y_3-4\\ 2y_2-y_3 &\geq 0. \end{aligned} \tag{4}$$

Thus (3) implies (4), where y_1 appears with zero coefficient in all inequalities. We say that y_1 has been "eliminated". The process is repeated with the new system except now y_2 is eliminated.

We now apply the Fourier-Motzkin elimination to our system (2). First rewrite it in the form

$$
(5)\qquad \begin{aligned}
a_6 - 1 - y_3 &\geq y_4\\
y_4 &= d - y_1 - y_2 - y_3\\
y_4 &\geq a_3 - y_1\\
y_4 &\geq a_5 - y_2\\
y_4 &\geq 0\\
y_1 + y_3 &\geq a_2\\
y_1 + y_2 &\leq a_1 - 1\\
y_2 + y_3 &\geq a_4\\
y_1, y_2, y_3 &\geq 0.
\end{aligned}
$$

Eliminating y_4 we then get

$$
(6)\qquad \begin{aligned}
d - a_3 - y_2 &\geq y_3\\
d - a_5 - y_1 &\geq y_3\\
d - y_1 - y_2 &\geq y_3\\
y_3 &\geq 0\\
y_3 &\geq a_2 - y_1\\
y_3 &\geq a_4 - y_2\\
y_1 + y_2 + a_6 - d - 1 &\geq 0\\
y_1 + y_2 &\leq a_1 - 1\\
y_1, y_2 &\geq 0.
\end{aligned}
$$

Eliminating y_3 we now obtain

$$
(7)\qquad \begin{aligned}
d - a_3 &\geq y_2 && (7.1)\\
d - a_2 - a_3 + y_1 &\geq y_2 && (7.2)\\
d - y_1 &\geq y_2 && (7.3)\\
d - a_2 &\geq y_2 && (7.4)\\
a_1 - 1 - y_1 &\geq y_2 && (7.5)\\
y_2 &\geq 0 && (7.6)\\
y_2 &\geq a_4 + a_5 - d + y_1 && (7.7)\\
y_2 &\geq d + 1 - a_6 - y_1 && (7.8)\\
d - a_5 &\geq y_1 && (7.9)\\
d - a_4 &\geq y_1 && (7.10)\\
y_1 &\geq 0 && (7.11)\\
d &\geq a_3 + a_4 && (7.12)\\
d &\geq a_2 + a_5. && (7.13)
\end{aligned}
$$

By eliminating y_2 we get a system of 20 constraints. However 7 of them are redundant: $(7.12) \Rightarrow (7.1, 7.6)$; $(7.9) + (7.12) \Rightarrow (7.1, 7.7)$; $(7.12) + (7.13) \Rightarrow (7.2, 7.7)$; $(7.9) + (7.10) \Rightarrow (7.3, 7.6), (7.3, 7.7), (7.4, 7.7)$ and $(7.13) \Rightarrow (7.4, 7.6)$.

Deleting these redundant constraints we get

$$(8)\qquad \begin{array}{rll}
d-a_4 & \geq y_1 & (8.1)\\
d-a_5 & \geq y_1 & (8.2)\\
a_1-1 & \geq y_1 & (8.3)\\
\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor & \geq y_1 & (8.4)\\
y_1 & \geq 0 & (8.5)\\
y_1 & \geq \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil & (8.6)\\
y_1 & \geq a_2+a_3-d & (8.7)\\
y_1 & \geq a_3-a_6+1 & (8.8)\\
y_1 & \geq a_2-a_6+1 & (8.9)\\
a_1+a_6-2 & \geq d & (8.10)\\
d & \geq a_2+a_5 & (8.11)\\
d & \geq a_3+a_4 & (8.12)\\
a_6 & \geq 1. & (8.13)
\end{array}$$

Here, for a real number a, we set

$$\lceil a\rceil = \min\{n\in\mathbb{Z}|\ n\geq a\} \text{ and } \lfloor a\rfloor = \max\{n\in\mathbb{Z}|\ n\leq a\}.$$

