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ATYPICAL VALUES AT INFINITY OF POLYNOMIAL AND RATIONAL FUNCTIONS ON AN ALGEBRAIC SURFACE IN $\mathbb{R}^n$

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. We consider polynomial and rational functions defined on an algebraic surface in $\mathbb{R}^n$ and we characterize values which are atypical at infinity due to singularities at infinity. This characterization gives an upper bound of the number of atypical values at infinity.

1. Introduction

Let $f : \mathbb{K}^n \to \mathbb{K}$ be a polynomial function, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then f defines the global Milnor fibration

$$f : \mathbb{K}^n \backslash f^{-1}(B) \to \mathbb{K} \backslash B.$$

The set B of bifurcation values consists of critical values of f and its atypical values at infinity (or, critical values of the singularities at infinity).

Problem. *What we can tell about if a given $\lambda \in \mathbb{K}$ is an atypical value at infinity of f?*

Let us mention here some previous results on this problem.

- $n = 2$, $\mathbb{K} = \mathbb{C}$: In [5], Hà Huy Vui and Lê Dũng Tráng proved that a regular value λ of a complex polynomial function is atypical at infinity if and only if $\chi[f^{-1}(\lambda)] \neq \chi[f^{-1}(t)]$, where $f^{-1}(t)$ is a general fiber of f and χ denotes the Euler characteristic. In [6], Hà Huy Vui and Nguyên Lê Anh showed an equivalent form of the above criterion as follows: A value λ of a complex polynomial function is atypical at infinity if and only if there is at least one ramification point of the curve $f^{-1}(t)$ (with respect to a general projection) which tends to infinity as t tends to λ.

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- $n = 2$, $\mathbb{K} = \mathbb{R}$: In [4] and [3], the arguments of [6] are adapted for the real case. The authors gave a characterization of atypical values at infinity of a polynomial function f on $\mathbb{R}^2$. Moreover, it provides an algorithmic way to determine effectively the atypical values at infinity of f.
- Tibăr and Zaharia considered in [7] the following situation: Let V be a smooth non-compact algebraic set in $\mathbb{R}^n$, and $f_V := f|_V$ be the restriction of the polynomial function f to V. The map f_V defines a fibration

$$f_V : V \backslash f_V^{-1}(B) \to \mathbb{R} \backslash B,$$

where the set B of bifurcation values consists of critical values of f_V and its atypical values at infinity. It is proved that a regular value λ of f_V is typical at infinity if and only if the Euler characteristic of the level curve $f_V^{-1}(t)$ is constant for t in a neighborhood of λ and there is no connected component of $f_V^{-1}(t)$ which is vanishing at infinity as t tends to λ. Unfortunately, the method of [7] does not tell us when there is a vanishing at infinity component or when the Euler characteristic is not constant.
- Recently, in [2], Bodin and Pichon considered meromorphic functions of two complex variables and gaved some characterization of atypical values.

In this paper, we consider a rational function $(f/g)_V := (f/g)|_V$ on a smooth non-compact algebraic surface $V = F^{-1}(0)$, where $f, g : \mathbb{R}^n \to \mathbb{R}$ are two polynomial functions and $F = (f_1, \dots, f_{n-2}) : \mathbb{R}^n \to \mathbb{R}^{n-2}$ is a polynomial mapping. We will give a characterization of atypical values at infinity of $(f/g)_V$ on V. The main advantage is that the characterization provides a way to recognize algorithmically when there is a connected component of the fiber which is vanishing or cleaving at infinity. The method of [3] is to use the polar curve for characterizing atypical values at infinity of a polynomial function f on $\mathbb{R}^2$. In this case, it is necessary to consider a linear function $L : \mathbb{R}^2 \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$, the restriction function $L : f_V^{-1}(D_\delta) \to \mathbb{R}$ is proper, where $D_\delta = \{t \in \mathbb{R} : |t - \lambda| \le \delta\}$. However, in general, such a linear function does not exist if $\mathbb{R}^2$ is replaced by an algebraic surface (see Sec. 5). To overcome this difficulty, we replace the polar curve with the tangency curve, and with the use of the latter, we show that the arguments of [3] still work for the surface case. Besides, based on our characterization, one can give an upper bound of the number of atypical values at infinity in terms of degrees of f, g and V.

