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UNIFORM ATTRACTORS FOR A NON-AUTONOMOUS PARABOLIC EQUATION INVOLVING GRUSHIN OPERATOR

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Abstract. In this paper, we consider the first initial boundary value problem for a non-autonomous semilinear degenerate parabolic equation involving the Grushin operator, in which the nonlinearity satisfies the polynomial condition of arbitrary order and the external force is normal. Using the asymptotic *a priori* estimate method, we prove the existence of uniform attractors for this problem. The obtained results, in particular, improve some recent ones for the non-autonomous Laplacian equation.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for dissipative dynamical systems is to consider its attractor. In the last decades, the existence of the attractor has been proved for a large class of partial differential equations, in both the autonomous and non-autonomous cases (see e.g. [10, 24] and references therein). However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions of degenerate equations.

One of classes of degenerate equations that has been studied widely in recent years is the class of equations involving an operator of Grushin type

$$G_s u = \Delta_{x_1} u + |x_1|^{2s}\Delta_{x_2} u, \quad (x_1, x_2) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, s \geq 0.$$

This operator was first introduced in [15]. Noting that $G_0 = \Delta$ and G_s, when $s > 0$, is not elliptic in domains in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ intersecting with the hyperplane $\{x_1 = 0\}$. The local properties of G_s were investigated in [15, 6]. The existence and nonexistence results for the elliptic equation

$$\begin{aligned} -G_s u + f(u) &= 0, \ x \in \Omega, \\ u &= 0, \ x \in \partial\Omega, \end{aligned}$$

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were proved in [25]. Furthermore, the semilinear elliptic systems with the Grushin type operator, which are in the Hamilton form or in the potential form, were also studied in [11, 12, 17].

In order to study boundary value problems for equations involving the Grushin operator, we have usually used the natural energy space $S_0^1(\Omega)$ defined as the completion of $C_0^1(\bar{\Omega})$ in the following norm

$$\|u\|_{S_0^1(\Omega)} = \Big(\int\limits_\Omega \big(|\nabla_{x_1} u|^2 + |x_1|^{2s}|\nabla_{x_2} u|^2\big)dx\Big)^{1/2}.$$

We have the continuous embedding $S_0^1(\Omega) \hookrightarrow L^r(\Omega)$, for $1 \le r \le 2_s^* = \frac{2N(s)}{N(s)-2}$, where $N(s) = N_1 + (s+1)N_2$. Moreover, this embedding is compact if $1 \le r < 2_s^*$ (for more details, see [25]).

In this paper, we study the following non-autonomous semilinear degenerate parabolic equation in a bounded domain $\Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} (N_1, N_2 \ge 1)$,

$$\begin{aligned} \frac{\partial u}{\partial t} - G_s u + f(u) &= g(t), \quad x \in \Omega, t > \tau, \\ u|_{t=\tau} &= u_\tau(x), \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{1.1}$$

where $u_\tau \in L^2(\Omega)$ is given, $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$C_1|u|^p - k_1 \le f(u)u \le C_2|u|^p + k_2,\ p \ge 2, \tag{1.2}$$

$$f'(u) \ge -\ell,\ \text{for some } \ell > 0, \tag{1.3}$$

and the external force $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfying some conditions specified later.

The existence and long-time behavior of solutions to problem type (1.1) in the autonomous case, that is the case in which $g(x) \in L^2(\Omega)$, have been studied in [3, 4]. In [3], the authors considered problem (1.1) with $u_0 \in S_0^1(\Omega)$, $g \in L^2(\Omega)$ given, and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(u) - f(v)| \le C_0|u - v|(1 + |u|^\gamma + |v|^\gamma)\ , 0 \leqslant \gamma < \frac{4}{N(s) - 2},$$

$$F(u) \ge -\frac{\mu}{2}u^2 - C_1,$$

$$f(u)u \ge -\mu u^2 - C_2,$$

where $C_0, C_1, C_2 \ge 0$, F is the primitive $F(y) = \int\limits_0^y f(s)ds$ of f, $\mu < \lambda_1$, λ_1 is the first eigenvalue of the operator $-G_s$ in Ω with homogeneous Dirichlet condition. Under the above assumptions of f, the authors proved that problem (1.1) defines a semigroup $S(t) : S_0^1(\Omega) \to S_0^1(\Omega)$, which possesses a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $S_0^1(\Omega)$. Furthermore, for each $u_0 \in S_0^1(\Omega)$, the corresponding solution $u(t)$ tends to the set E of equilibrium points in $S_0^1(\Omega)$ as

$t \to +\infty$. The basic tool for the approach in this case is the following Lyapunov functional

$$\Phi(u) = \frac{1}{2}\|u\|^2_{S^1_0(\Omega)} + \int_\Omega (F(u) + gu)dx.$$

Noting that the critical exponent of the embedding $S^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is $2^*_s = \frac{2N(s)}{N(s)-2}$, so the condition $0 \leqslant \gamma < \frac{4}{N(s)-2}$ is necessary to prove the existence of a mild solution by the fixed point method and to ensure the existence of the Lyapunov functional Φ. In [4], under conditions (1.2)-(1.3), the authors proved the existence of a global attractor in $L^2(\Omega)$ of problem (1.1) in the autonomous case. Moreover, the upper semicontinuity of the global attractor with respect to the nonlinearity and the shape of the domain where the problem is posed was also investigated in [4]. For long-time behavior of solutions to other autonomous degenerate parabolic equations, we refer the reader to some recent works [1, 2, 5, 13, 14, 21].