Eliminating y_1 we get a system of 24 constraints. Among them 14 are redundant: (8.12) $\Rightarrow$ (8.1, 8.5); (8.1, 8.9) + (8.12) $\Rightarrow$ (8.1, 8.6); (8.11) + (8.12) $\Rightarrow$ (8.1, 8.7); (8.12) + 8.13) $\Rightarrow$ (8.1, 8.8); (8.11) $\Rightarrow$ (8.2, 8.5); (8.2, 8.8) + (8.11) $\Rightarrow$ (8.2, 8.6); (8.11) + (8.12) $\Rightarrow$ (8.2, 8.7); (8.11) + (8.13) $\Rightarrow$ (8.2, 8.9); (8.3, 8.7) + (8.10) $\Rightarrow$ (8.3, 8.6); (8.10)+(8.12) $\Rightarrow$ (8.3, 8.8); (8.10)+(8.11) $\Rightarrow$ (8.3, 8.9); (8.11)+(8.12)+(8.3, 8.7) $\Rightarrow$ (8.4, 8.7); (8.10)+(8.12)+(8.2, 8.8) $\Rightarrow$ (8.4, 8.8) and (8.10)+(8.11) + (8.1, 8.9) $\Rightarrow$ (8.4, 8.9). Deleting these redundant constraints, we finally get the system

$$(9)\qquad \begin{array}{rl}
a_1+a_6-2 & \geq d\\
d & \geq a_2+a_5\\
d & \geq a_3+a_4\\
d & \geq a_2+a_4-a_6+1\\
d & \geq a_3+a_5-a_6+1\\
d & \geq a_2+a_3-a_1+1\\
d & \geq a_4+a_5-a_1+1\\
\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor & \geq \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil\\
a_1, a_6 & \geq 1.
\end{array}$$

Lemma 2.1. *Assume that* $a_1+a_6-2\geq d\geq \max\{a_2+a_5, a_3+a_4\}$. *Then* $\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor < \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil$ *if and only if* $a_2+a_3-a_6$ *is even and* $a_1+a_6-2=a_2+a_5=a_3+a_4$.

Proof. If $a_2+a_3-a_6$ is odd, then

$$\lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil = \frac{1}{2}(a_2+a_3-a_6+1).$$

Since $a_2+a_5+a_3+a_4 \leq d+a_1+a_6-2$, we get $d+a_1-a_4-a_5-1 \geq a_2+a_3-a_6+1$, which yields

$$\frac{1}{2}(d+a_1-a_4-a_5-1) \geq \frac{1}{2}(a_2+a_3-a_6+1).$$

Hence

$$\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor \geq \frac{1}{2}(a_2+a_3-a_6+1) = \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil.$$

If $a_2+a_3-a_6$ is even, then

$$\lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil = \frac{1}{2}(a_2+a_3-a_6)+1.$$

In the case $a_1+a_6-2 > \min\{a_2+a_5, a_3+a_4\}$, we have $a_2+a_5+a_3+a_4 \leq d+a_1+a_6-3$. Hence $d+a_1-a_4-a_5-1 \geq a_2+a_3-a_6+2$, which implies

$$\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor \geq \frac{1}{2}(a_2+a_3-a_6)+1 = \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil.$$

The left case is $a_1+a_6-2 = \min\{a_2+a_5, a_3+a_4\}$. Since $a_1+a_6-2 \geq d \geq \max\{a_2+a_5, a_3+a_4\}$, we must have $d = a_2+a_5 = a_3+a_4 = a_1+a_6-2$. Then $d+a_1-a_4-a_5-1 = a_2+a_3-a_6+1$ is an odd number. Therefore

$$\lfloor\frac{1}{2}(d+a_1-a_4-a_5-1)\rfloor < \lceil\frac{1}{2}(a_2+a_3-a_6+1)\rceil.$$

This completes the proof of the lemma. □

Going back from (9) to (5), the Fourier-Motzkin algorithm gives us in general only a rational solution of (2) if (9) holds. However, in our concrete situation we can already find an integer solution.