2. Statement of the results

Let $F = (f_1 \dots, f_{n-2}) : \mathbb{R}^n \to \mathbb{R}^{n-2}$ be a polynomial mapping, $V = F^{-1}(0)$ be a smooth non-compact algebraic surface. Let f, $g : \mathbb{R}^n \to \mathbb{R}$ be polynomial functions and $(f/g)_V := (f/g)|_V$ be the restriction of f/g to V, where $f/g : \mathbb{R}^n \to \mathbb{R}$ denotes the rational map . The function $(f/g)_V$ is well-defined on the set $V^0 := V \backslash (g = 0)$.

Definition 2.1. We say that $(f/g)_V$ is trivial at infinity over the interval (α,β) if there exists a compact subset K of V such that the restriction function $(f/g)_V : (f/g)_V^{-1}[(\alpha,\beta)]\backslash K \to (\alpha,\beta)$ is a C^∞-trivial fibration. A value $\lambda \in \mathbb{R}$ is called a *typical value at infinity of* $(f/g)_V$ if $(f/g)_V$ is trivial at infinity over some open interval (α,β) containing λ. Otherwise, λ will be called an *atypical value at infinity of* $(f/g)_V$.

Remark 2.1. If there is a non-compact connected component W of V^0 such that the restriction of f/g to W is constant, then $(f/g)(W)$ is an atypical value at infinity of $(f/g)_V$.

By Remark 2.1, from now on, we can restrict our consideration to the case when f/g is non-constant on each connected component of V^0.

Also, we assume that the set $V \cap (f=0) \cap (g=0)$ is bounded. In this case, for every $\lambda \in \mathbb{R}$, the restriction function $Q_a : (f/g)_V^{-1}(D_\delta)\backslash U \to \mathbb{R}$, $x \mapsto \sum_{i=1}^n (x_i - a_i)^2$ is proper, where U is a bounded open set containing the set $V \cap (f=0) \cap (g=0)$.

For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, put

$$H_a := \det \begin{pmatrix} \partial(f/g)/\partial x_1 & \cdots & \partial(f/g)/\partial x_n \\ \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_1 & \cdots & \partial f_{n-2}/\partial x_n \\ x_1 - a_1 & \cdots & x_n - a_n \end{pmatrix}.$$

Lemma 2.1. *For almost every $a \in \mathbb{R}^n$, $H_a^{-1}(0) \cap V$ is a semi-algebraic set of dimension at most 1.*

Proof. Let us consider $H_a(x)$ as a function $\widetilde{H}$ in $2n$ variables (x,a). Put

$$X = \{(x,a) \in \mathbb{R}^n \times \mathbb{R}^n : F(x) = 0,\ \widetilde{H}(x,a) = 0\}.$$

Since $(f/g)_V$ is non-constant on each connected component of V^0, the set of all singular points of $(f/g)_V$, say $S[(f/g)_V]$, is a semi-algebraic subset of V of dimension at most 1. Moreover, since V is smooth, $\operatorname{rank}(\frac{\partial f_i}{\partial x_j})(x) = n-2$ for all $x \in V$. Therefore, for all $x \in V^0\backslash S[(f/g)_V]$, we have

$$\operatorname{rank} \begin{pmatrix} \partial(f/g)/\partial x_1 & \cdots & \partial(f/g)/\partial x_n \\ \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_1 & \cdots & \partial f_{n-2}/\partial x_n \end{pmatrix}(x) = n-1.$$