In this paper, we continue studying the long-time behavior of solutions to problem (1.1) by allowing the external force g depending on time t and prove the existence of attractors in more regular spaces. Non-autonomous equations appear in many applications in the natural sciences, so they are also of great importance and interest. The long-time behavior of solutions of such equations have been studied extensively in the last years. In the book [16], Haraux considered some special classes of such systems and studied the notion of a uniform attractor. Later on, Chepyzhov and Vishik [8] presented a general approach that is well suited to studying many equations arising in mathematical physics; see also a recent work of Lu et al. [19] where a new class of symbols was introduced.

The main aim of this paper is to study the existence of an $(L^2(\Omega), S^1_0(\Omega) \cap L^p(\Omega))$-uniform attractor for a family of processes generated by problem (1.1). Let us describe the methods used in the paper. First, we use the compactness method [18] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of an $(L^2(\Omega), S^1_0(\Omega) \cap L^p(\Omega))$-uniformly absorbing set B_0 for the family of processes. By the compactness of the embedding $S^1_0(\Omega) \hookrightarrow L^2(\Omega)$, the family of processes is uniformly asymptotically compact in $L^2(\Omega)$. This immediately implies the existence of an $(L^2(\Omega), L^2(\Omega))$-uniform attractor. When proving the existence of an $(L^2(\Omega), L^p(\Omega))$-uniform attractor and an $(L^2(\Omega), S^1_0(\Omega) \cap L^p(\Omega))$-uniform attractor, to overcome the difficulty arising because lack of embbeding results, we use the asymptotic *a priori* estimate method initiated in [20] for autonomous equations and developed in [19] for non-autonomous equations. This method has been applied successfully for some classes of partial differential equations (see e.g. [19, 20, 22, 23, 26, 28]). One of the main new features in our paper is that the existence of uniform attractors is proved for a class of semilinear degenerate parabolic equations involving the Grushin operator. It is also worth noticing that, when $s = 0$, our results improve the recent ones in [22] for the non-autonomous Laplacian equation.

The rest of this paper is organized as follows. In Section 2, for the convenience of the reader, we recall some concepts and results on function spaces and uniform attractors which we will use. Section 3 is devoted to the proof of the existence and uniqueness of a global weak solution to problem (1.1). In Section 4, we prove the existence of uniform attractors in various spaces by using the asymptotic *a priori* estimate method. Some further remarks are also given.

2. Preliminaries

2.1. **Function Spaces and Operator.** The natural energy space for problem (1.1) involves the space $S_0^1(\Omega)$, defined as the closure of $C_0^1(\bar{\Omega})$ with respect to the norm

$$\|u\|_{S_0^1(\Omega)} := \left(\int\limits_{\Omega} (|\nabla_{x_1} u|^2 + |x_1|^{2s}|\nabla_{x_2} u|^2)dx\right)^{\frac{1}{2}}.$$

The space $S_0^1(\Omega)$ is a Hilbert space with respect to the scalar product

$$(u,v) := \int\limits_{\Omega} (\nabla_{x_1} u \nabla_{x_1} v + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2} v)dx.$$

The following lemma comes from [25].

Lemma 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ ($N_1, N_2 \geq 2$). Then the following embeddings hold:*
(i) $S_0^1(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ *continuously,*
(ii) $S_0^1(\Omega) \hookrightarrow L^r(\Omega)$ *compactly if* $r \in [1, 2_s^*)$,
where $2_s^* = \frac{2N(s)}{N(s)-2}$, $N(s) = N_1 + (s+1)N_2$.

We consider the boundary value problem

$$-G_s u = h(x),\ x \in \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.1}$$

In order to apply the Friedrichs extension of symmetric operators, we set

$$X = L^2(\Omega), \qquad D(\tilde{A}) = C_0^\infty(\Omega), \qquad \tilde{A}u = -G_s u.$$

The problem (2.1) corresponds to the operator equation

$$\tilde{A}u = h, \qquad u \in C_0^\infty(\Omega), h \in X.$$

For every $u, v \in C_0^\infty(\Omega)$, we have

$$(\tilde{A}u, v) = \int\limits_{\Omega} (\nabla_{x_1} u \nabla_{x_1} v + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2} v)dx = (u, \tilde{A}v).$$

It follows from Lemma 2.1 that there exists a constant $C > 0$ such that

$$(\tilde{A}u, u) \geq C||u||_X^2, \qquad \text{for any } u \in C_0^\infty(\Omega).$$