Lemma 2.2. *Let*

$$\begin{aligned}\mathcal{A} = \max\{&a_2+a_5,\ a_3+a_4,\ a_2+a_4-a_6+1,\ a_3+a_5-a_6+1,\\ &a_2+a_3-a_1+1,\ a_4+a_5-a_1+1\}.\end{aligned}$$

The system (2) has an integer solution if and only if $a_1, a_6 \geq 1$ and one of the following conditions holds:

(i) $a_1+a_6-2 > \mathcal{A}$ *and* $a_1+a_6-2 \geq d \geq \mathcal{A}$.
(ii) $a_1+a_6-2 = \mathcal{A} = d$ *and* $a_1+a_6-2 > \min\{a_2+a_5,\ a_3+a_4\}$.
(iii) $a_1+a_6-2 = a_2+a_5 = a_3+a_4 = \mathcal{A} = d$ *and* $a_2+a_3-a_6$ *is odd.*

Proof. If (2) has an integer solution, then by Fourier-Motzkin algorithm, (9) holds. Using Lemma 2.1 we get the necessity.

Assume that $a_1, a_6 \geq 1$ and one of the above conditions (i)-(iii) holds. Then for any d such that $\mathcal{A} \leq d \leq a_1+a_6-2$, the system (9) holds by Lemma 2.1. Fix such an integer d. Denote by $\mathcal{L}_8$ the minimum of integers in the left sides of (8.1) – (8.4) and $\mathcal{R}_8$ the maximum of integers in the right sides of (8.5) – (8.9). Then from (9) it follows that $\mathcal{L}_8 \geq \mathcal{R}_8$. Hence $y_1 = \mathcal{R}_8$ is an integer solution of (8). Putting $y_1 = \mathcal{R}_8$ into (7)-(5) and repeating this process, we can similarly define

$\mathcal{L}_7 \geq \mathcal{R}_7,\ \mathcal{L}_6 \geq \mathcal{R}_6,\ \mathcal{L}_5 \geq \mathcal{R}_5$ such that $y_1 = \mathcal{R}_8,\ y_2 = \mathcal{R}_7,\ y_3 = \mathcal{R}_6,\ y_4 = \mathcal{R}_5$ is an integer solution of (5), which is equivalent to (2). □

3. Structure of the first local cohomology module

In this section we describe the first local cohomology module of R/I. From now on, w.l.o.g., we always assume that $a_1 + a_6$ is the maximum among the sums $a_1 + a_6,\ a_2 + a_5,\ a_3 + a_4$. In other words we may assume that the following holds:

$$(*) \qquad a_1 + a_6 \geq \max\{a_2 + a_5,\ a_3 + a_4\}.$$

Lemma 3.1. *Under the assumption* $(*)$ *there exists no* $\underline{\alpha} \in \mathbb{Z}^4$ *such that* $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,3\},\ \{2,4\}\}$ *or* $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,4\},\ \{2,3\}\}$.

Proof. Assume, w.l.o.g., the existence of $\underline{\alpha} \in \mathbb{Z}^n$ such that $\mathrm{Max}(\Delta_{\underline{\alpha}}) = \{\{1,3\}, \{2,4\}\}$. Then applying Lemma 1.4 and Lemma 2.2 to this situation we would get $a_2 + a_5 - 2 \geq a_1 + a_6$, a contradiction to (*). □

We can now explicitly determine all arithmetically Cohen-Macaulay tetrahedral curves in terms of a_i. This result recovers the main theorem in [3].