Without loss of generality, assume that

$$\det \begin{pmatrix} \partial(f/g)/\partial x_2 & \cdots & \partial(f/g)/\partial x_n \\ \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_2 & \cdots & \partial f_{n-2}/\partial x_n \end{pmatrix}(x) \neq 0.$$

This determinant is nothing else but $(-1)^n \frac{\partial \widetilde{H}}{\partial a_1}(x,a)$, where $a \in \mathbb{R}^n$. Thus, if $a \in V^0 \backslash S[(f/g)_V]$, then the point $(a,a) \in X$, $\partial \widetilde{H}/\partial a_1(a,a) \neq 0$, and hence the maximal rank of the matrix

$$\begin{pmatrix} \partial \widetilde{H}/\partial x_1 & \cdots & \partial \widetilde{H}/\partial x_n & \partial \widetilde{H}/\partial a_1 & \cdots & \partial \widetilde{H}/\partial a_n \\ \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_1 & \cdots & \partial f_{n-2}/\partial x_n & 0 & \cdots & 0 \end{pmatrix}$$

is equal to $n-1$. This implies that $\dim X = n+1$. Let X^r be the set of regular points of X and $\pi : X^r \to \mathbb{R}^n$ be the map $\pi(x,a) = a$. By Sard's lemma, almost every $a \in \mathbb{R}^n$ is a regular value of π and $\pi^{-1}(a) \subset X^r$ is a one-dimensional manifold, if it is non-empty. Consequently,

$$\dim H_a^{-1}(0) \cap V \leq 1,$$

because $H_a^{-1}(0) \cap V$ is the union of two semi-algebraic sets $S[(f/g)_V]$ and $\pi_1[\pi^{-1}(a)]$ of dimension at most 1, where $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the first n coordinates. □

Let $\Gamma[a, (f/g)_V] = H_a^{-1}(0) \cap V$. If $\dim \Gamma[a,(f/g)_V] = 1$, we call it the *tangency curve* of $(f/g)_V$ with respect to $a \in \mathbb{R}^n$.

We now can assume that the point $0 \in \mathbb{R}^n$ is generic, i.e., $\Gamma[(f/g)_V] := \Gamma[0,(f/g)_V]$ is the tangency curve of $(f/g)_V$. Then, for $R > 0$ large enough, the set $\Gamma[(f/g)_V] \backslash \mathbb{B}_R$ consists of a finite number of one-dimensional connected components which will be called *half-branches at infinity* of $\Gamma[(f/g)_V]$. Let C be a half-branch of $\Gamma[(f/g)_V]$. Then, the function f/g is monotonic along C. Hence it has a limit $\lambda = \lim_C f/g$. We denote $f/g \nearrow_C \lambda$, $f/g \searrow_C \lambda$ and $f/g =_C \lambda$ respectively for the three possibilities (increasing, decreasing and constant). Moreover, there exists a Nash function $h : (R, +\infty) \to \mathbb{R}^n$, $t \mapsto h(t)$, such that C is the curve $x = h(t)$.

Claim 2.1. *For $R > 0$ large enough, $V^0 \cap \mathbb{S}_R^{n-1}(a)$ is a smooth manifold of dimension 1.*

Proof. Let $Q_a = \sum_{i=1}^n (x_i - a_i)^2$, $Q_{a,V^0} = Q_a|_{V^0}$. Since the set of all critical values of Q_{a,V^0} is finite, $V^0 \cap \mathbb{S}_R^{n-1}(a) = Q_{a,V^0}^{-1}(R^2)$ is a smooth manifold of dimension 1 for R large enough. □

Taking $R_0 > 0$ large enough, we consider a vector field v on $V^0 \backslash \mathbb{B}_{R_0}(a)$ which is tangential to the curve $V_0 \cap \mathbb{S}_R^{n-1}(a)$ for all $R > R_0$. Let $x = \rho(\tau)$ be the solution of the differential equation $\frac{d}{d\tau}\rho(\tau) = v[\rho(\tau)]$. We will denote it by $x = \rho(q;\tau)$ if $\rho(0) = q \in V_0 \backslash \mathbb{B}_{R_0}(a)$.