Hence, $\tilde{A}$ is symmetric and strongly monotone. Applying the Friedrichs extension theorem [27], we find that the energy space X_E equals $S_0^1(\Omega)$ since X_E is the completion of $D(\tilde{A}) = C_0^\infty(\Omega)$ with respect to the scalar product $(u,v) = \int\limits_{\Omega}(\nabla_{x_1} u \nabla_{x_1} v + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2} v)dx$, and the extensions satisfy

$$\tilde{A} \subset A \subset A_E,$$

where $A_E : S_0^1(\Omega) \to S^{-1}(\Omega)$ is the energetic extension ($S^{-1}(\Omega)$ is the dual space of $S_0^1(\Omega)$), and $A = -G_s$ is the Friedrichs extension of $\tilde{A}$ with the domain of definition

$$D(A) = \{u \in S_0^1(\Omega) : Au \in X\}.$$

Noticing that $2_s^* > 2$, we have an evolution triple

$$S_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow S^{-1}(\Omega)$$

with compact and dense embbedings. Hence, there exists a complete orthonormal system of eigenvectors $\{e_j\}$ corresponding to the eigenvalues $\{\lambda_j\}$ such that

$$(e_j, e_k) = \delta_{jk} \qquad \text{and} \qquad - G_s e_j = \lambda_j e_j, \quad j, k = 1, 2, \dots,$$
$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots, \qquad \lambda_j \to +\infty \text{ as } j \to \infty.$$

2.2. **The translation normal functions.**

Definition 2.1. *Let $\mathcal{E}$ be a reflexive Banach space.*

(1) *A function $\varphi \in L^2_{loc}(\mathbb{R}; \mathcal{E})$ is said to be the translation bounded if*

$$\|\varphi\|^2_{L^2_b} = \|\varphi\|_{L^2_b(\mathbb{R};\mathcal{E})} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\varphi\|^2_{\mathcal{E}} ds < \infty.$$

(2) *A function $\varphi \in L^2_{loc}(\mathbb{R}; \mathcal{E})$ is said to be the translation normal if for any $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} \|\varphi\|^2_{\mathcal{E}} ds < \varepsilon.$$

(3) *A function $\varphi \in L^2_{loc}(\mathbb{R}; \mathcal{E})$ is said to be the translation compact if the closure of $\{\varphi(\cdot + h) | h \in \mathbb{R}\}$ is compact in $L^2_{loc}(\mathbb{R}; \mathcal{E})$.*

Denote by $L^2_b(\mathbb{R}; \mathcal{E})$, $L^2_n(\mathbb{R}; \mathcal{E})$ and $L^2_c(\mathbb{R}; \mathcal{E})$ the sets of all translation bounded, translation normal and translation compact functions in $L^2_{loc}(\mathbb{R}; \mathcal{E})$, respectively. It is well-known (see [19]) that

$$L^2_c(\mathbb{R}; \mathcal{E}) \subset L^2_n(\mathbb{R}; \mathcal{E}) \subset L^2_b(\mathbb{R}; \mathcal{E}).$$

Let $g \in L^2_b(\mathbb{R}; \mathcal{E})$. Denote by $\mathcal{H}_w(g)$ the closure of the set $\{g(\cdot + h) | h \in \mathbb{R}\}$ in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$ with the weak topology. The following results were proved in [10].

Lemma 2.2. [10, Chapter 5, Proposition 4.2]

(1) *For all $\sigma \in \mathcal{H}_w(g)$, $\|\sigma\|^2_{L^2_b} \le \|g\|^2_{L^2_b}$;*
(2) *The translation group $\{T(h)\}$ is weakly continuous on $\mathcal{H}_w(g)$;*
(3) *$T(h)\mathcal{H}_w(g) = \mathcal{H}_w(g)$ for $h \ge 0$;*
(4) *$\mathcal{H}_w(g)$ is weakly compact.*

Lemma 2.3. [19, Lemma 3.1] *If $g \in L^2_n(\mathbb{R}; \mathcal{E})$ then for any $\tau \in \mathbb{R}$,*

$$\lim_{\gamma \to +\infty} \sup_{t \ge \tau} \int_\tau^t e^{-\gamma(t-\tau)} \|\varphi\|^2_{\mathcal{E}} ds = 0,$$

for all $\varphi \in \mathcal{H}_w(g)$.

2.3. **Uniform attractors.** Let Σ be a parameter set, X, Y are two Banach spaces, $Y \subset X$ continuously. $\{U_\sigma(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, is said to be a family of processes in X, if each $\sigma \in \Sigma, \{U_\sigma(t,\tau)\}$ is a process, that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from X to X satisfies

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \forall t \geq s \geq \tau, \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = Id, \text{ the identity operator}, \tau \in \mathbb{R},$$

where Σ is called the symbol space, $\sigma \in \Sigma$ is the symbol. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X.