Theorem 3.2. *Let*

$$\mathcal{A} = \max\{a_2 + a_5,\ a_3 + a_4,\ a_2 + a_4 - a_6 + 1,\ a_3 + a_5 - a_6 + 1,\\ a_2 + a_3 - a_1 + 1,\ a_4 + a_5 - a_1 + 1\}.$$

Under the assumption $(*)$, *a tetrahedral curve* $C(a_1, ..., a_6)$ *is arithmetically Cohen-Macaulay if and only if one of the following conditions holds:*

(i) $a_1 = 0$ *or* $a_6 = 0$;
(ii) $a_1 + a_6 - 2 < \mathcal{A}$;
(iii) $a_1 + a_6 - 2 = a_2 + a_5 = a_3 + a_4 = \mathcal{A}$ *and* $a_2 + a_3 - a_6$ *is even.*

Proof. By Lemma 3.1, $C = C(a_1, ..., a_6)$ is arithmetically Cohen-Macaulay if and only if there is no d such that the system (2) has an integer solution. Hence the statement follows from Lemma 2.2. □

Remark. In [6], Question 7.4(5), Migliore and Nagel asked whether an arithmetically Cohen-Macaulay tetrahedral curve $C = C(a_1, ..., a_6)$ can be explicitly identified by the 6-tuples $a_1, ..., a_6$. This question was solved by Francisco in [3]. His main result says that under the assumption (*), $C(a_1, ..., a_6)$ is arithmetically Cohen-Macaulay if and only if one of the following conditions holds:

(a) $a_1 = 0$ or $a_6 = 0$;
(b) $a_1 + a_6 = \epsilon + \max\{a_2 + a_5,\ a_3 + a_4\}$, where $\epsilon \in \{0, 1\}$.
(c) $2a_1 < a_2 + a_3 - a_6 + 3$ or $2a_1 < a_4 + a_5 - a_6 + 3$ or $2a_6 < a_2 + a_4 - a_1 + 3$ or $2a_6 < a_3 + a_5 - a_1 + 3$;
(d) All inequalities of (c) fail, $a_1 + a_6 = a_2 + a_5 + 2 = a_3 + a_4 + 2$ and $a_1 + a_3 + a_5$ is even.

One can easily check that this statement is equivalent to that of Theorem 3.2.

Assume now that C is not arithmetically Cohen-Macaulay. Then $a_1, a_6 \geq 1$ and one of three conditions in Lemma 2.2 is satisfied. In particular $\mathcal{A} \leq a_1 + a_6 - 2$. Let

$$\begin{aligned} T_1 &= \{\underline{y} \in \mathbb{N}^4 |\ y_1 + y_3 \geq a_2,\ y_1 + y_4 \geq a_3,\ y_2 + y_3 \geq a_4,\ y_2 + y_4 \geq a_5\}, \\ T_2 &= \{\underline{y} \in T_1 |\ y_1 + y_2 \geq a_1\}, \end{aligned}$$

and

$$T_3 = \{\underline{y} \in T_1 |\ y_3 + y_4 \geq a_6\}.$$

Let $S = T_1 \setminus (T_2 \cup T_3)$. Then the set S_d of all elements of degree d of S is the set of all solutions of the system (2). As usual we identify $K[T_i],\ i \leq 3$, and $K[S]$ with subsets of $R = K[x_1, ..., x_4]$. Note that $K[T_i],\ i \leq 3$, are ideals of R. Hence we may consider $K[S]$ as a factor module $K[T_1]/K[T_2] + K[T_3]$. Thus, the module structure on $K[S]$ over R is defined as follows: for $\underline{\alpha} \in S$ and $\underline{\beta} \in \mathbb{N}^4$,

$$\underline{x}^{\underline{\beta}} \cdot \underline{x}^{\underline{\alpha}} = \begin{cases} \underline{x}^{\underline{\beta}+\underline{\alpha}} & \text{if } \underline{\beta} + \underline{\alpha} \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The following result describes the module structure of $H^1_{\mathfrak{m}}(R/I)$.