By the conic structure theorem at infinity [1], for $R > R_0$, the set $V_0 \backslash \mathbb{B}_R(a)$ consists of a finite number of connected components which will be denoted by $V_1, \ldots, V_m$. Notice that $V_0 \cap \mathbb{S}_R^{n-1}(a) = \cup_{i=1}^m K_i$ with $K_i = (\partial V_i) \backslash (g = 0)$. Moreover, each K_i can be parameterized by $x = \rho(q;\tau)$, $q \in K_i$.

Definition 2.2. Let C be a half-branch at infinity of $\Gamma[(f/g)_V]$ parameterized by $h:(R,+\infty)\to\mathbb{R}^n$, $t\mapsto h(t)$. We say that $H:=H_0$ *changes sign* along C if for $\epsilon>0$ small enough, we have

$$H[\rho(h(t);-\epsilon)].H[\rho(h(t);+\epsilon)]<0.$$

We shall consider all half-branches at infinity of $\Gamma[(f/g)_V]$, starting from one connected component K of $V^0\cap\mathbb{S}^{n-1}_R(a)$. There are two cases:

CASE 1: K is a non-compact connected component of $V^0\cap\mathbb{S}^{n-1}_R(a)$. Assume that

$$K\cap\Gamma[(f/g)_V]=\{p_1,\dots,p_k\},$$

where $p_i=\rho(\tau_i)$ (where $x=\rho(\tau)$ is the parameterization of K) and $\tau_1<\cdots<\tau_k$ are consecutive roots of the equation $H[\rho(\tau)]=0$. Every point p_i is the boundary of the unique half-branch of $\Gamma[(f/g)_V]$, which is denoted by C_i. The order $\tau_1<\cdots<\tau_k$ induces an order $C_1<\cdots<C_k$.

Definition 2.3 (see [3])**.** Let $C_1<\cdots<C_k$ be the half-branches at infinity of $\Gamma[(f/g)_V]$, starting from K. A sequence of consecutive half-branches $C_r<\cdots<C_s$ is said to be a *critical cluster* belonging to $\lambda\in\mathbb{R}$ if there is a symbol $\succ$ in $\{\nearrow,\searrow,=\}$ such that:

(i) for every $i=r,\dots,s$, one has $f/g\succ_{C_i}\lambda$,

(ii) $f/g\succ_{C_{r-1}}\lambda$ does not hold (or $r=1$),

(iii) $f/g\succ_{C_{s+1}}\lambda$ does not hold (or $s=k$).

Definition 2.4. Let $C_1<\cdots<C_k$ be the half-branches at infinity of $\Gamma[(f/g)_V]$ starting from K. Assume that $h^1,\dots,h^k:(R,+\infty)\to\mathbb{R}^n$ are the parameterization of $C_1,\cdots,C_k$ such that $\|h^1(t)\|=\ldots=\|h^k(t)\|$ for all $t\in(R,+\infty)$, and $x=\rho_t(\tau)$ is the parameterization of the connected component of $\mathbb{S}^{n-1}_{\|h^i(t)\|}\cap V^0$ such that $\rho_t(\tau_i)=h^i(t)$, $i=1,\dots,k$, $(\tau_1<\cdots<\tau_k)$. We define *bands*

$$(C_i,C_{i+1}):=\{\rho_t(\tau):R<t<+\infty\text{ and }\tau_i<\tau<\tau_{i+1}\},\quad\text{for } i=1,\dots,k-1.$$

For simplicity, we also write

$$(C_0,C_1):=\{\rho_t(\tau):R<t<+\infty\text{ and }\tau<\tau_1\},$$

$$(C_k,C_{k+1}):=\{\rho_t(\tau):R<t<+\infty\text{ and }\tau_k<\tau\},$$

and also call them *bands*. Note that C_0 and C_{k+1} are not half-branches.