Definition 2.2. *A set $B_0 \in \mathcal{B}(Y)$ is said to be an (X,Y)-uniformly absorbing set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if for any $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(X)$, there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$.*

Definition 2.3. *A closed set $\mathcal{A}_\Sigma \subset Y$ is said to be an (X,Y)-uniformly attractor set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if it is (X,Y)-uniformly attracting set, i.e., $\lim_{t\to+\infty}(\sup dist_Y(U_\sigma(t,\tau)B, \mathcal{A}_\Sigma)) = 0$ for any $B \in \mathcal{B}(X)$, and it is contained in any closed (X,Y)-uniformly attracting set $\mathcal{A}'$ for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$.*

Theorem 2.4. [7, Theorem 3.9] *Let $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ be a family of processes acting on X such that*

1. *$U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau)$, where $\{T(h) | h \geq 0\}$ is a family of operators acting on Σ and satisfies $T(h)\Sigma = \Sigma, \forall h \in \mathbb{R}^+$;*
2. *Σ is a weakly compact set and $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is $(X \times \Sigma, Y)$-weakly continuous, i.e., for any fixed $t \geq \tau, \tau \in \mathbb{R}$, the mapping $(u,\tau) \mapsto U_\sigma(t,\tau)u$ is weakly continuous from $X \times \Sigma$ to Y;*
3. *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is (X,Y)-uniformly asymptotically compact, i.e., it possesses a compact (X,Y)-uniformly attracting set.*

Then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ possesses an (X,Y)-uniform attractor $\mathcal{A}_\Sigma$, which is compact in Y and attracts all bounded subsets of X in the topology of Y. Moreover,

$$\mathcal{A}_\Sigma = \omega_{\tau,\Sigma}(B_0) = \bigcap_{t\geq\tau} \overline{\bigcup_{\sigma\in\Sigma} \bigcup_{s\geq t} U_\sigma(s,\tau)B_0},$$

where B_0 is a bounded (X,Y)-uniformly absorbing set of $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$.

3. Existence of global weak solutions

We denote

$$V = L^2(\tau, T; S_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)),$$
$$V^* = L^2(\tau, T; S^{-1}(\Omega)) + L^{p'}(\tau, T; L^{p'}(\Omega)),$$

where p' is the conjugate index of p. In what follows, we assume that $u_\tau \in L^2(\Omega)$ is given.

Definition 3.1. *A function $u(x,t)$ is called a weak solution of* (1.1) *on* (τ, T) *if and only if*

$$u \in V, \quad \frac{\partial u}{\partial t} \in V^*,$$
$$u|_{t=\tau} = u_\tau \ \ a.e. \ in \ \ \Omega,$$

and

$$\int_\tau^T \int_\Omega \Big(\frac{\partial u}{\partial t}\varphi + \nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2s}\nabla_{x_2} u \nabla_{x_2}\varphi + f(u)\varphi\Big) dxdt = \int_\tau^T \int_\Omega g\varphi dxdt,$$

for all test functions $\varphi \in V$.

It follows from Theorem 1.8 in [10, p. 33] that if $u \in V$ and $\dfrac{du}{dt} \in V^*$ then $u \in C([\tau, T]; L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful.

Theorem 3.1. *Suppose that f satisfies* (1.2)-(1.3) *and that $g \in L^2_b(\mathbb{R}; L^2(\Omega))$. Then for any $\tau, T \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$ given, the problem* (1.1) *has a unique weak solution on* (τ, T).

Proof. The proof is classical, so it is omitted here (see, for example, [24, p. 91-93] for a similar proof). We only prove an inequality which implies that the solution u is global. We have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{S^1_0(\Omega)} + \int_\Omega f(u)u = \int_\Omega g(t)u.$$

Using hypothesis (1.2) and the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{S^1_0(\Omega)} + C_1\|u\|^p_{L^p(\Omega)} - k_1|\Omega| \le \frac{1}{2\lambda_1}\|g(t)\|^2_{L^2(\Omega)} + \frac{\lambda_1}{2}\|u\|^2_{L^2(\Omega)},$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-G_s$ in Ω with the homogeneous Dirichlet condition. Noting that $\|u\|^2_{S^1_0(\Omega)} \ge \lambda_1\|u\|^2_{L^2(\Omega)}$, we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{S^1_0(\Omega)} + 2C_1\|u\|^p_{L^p(\Omega)} \le \frac{1}{\lambda_1}\|g(t)\|^2_{L^2(\Omega)} + 2k_1|\Omega|. \tag{3.1}$$

Hence,

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \lambda_1\|u\|^2_{L^2(\Omega)} \le \frac{1}{\lambda_1}\|g(t)\|^2_{L^2(\Omega)} + 2k_1|\Omega|.$$

By the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \leqslant \|u(\tau)\|^2_{L^2(\Omega)} e^{-\lambda_1(t-\tau)} + \frac{2}{\lambda_1}k_1|\Omega|\Big(1 - e^{-\lambda_1(t-\tau)}\Big)$$
$$+ \frac{1}{\lambda_1}\int_\tau^t e^{-\lambda_1(t-s)}\|g(s)\|^2_{L^2(\Omega)}ds$$