Proposition 3.3. *Under the assumption* $(*)$*,*

$$H^1_{\mathfrak{m}}(R/I) \cong K[S]$$

as graded modules over R*.*

Proof. Let

$$\mathcal{C}^\bullet : 0 \to R/I \to \oplus_{i=1}^4 (R/I)_{x_i} \to \cdots \to (R/I)_{x_1x_2x_3x_4} \to 0,$$

be the Čech complex of R/I. Then $H^1_{\mathfrak{m}}(R/I) \cong H^1(\mathcal{C}^\bullet)$. By [9], Lemma 2, for all $\underline{\alpha} \in \mathbb{Z}^4$ there is an isomorphism of complexes

$$(\mathcal{C}^\bullet_{\underline{\alpha}}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\underline{\alpha}})[-j-1], K),$$

where $j = |G_{\underline{\alpha}}|$ and $\mathcal{C}(\Delta_{\underline{\alpha}})[-j-1]$ means the shifting of the augmented oriented chain complex $\mathcal{C}(\Delta_{\underline{\alpha}})$ by $-j-1$. Denote by π the simplicial complex on $\{1, 2, 3, 4\}$ with $\mathrm{Max}(\pi) = \{\{1, 2\},\ \{3, 4\}\}$. By Lemmas 1.3, 1.4 and 3.1 it follows that $H^1(\mathcal{C}^\bullet_{\underline{\alpha}}) \neq 0$ if and only if $\Delta_{\underline{\alpha}} = \pi$, $G_{\underline{\alpha}} = \emptyset$ and $\underline{\alpha} \in S$. Moreover, in this case $H^1(\mathcal{C}^\bullet_{\underline{\alpha}}) \cong K\underline{x}^{\underline{\alpha}}$. From this we get $H^1_{\mathfrak{m}}(R/I) \cong K[S]$, as required. □

The above description of S allows us to describe the module structure of $K[S]$ in an obvious way. Of course, S can be written as:

$$\begin{aligned} S = \{\underline{y} \in \mathbb{N}^4 |\quad & y_1 + y_3 \geq a_2,\ y_1 + y_4 \geq a_3,\ y_2 + y_3 \geq a_4,\ y_2 + y_4 \geq a_5, \\ & y_1 + y_2 < a_1,\ y_3 + y_4 < a_6\}. \end{aligned}$$

It is easy to write a program to compute this set S. Hence the module structure of $H^1_{\mathfrak{m}}(R/I)$ is known once $a_1, ..., a_6$ are given.

We say that a non-zero $\mathbb{Z}$-graded module M has no gap if $M_i \neq 0$ and $M_j \neq 0$ for some $i \leq j$, then $M_k \neq 0$ for all $i \leq k \leq j$. Recall that the diameter of a module M of finite length is defined as

$$\operatorname{diam}(M) = \operatorname{end}(M) - \operatorname{beg}(M) + 1,$$

where $\operatorname{beg}(M) = \min\{i|\ M_i \neq 0\}$ and $\operatorname{end}(M) = \max\{i|\ M_i \neq 0\}$ (if $M = 0$ we set $\operatorname{diam}(M) = 0$).

Theorem 3.4. *Let*

$$\begin{aligned}\mathcal{A} = \max\{ \quad & a_2 + a_5,\ a_3 + a_4,\ a_2 + a_4 - a_6 + 1,\ a_3 + a_5 - a_6 + 1, \\ & a_2 + a_3 - a_1 + 1,\ a_4 + a_5 - a_1 + 1\}.\end{aligned}$$

Assume that (*) holds and the tetrahedral curve C is not arithmetically Cohen-Macaulay. Then $a_1 + a_6 - 2 \geq \mathcal{A}$ and

$$k(R/I) = \operatorname{diam}(H^1_{\mathfrak{m}}(R/I)) = a_1 + a_6 - \mathcal{A} - 1.$$

In particular, $H^1_{\mathfrak{m}}(R/I)$ has no gap.