It is easy to see that H is everywhere positive or everywhere negative on a band (C_i,C_{i+1}), $i=0,\dots,k$.

Definition 2.5. A half-branch C_i along which H changes sign is called a *valley* if $H<0$ on the band (C_{i-1},C_i) and $H>0$ on the band (C_i,C_{i+1}); and a *crest* if $H>0$ on the band (C_{i-1},C_i) and $H<0$ on the band (C_i,C_{i+1}).

We set

- $\widetilde{\theta}(C_i) := \begin{cases} -1 & \text{if } C_i \text{ is a valley,} \\ +1 & \text{if } C_i \text{ is a crest,} \\ 0 & \text{if } H \text{ does not change sign along } C_i. \end{cases}$
- $\theta(C_i) := \begin{cases} -1 & \text{if } f/g \searrow_{C_i} \lambda, \\ +1 & \text{if } f/g \nearrow_{C_i} \lambda, \\ 2 & \text{if } f/g =_{C_i} \lambda. \end{cases}$
- $\delta(C_i) := \theta(C_i).\widetilde{\theta}(C_i)$.
- Let $\mathfrak{C} = \{C_r < \cdots < C_s\}$ be a critical cluster belonging to λ. Set

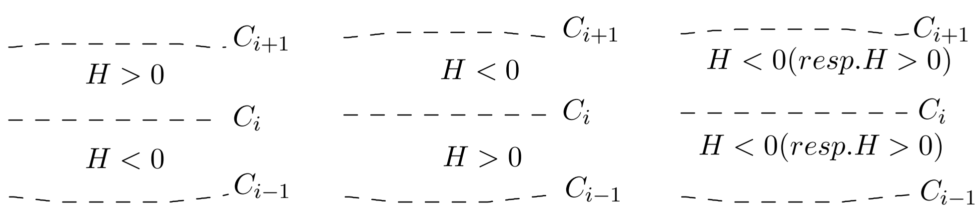
$$\Delta(\mathfrak{C}) := \sum_{i=r}^{s} \delta(C_i), \quad \theta(\mathfrak{C}) := \theta(C_i).$$

Clearly, $\delta(C_i)$ and $\Delta(\mathfrak{C})$ take values in $\{0, \pm 1, \pm 2\}$; $\theta(\mathfrak{C})$ takes values in $\{\pm 1, 2\}$.

We explain the definitions on the pictures (see Figures 1 and 2).

$(a) : \theta(C_i) = -1.$ $(b) : \theta(C_i) = +1.$

Figure 1:



$(a) : \widetilde{\theta}(C_i) = -1.$ $(b) : \widetilde{\theta}(C_i) = +1.$ $(c) : \widetilde{\theta}(C_i) = 0.$

Figure 2:

Notation for all figures. An arrow on a half-branch or a segment-arc indicates the sense along which f/g is increasing.

CASE 2: K is a compact component of $V^0 \cap \mathbb{S}_R^{n-1}(a)$. Let

$$K \cap \Gamma[(f/g)_V] = \{p_1, \ldots, p_k\},$$

where, as before, $p_i = \rho(\tau_i)$ and $\tau_1, \cdots, \tau_k$ are consecutive roots of the equation $H[\rho(\tau)] = 0$ $(\tau_1 < \cdots < \tau_k)$. Denote by $C_{[i]}$ the boundary of the unique half-branch of $\Gamma[(f/g)_V]$ with $\partial C_{[i]} = \{p_i\}$. It is similar to the non-compact case, we introduce the following notations.