(3.2)

$$\leqslant \|u(\tau)\|^2_{L^2(\Omega)} e^{-\lambda_1(t-\tau)} + \frac{2k_1|\Omega|}{\lambda_1}(1 - e^{-\lambda_1(t-\tau)}) + \frac{1}{\lambda_1(1 - e^{-\lambda_1})}\|g\|^2_{L^2_b},$$

where we have used the fact that

$$\begin{aligned}\int_{\tau}^{t} e^{-\lambda_1(t-s)}\|g(s)\|^2_{L^2(\Omega)}ds &\leqslant \int_{t-1}^{t} e^{-\lambda_1(t-s)}\|g(s)\|^2_{L^2(\Omega)}ds \\ &\quad + \int_{t-2}^{t-1} e^{-\lambda_1(t-s)}\|g(s)\|^2_{L^2(\Omega)}ds + ... \\ &\leqslant \int_{t-1}^{t} \|g(s)\|^2_{L^2(\Omega)}ds + e^{-\lambda_1}\int_{t-2}^{t-1} \|g(s)\|^2_{L^2(\Omega)}ds + ... \\ &\leqslant (1+e^{-\lambda_1}+e^{-2\lambda_1}+...)\|g\|^2_{L^2_b} = \frac{1}{1-e^{-\lambda_1}}\|g\|^2_{L^2_b}.\end{aligned}$$

The inequality (3.2) implies that the solution exists globally in time. □

4. Existence of uniform attractors

From Theorem 3.1, we can define a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ acting on $L^2(\Omega)$ as $U_\sigma(t,\tau)u_\tau = u(t)$, where $u(t)$ is the unique weak solution of problem (1.1) with initial condition u_τ and the external force σ. Because of the uniqueness of weak solutions, one has

$$U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau), \forall\sigma\in\mathcal{H}_w(g), t\geq\tau, \tau\in\mathbb{R}, h\geq 0.$$

The following result is proved similarly as in [7, Theorem 4.2].

Proposition 4.1. *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to* (1.1) *is $(L^2(\Omega)\times\mathcal{H}_w(g), L^2(\Omega))$-weakly continuous, and $(L^2(\Omega)\times\mathcal{H}_w(g), L^p(\Omega)\cap S^1_0(\Omega))$-weakly continuous.*

Proposition 4.2. *Suppose that f satisfies* (1.2)-(1.3) *and that $g\in L^2_b(\mathbb{R};L^2(\Omega))$. Then $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has a bounded $(L^2(\Omega), S^1_0(\Omega)\cap L^p(\Omega))$-uniformly absorbing set B_0.*

Proof. Suppose that $B\subset L^2(\Omega)$ is bounded, $u_\tau\in B, \sigma\in\mathcal{H}_w(g)$ and $u = U_\sigma(t,\tau)u_\tau$ is the weak solution of problem (1.1) when $g(t)$ is replaced by $\sigma(t)$. From (3.1) we obtain

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \lambda_1\|u\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1}\|\sigma(t)\|^2_{L^2(\Omega)} + 2k_1|\Omega|.$$

Hence, similarly to (3.2), we get

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(\tau)\|^2_{L^2(\Omega)}e^{-\lambda_1(t-\tau)} + \frac{2k_1|\Omega|}{\lambda_1}\Big(1-e^{-\lambda_1(t-\tau)}\Big) + \frac{1}{\lambda_1(1-e^{-\lambda_1})}\|g\|^2_{L^2_b}.$$

It follows that there exists $T_1 = T_1(B,\tau)$ such that

$$\|u(t)\|^2_{L^2(\Omega)} \leq \rho_0, \text{ for all } t\geq T_1, u_\tau\in B. \tag{4.1}$$

By (3.1) and (4.1), we have

$$\int_t^{t+1}\left(\|u\|^2_{S^1_0(\Omega)}+\|u\|^p_{L^p(\Omega)}\right)\le C_4 \text{ for any } t\ge T_1. \tag{4.2}$$

Meanwhile, let $F(s)=\int_0^s f(\xi)d\xi$, then by (1.2), we deduce that

$$\tilde{C}_1|u|^p-\tilde{k}_1\le F(u)\le \tilde{C}_2|u|^p+\tilde{k}_2,$$

$$\tilde{C}_1\|u\|^p_{L^p(\Omega)}-\tilde{k}_1|\Omega|\le\int_\Omega F(u)\le \tilde{C}_2\|u\|^p_{L^p(\Omega)}+\tilde{k}_2|\Omega|.$$

Combining with (4.2), we get

$$\int_t^{t+1}\left(\|u\|^2_{S^1_0(\Omega)}+\int_\Omega F(u)\right)\le C_5 \text{ for any } t\ge T_1. \tag{4.3}$$

On the other hand, multiplying (1.1) by u_t, we obtain

$$\|u_t\|^2_{L^2(\Omega)}+\frac{d}{dt}\left(\|u\|^2_{S^1_0(\Omega)}+\int_\Omega F(u)\right)\le\frac{1}{2}\|\sigma(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|u_t\|^2_{L^2(\Omega)}. \tag{4.4}$$

Thus

$$\frac{d}{dt}\left(\|u\|^2_{S^1_0(\Omega)}+\int_\Omega F(u)\right)\le\frac{1}{2}\|\sigma(t)\|^2_{L^2(\Omega)}. \tag{4.5}$$

From (4.3) and (4.5), by virtue of the uniform Gronwall lemma [24, p. 91], we get

$$\|u\|^2_{S^1_0(\Omega)}+\int_\Omega F(u)\le C_6 \text{ for any } t\ge T_1.$$

Hence,

$$\|u(t)\|^2_{S^1_0(\Omega)}+\|u(t)\|^p_{L^p(\Omega)}\le C_7 \text{ for any } t\ge T_1.$$

This inequality implies the existence of a bounded $(L^2(\Omega), S^1_0(\Omega)\cap L^p(\Omega))$-uniformly absorbing set B_0. □

The set B_0 is also of course a bounded $(L^2(\Omega), L^2(\Omega))$- and $(L^2(\Omega), L^p(\Omega))$-uniformly absorbing set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$. By Theorem 2.4, to prove the existence of a uniform attractor, we only need to verify that $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ is uniformly asymptotically compact.