Proof. Since R/I is not a Cohen-Macaulay ring, by Theorem 3.2, $a_1 + a_6 - 2 \geq \mathcal{A}$ and $a_1, a_6 \geq 1$. By Lemma 2.2, for each d such that $\mathcal{A} \leq d \leq a_1 + a_6 - 2$ we have $S_d \neq \emptyset$. Hence, by Proposition 3.3, $H^1_{\mathfrak{m}}(R/I)$ has no gap, $\operatorname{beg}(H^1_{\mathfrak{m}}(R/I)) = \mathcal{A}$ and $\operatorname{end}(H^1_{\mathfrak{m}}(R/I)) = a_1 + a_6 - 2$, which implies $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I)) = a_1 + a_6 - \mathcal{A} - 1$.

Further, let $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in S_{\mathcal{A}}$ be a fixed element. Then $\alpha_1 + \alpha_2 \leq a_1 - 1$ and $\alpha_3 + \alpha_4 \leq a_6 - 1$. Let $\underline{\alpha}^* = (\alpha_1, a_1 - 1 - \alpha_1, \alpha_3, a_6 - 1 - \alpha_3)$. Since $a_1 - 1 - \alpha_1 \geq \alpha_2$ and $a_6 - 1 - \alpha_3 \geq \alpha_4$, the condition $\underline{\alpha} \in T_1$ implies $\underline{\alpha}^* \in T_1$ too. On the other hand $\underline{\alpha}^* \notin T_1 \cup T_2$. Hence $\underline{\alpha}^* \in S_{a_1+a_6-2}$. Note that $\underline{\alpha}^* = \underline{\alpha} + \underline{\beta}$, where $\underline{\beta} = (0, a_1 - 1 - \alpha_1 - \alpha_2, 0, a_6 - 1 - \alpha_3 - \alpha_4) \in \mathbb{N}^4$ and $\deg(\underline{\beta}) = a_1 + a_6 - \mathcal{A} - 2$. Therefore, by Proposition 3.3,

$$\underline{x}^{\underline{\beta}} H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}} \cong H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}+\underline{\beta}} = H^1_{\mathfrak{m}}(R/I)_{\underline{\alpha}^*} \neq 0,$$

which yields

$$k(R/I) \geq a_1 + a_6 - \mathcal{A} - 1 = \operatorname{diam}(H^1_{\mathfrak{m}}(R/I)).$$

Since $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I)) \geq k(R/I)$, we finally get $k(R/I) = \operatorname{diam}(H^1_{\mathfrak{m}}(R/I))$, as required. $\square$

In the above proof we already showed:

Corollary 3.5. *Assume that (*) holds and the tetrahedral curve C is not arithmetically Cohen-Macaulay. Then $a_1 + a_6 - 2 \geq \mathcal{A}$ and $\operatorname{end}(H^1_{\mathfrak{m}}(R/I)) = a_1 + a_6 - 2$.*

Recall that C is arithmetically Buchsbaum if and only if $k(R/I) \leq 1$. As an immediate consequence of Theorem 3.4 we recover Corollary 4 in [6].

Corollary 3.6. *A tetrahedral curve C is arithmetically Buchsbaum if and only if*

$$H^1_{\mathfrak{m}}(R/I) \cong K^m(t),$$

for some non-negative integers m, t.

Migliore and Nagel found all arithmetically Buchsbaum tetrahedral curves which are so-called minimal (see Corollary 3.8 below). Using Theorems 3.4 and 3.2 we are able to determine all arithmetically Buchsbaum tetrahedral curves which are not necessarily minimal.

Theorem 3.7. *Under the assumption* $(*)$, *a tetrahedral curve* C *is arithmetically Buchsbaum if and only if one of the following conditions is satisfied:*

(i) $a_1 = 0$ *or* $a_2 = 0$;

(ii) $a_1 + a_6 - 2 \leq \mathcal{A}$.