Definition 2.6. Let $C_{[1]}, \cdots, C_{[k]}$ be the half-branches at infinity of $\Gamma[(f/g)_V]$ starting from K. A sequence of consecutive half-branches $C_{[r]} < \cdots < C_{[s]}$ (with $[s] \neq [r]$) is said to be a *critical cluster* belonging to $\lambda \in \mathbb{R}$ if there is a symbol $\succ$ in $\{\nearrow, \searrow, =\}$ such that:

(i) for every $[i] = [r], \ldots, [s]$, one has $f/g \succ_{C_{[i]}} \lambda$,

(ii) $f/g \succ_{C_{[r-1]}} \lambda$ and $f/g \succ_{C_{[s+1]}} \lambda$ do not hold (or $[s+1] = [r]$).

Definition 2.7. Let $C_{[1]} < \cdots < C_{[k]}$ be the half-branches at infinity of $\Gamma[(f/g)_V]$ starting from K. Assume that $h^1, \ldots, h^k : (R, +\infty) \to \mathbb{R}^n$ are the parameterization of the connected component of $\mathbb{S}^{n-1}_{\|h^i(t)\|} \cap V^0$ such that $\|h^1(t)\| = \ldots = \|h^k(t)\|$ for all $t \in (R, +\infty)$, and $x = \rho_t(\tau)$ is the parameterization of the connected component of $\mathbb{S}^{n-1}_{\|h^i(t)\|} \cap V^0$ with $\rho_t(\tau_i) = h^i(t)$, $i = 1, \ldots, k$, $\rho_t(\tau_{k+1}) = h^1(t)$. We define the *bands*

$$(C_{[i]}, C_{[i+1]}) = \{\rho_t(\tau) : R < t < +\infty \text{ and } \tau_i < \tau < \tau_{i+1}\},\ i = 1, \ldots, k.$$

Definition 2.8. A half-branch $C_{[i]}$ is called a *valley* if $H < 0$ on the band $(C_{[i-1]}, C_{[i]})$ and $H > 0$ on the band $(C_{[i]}, C_{[i+1]})$; and a *crest* if $H > 0$ on the band $(C_{[i-1]}, C_{[i]})$ and $H < 0$ on the band $(C_{[i]}, C_{[i+1]})$.

We set

- $\widetilde{\theta}(C_{[i]}) := \begin{cases} -1 & \text{if } C_{[i]} \text{ is a valley,} \\ +1 & \text{if } C_{[i]} \text{ is a crest,} \\ 0 & \text{if } H \text{ does not change sign along } C_{[i]}. \end{cases}$
- $\theta(C_{[i]}) := \begin{cases} -1 & \text{if } f/g \searrow_{C_{[i]}} \lambda, \\ +1 & \text{if } f/g \nearrow_{C_{[i]}} \lambda, \\ 2 & \text{if } f/g =_{C_{[i]}} \lambda. \end{cases}$
- $\delta(C_{[i]}) := \theta(C_{[i]}).\widetilde{\theta}(C_{[i]})$.
- Let $\mathfrak{C} := \{C_{[r]} < \cdots < C_{[s]}\}$ be a critical cluster belonging to λ. Set

$$\Delta(\mathfrak{C}) := \sum_{i=r}^{s} \delta(C_{[i]}), \quad \theta(\mathfrak{C}) := \theta(C_{[i]}).$$

Clearly, $\delta(C_{[i]})$ and $\Delta(\mathfrak{C})$ take values in $\{0, \pm 1, \pm 2\}$; $\theta(\mathfrak{C})$ takes values in $\{\pm 1, 2\}$.

Now, we can state the results of this paper.

Theorem 2.1. *A real number λ is an atypical value at infinity of $(f/g)_V$ if and only if there exists a critical cluster $\mathfrak{C}$ belonging to λ such that $\Delta(\mathfrak{C}) \in \{\pm 1, \pm 2\}$.*

The following result gives an upper bound of the number of atypical values at infinity.