4.1. $(L^2(\Omega), L^2(\Omega))$ **- uniform attractor.** Because $S^1_0(\Omega)\hookrightarrow L^2(\Omega)$ compactly, we immediately get the following result.

Theorem 4.3. *Suppose that f satisfies (1.2)-(1.3) and that $g\in L^2_b(\mathbb{R};L^2(\Omega))$. Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem (1.1) has an $(L^2(\Omega), L^2(\Omega))$-uniform attractor $\mathcal{A}_2$, which is compact in $L^2(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^2(\Omega)$. Moreover,*

$$\mathcal{A}_2=w_{\tau,\mathcal{H}_w(g)}(B_0),$$

where B_0 is the $(L^2(\Omega), L^2(\Omega))$-uniformly absorbing set.

4.2. $(L^2(\Omega), L^p(\Omega))$ **- uniform attractor.** To prove that the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ is $(L^2(\Omega), L^p(\Omega))$-uniformly asymptotically compact, we use the following result (see [7, Corollary 3.12]).

Lemma 4.4. *Let $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ be a family of processes on $L^2(\Omega)$ and be $(L^2(\Omega), L^2(\Omega))$-uniformly (with respect to $\sigma\in\Sigma$) asymptotically compact. Then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is $(L^2(\Omega), L^p(\Omega))$-uniformly asymptotically compact, $2\le p<\infty$, if*

(1) *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has a bounded $(L^2(\Omega), L^p(\Omega))$-uniformly absorbing set B_0;*
(2) *for any $\varepsilon>0, \tau\in\mathbb{R}$ and any bounded subset $B\subset L^2(\Omega)$, there exist positive constants $M=M(\varepsilon,B)$ and $T=T(\varepsilon,B,\tau)$, such that*

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\ge M)} |U_\sigma(t,\tau)u_\tau|^p<\varepsilon \quad \text{for any } u_\tau\in B, t\ge T, \sigma\in\Sigma.$$

We are now in a position to prove the following

Theorem 4.5. *Suppose that f satisfies (1.2)-(1.3) and that $g\in L^2_n(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem (1.1) has an $(L^2(\Omega), L^p(\Omega))$- uniform attractor $\mathcal{A}_p$, which is compact in $L^p(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^p(\Omega)$. Moreover,*

$$\mathcal{A}_p = w_{\tau,\mathcal{H}_w(g)}(B_0),$$

where B_0 is the $(L^2(\Omega), L^p(\Omega))$-uniformly absorbing set.

Proof. By Lemma 4.4 and Theorem 4.3, it is sufficient to show that: for any $\varepsilon>0, \tau\in\mathbb{R}$ and any bounded subset $B\subset L^2(\Omega)$, there exist two positive constants $T=T(B,\varepsilon,\tau)$ and $M=M(B,\varepsilon)$ such that

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\ge M)} |U_\sigma(t,\tau)u_\tau|^p dx<\varepsilon \text{ for any } u_\tau\in B, t\ge T, \sigma\in\mathcal{H}_w(g).$$

In fact, take M large enough such that $C_1|u|^{p-1}\le f(u)$ in $\Omega_1=\Omega(u(t)\ge M)=\{x\in\Omega: u(x,t)\ge M\}$ and denote

$$(u-M)_+ = \begin{cases} u-M, & u\ge M\\ 0, & u\le M.\end{cases}$$

In Ω_1, we see that

$$\begin{aligned}\sigma(t)(u-M)_+^{p-1} &\le \frac{C_1}{2}(u-M)_+^{2p-2}+\frac{1}{2C_1}|\sigma(t)|^2\\ &\le \frac{C_1}{2}(u-M)_+^{p-1}|u|^{p-1}+\frac{1}{2C_1}|\sigma(t)|^2,\end{aligned}$$

and

$$\begin{aligned} f(u)(u-M)_+^{p-1} &\ge C_1|u|^{p-1}(u-M)_+^{p-1}\\ &\ge \frac{C_1}{2}(u-M)_+^{p-1}|u|^{p-1}+\frac{C_1}{2}|u|^{p-2}(u-M)_+^p\\ &\ge \frac{C_1}{2}(u-M)_+^{p-1}|u|^{p-1}+\frac{C_1M^{p-2}}{2}(u-M)_+^p.\end{aligned}$$

Multiplying equation (1.1) by $|(u-M)_+|^{p-1}$ and using the above inequalities, we deduce that

$$\frac{2}{p}\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\Omega)}+(p-1)\int_{\Omega_1}(|\nabla_{x_1}(u-M)_+|^2+|x_1|^{2s}|\nabla_{x_2}(u-M)_+|^2)$$
$$|(u-M)_+|^{p-2}+C_1M^{p-2}\int_{\Omega_1}|(u-M)_+|^p\le\int_{\Omega_1}\frac{1}{C_1}|\sigma|^2.$$