Proof. If C is arithmetically Cohen-Macaulay, by Theorem 3.2, one of the above condition holds. Assume that C is not arithmetically Cohen-Macaulay and arithmetically Buchsbaum. Then $k(R/I) = 1$. By Theorem 3.4, $a_1, a_6 \geq 1$ and $a_1 + a_6 - 2 = \mathcal{A}$. Conversely, by Theorem 3.2 we may assume from the beginning that $a_1, a_6 \geq 1$. Under these conditions, again by Theorem 3.4,we immediately have $k(R/I) \leq 1$, i.e. C is arithmetically Buchsbaum. □

Migliore and Nagel introduced the following notion: Assume that $a_6 = \max\{a_1, ..., a_6\}$. A tetrahedral curve C is said to be *minimal* if $a_1 > \max\{a_2 + a_4,\ a_3 + a_5\}$ and $a_6 > \max\{a_2 + a_3,\ a_4 + a_5\}$ (see [6], Definition 3.4 and Corollary 3.5). Note that in this case we already have $a_1, a_6 \geq 1$ and $a_1 + a_6 - 2 \geq \mathcal{A}$.

Corollary 3.8. ([6], Corollary 4.3 and Corollary 5.4). *Assume that* $a_6 = \max\{a_1, ..., a_6\}$ *and* C *is a minimal tetrahedral curve. Then*

(i) C *is not arithmetically Cohen-Macaulay.*

(ii) C *is arithmetically Buchsbaum if and only if either* $a_2 = a_5 = 0$ *and* $a_1 = a_6 = a_3 + 1 = a_4 + 1$ *or* $a_3 = a_4 = 0$ *and* $a_1 = a_6 = a_2 + 1 = a_5 + 1$.

Proof. Since $a_1 > \max\{a_2 + a_4,\ a_3 + a_5\}$ and $a_6 > \max\{a_2 + a_3,\ a_4 + a_5\}$, we have

$$a_1 + a_6 - 2 \geq \max\{\begin{array}{l} a_2 + a_5 + 2a_4,\ a_2 + a_5 + 2a_3, \\ a_3 + a_4 + 2a_2,\ a_3 + a_4 + 2a_5\} \geq \mathcal{A}. \end{array} \tag{10}$$

If C is arithmetically Buchsbaum, then since $a_1, a_6 \geq 1$, by Theorem 3.2 and Theorem 3.7, we must have $a_1 + a_6 - 2 = \mathcal{A}$. Combining with (10) this implies that either $a_2 = a_5 = 0$ or $a_3 = a_4 = 0$. W.l.o.g. assume that $a_2 = a_5 = 0$. Then $\mathcal{A} = a_3 + a_4$ and $a_1 + a_6 - 2 = a_3 + a_4$. Since $a_1, a_6 > \max\{a_3,\ a_4\}$, the latter equality gives $a_1 = a_6 = a_3 + 1 = a_4 + 1$. In this case $a_2 + a_3 - a_6 = -1$ is odd, so C is not arithmetically Cohen-Macaulay. Thus we have proved (i) and the necessity of (ii). The sufficiency of (ii) immediately follows from Theorem 3.7. □

Similarly, using Theorem 3.4, we can quickly get

Corollary 3.9. ([6, Lemma 6.2]). *Assume that* $a_6 = \max\{a_1, ..., a_6\}$ *and* C *is a minimal tetrahedral curve. Then* $\operatorname{diam}(H^1_{\mathfrak{m}}(R/I)) = 2$ *if and only if after a* *suitable permutation of variables we have* $(a_1, ..., a_6) = (k, k-1, 0, 0, k-1, k+1),\ k \geq 1$ *or* $(a_1, ..., a_6) = (k, k-2, 0, 0, k-1, k),\ k \geq 2$.

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