Theorem 2.2. *Let f_i be polynomials of degrees d_i, $i = 1, \ldots, n-2$, f and g be polynomials of degrees c_1 and c_2, respectively. Assume that the set of singularities* *of $(f/g)_V$ is bounded. Denote $A_\infty[(f/g)_V]$ the set of atypical values at infinity of $(f/g)_V$. Then*

$$\sharp A_\infty[(f/g)_V] \leq 2d_1 \dots d_{n-2}[c_1 + c_2 + d_1 + \cdots + d_{n-2} + 2 - n].$$

Corollary 2.1. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two polynomial functions. Assume that f and g have no common factors. Then, the statement of Theorem 2.1 holds. Moreover, if the set of singularities of f/g is bounded, then*

$$\sharp A_\infty(f/g) \leq 2(c_1 + c_2),$$

where c_1 and c_2 are the degrees of f and g, respectively.

Corollary 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function and $V = F^{-1}(0)$ be a smooth non-compact algebraic surface, where $F = (f_1, \dots, f_{n-2}) : \mathbb{R}^n \to \mathbb{R}^{n-2}$ is a polynomial mapping. Then, the statement of Theorem 2.1 holds. Moreover, if the set of singularities of f is bounded, then*

$$\sharp A_\infty(f_V) \leq 2d_1 \dots d_{n-2}[c + d_1 + \cdots + d_{n-2} + 2 - n],$$

where c is the degree of f and d_i is the degree of f_i, $i = 1, \dots, n-2$.

3. Proof of Theorem 2.1

After the construction in Section 2, the proof goes essentially in the same lines as in [3]. We only treat critical clusters starting from compact connected components of $V^0 \cap \mathbb{S}^{n-1}_R(a)$. The other case is similar.

Lemma 3.1. *Let $C_{[i]}$ be a half-branch at infinity of $\Gamma[(f/g)_V]$. We assume that $\lim_{C_{[i]}} f/g = \lambda \in \mathbb{R}$ and $\theta(C_{[i]}) = +1$. Then, for every $\epsilon > 0$ sufficiently small, there is a unique $t_\epsilon > R$ such that $(f/g)[h(t_\epsilon)] = \lambda - \epsilon$, where $h : (R, +\infty) \to \mathbb{R}^n$ is the parameterization of $C_{[i]}$. Moreover,*

- (a) *if, in addition, $\widetilde{\theta}(C_{[i]}) = -1$, there exist δ_- and δ_+ in $[R, t_\epsilon)$ such that the intersection of $(f/g)_V^{-1}(\lambda - \epsilon)$ with the band $(C_{[i-1]}, C_{[i]})$ (resp. $(C_{[i]}, C_{[i+1]})$) is a continuous curve $\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n$ (resp. $(\delta_+, t_\epsilon) \to \mathbb{R}^n$) with $\|h(t)\| = \|\widetilde{h}(t)\|$ for all $t \in (\delta_-, t_\epsilon)$ (resp. (δ_+, t_ϵ)).*
- (b) *if $\widetilde{\theta}(C_{[i]}) = +1$, there exist δ_- and δ_+ in $[t_\epsilon, +\infty)$ such that the intersection of $(f/g)_V^{-1}(\lambda - \epsilon)$ with the band $(C_{[i-1]}, C_{[i]})$ (resp. $(C_{[i]}, C_{[i+1]})$) is a continuous curve $\widetilde{h} : (t_\epsilon, \delta_-) \to \mathbb{R}^n$ (resp. $(t_\epsilon, \delta_+) \to \mathbb{R}^n$) with $\|h(t)\| = \|\widetilde{h}(t)\|$ for all $t \in (t_\epsilon, \delta_-)$ (resp. (t_ϵ, δ_+)).*
- (c) *if $\widetilde{\theta}(C_{[i]}) = 0$, there exist δ_- in $[R, t_\epsilon)$ and δ_+ in $(t_\epsilon, +\infty)$ such that the intersection of $(f/g)_V^{-1}(\lambda - \epsilon)$ with the band $(C_{[i-1]}, C_{[i]})$ (resp. $(C_{[i]}, C_{[i+1]})$) is a continuous curve $\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n$ (resp. $(t_\epsilon, \delta_+) \to \mathbb{R}^n$) with $\|h(t)\| = \|\widetilde{h}(t)\|$ for all $t \in (\delta_-, t_\epsilon)$ (resp. (t_ϵ, δ_+)), or a continuous curve $\widetilde{h} : (t_\epsilon, \delta_+) \to \mathbb{R}^n$ (resp. $(\delta_-, t_\epsilon) \to \mathbb{R}^n$) with $\|h(t)\| = \|\widetilde{h}(t)\|$ for all $t \in (t_\epsilon, \delta_+)$ (resp. (δ_-, t_ϵ)).*