Therefore,

$$\frac{d}{dt}\|(u-M)_+\|^p_{L^p(\Omega)}+\frac{C_1M^{p-2}p}{2}\|(u-M)_+\|^p_{L^p(\Omega)}\le\int_{\Omega_1}\frac{p}{2C_1}|\sigma|^2.$$

Letting $k=T_B$, where T_B is chosen such that $\|U_\sigma(t,\tau)u_\tau\|^p_{L^p(\Omega)}\le\rho_p$, we deduce that

(4.6)

$$\begin{aligned}\|(u-M)_+(t)\|^p_{L^p(\Omega)}&\le\|(u-M)_+(k)\|^p_{L^p(\Omega)}e^{-\lambda(t-k)}+\int_k^t\left(e^{-\lambda(t-s)}\frac{p}{2C_1}\int_{\Omega_1}|\sigma|^2\right)\\&\le\|(u-M)_+(k)\|^p_{L^p(\Omega)}e^{-\lambda(t-k)}+\frac{p}{2C_1}\int_k^t e^{-\lambda(t-s)}\|\sigma\|^2_{L^2(\Omega)},\end{aligned}$$

where $\lambda=\frac{C_1M^{p-2}p}{2}$. By Lemma 2.3, we have

$$\frac{p}{2C_1}\int_k^t e^{-\lambda(t-s)}\|\sigma\|^2_{L^2(\Omega)}\le\frac{\varepsilon}{2^{p+2}},\quad\text{for }\sigma\in\mathcal{H}_w(g),M\ge M_1\text{ for some }M_1.\tag{4.7}$$

Letting $T_1=\frac{1}{\lambda}\ln(\frac{2^{p+3}\rho_p}{\varepsilon})+k$, then

$$\|(u-M)_+(k)\|^p_{L^p(\Omega)}e^{-\lambda(t-k)}\le\frac{\varepsilon}{2^{p+2}},\quad\text{for all }t>T_1.\tag{4.8}$$

From (4.6) - (4.8), we deduce that

$$\int_{\Omega(u(t)\ge M)}|(u-M)_+|^pdx\le\frac{\varepsilon}{2^{p+1}},\quad\text{for }t>T_1,\sigma\in\mathcal{H}_\omega(g),M\ge M_1.$$

Repeating the same steps above, just taking $|(u+M)_-|^{p-2}(u+M)_-$ instead of $|(u-M)_+|^{p-1}$, there exist M_2 and T_2 such that

$$\int_{\Omega(u(t)\le -M)}|(u+M)_-|^pdx\le\frac{\varepsilon}{2^{p+1}},\quad\text{for }t>T_2,\sigma\in\mathcal{H}_w(g),M\ge M_2,$$

where

$$(u+M)_-=\begin{cases}u+M, & u\le -M,\\ 0, & u\ge -M.\end{cases}$$

Taking $M_3=\max(M_1,M_2)$, we obtain

$$\int_{\Omega(|u(t)|\ge M_3)}|(|u(t)|-M_3)|^pdx\le 2\varepsilon,\quad\text{for }t>\max(T_1,T_2),\sigma\in\mathcal{H}_w(g).$$

Therefore,

$$
\begin{aligned}
\int_{\Omega(|u(t)|\geq 2M_3)} |u(t)|^p &= \int_{\Omega(|u(t)|\geq 2M_3)} ((|u(t)| - M_3) + M_3)^p \\
&\leq 2^{p-1} \int_{\Omega(|u(t)|\geq 2M_3)} (|u| - M_3)^p + \int_{\Omega(|u(t)|\geq 2M_3)} M_3^p \\
&\leq 2^{p-1} \int_{\Omega(|u(t)|\geq M_3)} (|u| - M_3)^p + \int_{\Omega(|u(t)|\geq M_3)} (|u| - M_3)^p \\
&\leq 2^{p+1} \frac{\varepsilon}{2^{p+1}} = \varepsilon.
\end{aligned}
$$

□

4.3. $(L^2(\Omega), S_0^1(\Omega) \cap L^p(\Omega))$ **- uniform attractor.** First, we prove the following lemma.

Lemma 4.6. *Suppose that f satisfies (1.2)-(1.3) and that $g, g' \in L_b^2(\mathbb{R}; L^2(\Omega))$. Then for any bounded subset $B \subset L^2(\Omega)$ and $\tau \in \mathbb{R}$, there exists a positive constant $T = T(B, \tau) \geq \tau$ such that*

$$
\|\frac{d}{dt}(U_\sigma(t,\tau)u_\tau)|_{t=s}\|^2_{L^2(\Omega)} \leq C \text{ for any } u_\tau \in B, s \geq T, \sigma \in \mathcal{H}_w(g),
$$

where C independent of B and σ.

Proof. We give some formal calculations, a rigorous proof is done by use of Galerkin approximations. Letting $u(t) = U_\sigma(t,\tau)u_\tau$ then differentiating (1.1) with the external force $\sigma(t)$ in time and setting $v = u_t$, we get

$$
\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} + \|v\|^2_{S_0^1(\Omega)} \leq \ell\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sigma'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)}.
$$

Therefore,

$$
\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq \left(\ell + \frac{1}{2}\right)\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sigma'(t)\|^2_{L^2(\Omega)}.
$$

From (4.3) and (4.4), we have

$$
\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \leq C, \quad \text{for } t \text{ large enough.}
$$

Now applying the uniform Gronwall lemma [24, p. 91], we get

$$
\int_\Omega |u_t|^2 dx \leq C,
$$

as t large enough, where C independent of B and σ. □

Theorem 4.7. *Suppose that f satisfies (1.2)-(1.3) and that $g \in L_n^2(\mathbb{R}; L^2(\Omega)), g' \in L_b^2(\mathbb{R}; L^2(\Omega))$. Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem (1.1) has an $(L^2(\Omega), S_0^1(\Omega) \cap L^p(\Omega))-$uniform attractor $\mathcal{A}$, which is compact in $S_0^1(\Omega) \cap L^p(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $S_0^1(\Omega) \cap L^p(\Omega)$. Moreover,*

$$
\mathcal{A} = w_{\tau,\mathcal{H}_w(g)}(B_0),
$$

where B_0 is the $(L^2(\Omega), S_0^1(\Omega)\cap L^p(\Omega))-$uniformly absorbing set.

Proof. Let B_0 be the $(L^2(\Omega), S_0^1(\Omega)\cap L^p(\Omega))$ - uniformly absorbing set, we need only to show that for any $u_{\tau_n}\in B_0, \sigma_n\in\mathcal{H}_\omega(g), t_n\to\infty, \{U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}\}$ is precompact in $S_0^1(\Omega)\cap L^p(\Omega)$. By Theorem 4.5, it is sufficient to verify that $\{U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}\}$ is precompact in $S_0^1(\Omega)$.

We will prove that $\{U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}\}$ is a Cauchy sequence in $S_0^1(\Omega)$. Using Theorems 4.3 and 4.5, we can assume that $\{U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}\}$ is a Cauchy sequence in both $L^p(\Omega)$ and $L^2(\Omega)$. Denote $u_n(t_n) = U_{\sigma_n}(t_n,\tau_n)u_{\tau_n}$, we have

$$\begin{aligned}
&\|u_n(t_n)-u_m(t_m)\|^2_{S_0^1(\Omega)} = -\langle G_s u_n(t_n) - G_s u_m(t_m), u_n(t_n)-u_m(t_m)\rangle\\
&= -\Big\langle \big(\partial_t u_n(t_n) - \partial_t u_m(t_m)\big) + \big(f(u_n(t_n)) - f(u_m(t_m))\big)\Big\rangle\\
&\quad - \Big\langle \big(\sigma_n(t_n)-\sigma_m(t_m)\big), u_n(t_n)-u_m(t_m)\Big\rangle\\
&\le \|\partial_t u_n(t_n)-\partial_t u_m(t_m)\|_{L^2(\Omega)}\|u_n(t_n)-u_m(t_m)\|_{L^2(\Omega)}\\
&\quad + \|f(u_n(t_n))-f(u_m(t_m))\|_{L^{p'}(\Omega)}\|u_n(t_n)-u_m(t_m)\|_{L^p(\Omega)}\\
&\quad + \|\sigma_n(t_n)-\sigma_m(t_m)\|_{L^2(\Omega)}\|u_n(t_n)-u_m(t_m)\|_{L^2(\Omega)}.
\end{aligned}$$

By Lemma 4.6 and since $\{f(u_n(t_n))\}$ is bounded in $L^{p'}(\Omega)$, the proof is complete. □

Remarks. i) In fact, in order to prove the existence of an $(L^2(\Omega), L^2(\Omega))$-uniform attractor and of an $(L^2(\Omega), L^p(\Omega))$-uniform attractor, we only need to assume that $f\in C(\mathbb{R})$ satisfying (1.2) and

$$(f(u)-f(v))(u-v)\ge -\ell|u-v|^2, \text{ for all } u,v\in\mathbb{R}.$$

Thus, in particular, our results improve the recent ones in [22] for the non-autonomous Laplacian equation.

ii) One can easily extend the results of this paper to the equation in the form

$$u_t - G_{\alpha_1,\dots,\alpha_m}u + f(u) = g(t), \quad x\in\Omega, t>0,$$

where Ω is a bounded domain in $\mathbb{R}^{N_0}\times\mathbb{R}^{N_1}\times\cdots\times\mathbb{R}^{N_m}$, $(x_0,x_1,\dots,x_m)\in\Omega$, $G_{\alpha_1,\dots,\alpha_m} = \Delta_{x_0} + |x_0|^{2\alpha_1}\Delta_{x_1}+\cdots+|x_0|^{2\alpha_m}\Delta_{x_m}$. Note that in this case we still are able to define the space $S_0^1(\Omega)$ and have the embedding theorems which are similar to those in Section 2.1 (for more details, see [25]). Here

$$N(s) = N_0 + (\alpha_1+1)N_1+\cdots+(\alpha_m+1)N_m.$$

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