Proof. The first assertion follows immediately from the fact that f/g is strictly increasing along the half-branch $C_{[i]}$ and it has the limit λ at the end.

Let us consider the tangent vector field on $V^0\backslash\mathbb{B}_{R_0}$: $v =$

$$\left(\begin{vmatrix} \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_2 & \cdots & \partial f_{n-2}/\partial x_n \\ x_2 - a_2 & \cdots & x_n - a_n \end{vmatrix}, \ldots, (-1)^{n+1} \begin{vmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_{n-1} \\ \vdots & \vdots & \vdots \\ \partial f_{n-2}/\partial x_1 & \cdots & \partial f_{n-2}/\partial x_{n-1} \\ x_1 - a_1 & \cdots & x_{n-1} - a_{n-1} \end{vmatrix}\right),$$

and let $x = \rho(\tau)$ be the solution of the differential equation $\frac{d}{d\tau}\rho(\tau) = v[\rho(\tau)]$. Then, we have

$$\begin{aligned} \frac{d}{d\tau}(f/g)[\rho(\tau)] &=< \mathrm{grad}(f/g)[\rho(\tau)], \frac{d}{d\tau}\rho(\tau) > \\ &=< \mathrm{grad}(f/g)[\rho(\tau)], v[\rho(\tau)] >= H[\rho(\tau)]. \end{aligned}$$

Consequently, $\frac{d}{d\tau}(f/g)[\rho(\tau)]$ is different from 0 on (τ_i, τ_{i+1}), and hence $(f/g)[\rho(\tau)]$ is strictly monotone on (τ_i, τ_{i+1}).

We now consider Case (a): Suppose that $\widetilde{\theta}(C_{[i]}) = -1$. Then $C_{[i]}$ is a valley and $H < 0$ on the band $(C_{[i-1]}, C_{[i]})$. Thus, by the above, $(f/g)(x_0) > \lambda - \epsilon$ for some $x_0 \in \mathbb{S}^{n-1}_{\|h(t_\epsilon)\|} \cap C_{[i-1]}$. Since f/g is monotone or constant along $C_{[i-1]}$, there exists $\delta_- \in [R, t_\epsilon)$ such that for $t \in (\delta_-, t_\epsilon)$ and $x \in \mathbb{S}^{n-1}_{\|h(t)\|} \cap C_{[i-1]}$, we have

$$(f/g)(x) > \lambda - \epsilon,$$

and for $t \in [R, \delta_-]$, $x \in \mathbb{S}^{n-1}_{\|h(t)\|} \cap C_{[i-1]}$, we have $(f/g)(x) < \lambda - \epsilon$. Since f/g is increasing along $C_{[i]}$, we obtain

$$(f/g)(x) < \lambda - \epsilon,$$

for $t \in (\delta_-, t_\epsilon)$ and $x \in \mathbb{S}^{n-1}_{\|h(t)\|} \cap C_{[i]}$. Thus, for all $t \in (\delta_-, t_\epsilon)$, the fiber $(f/g)^{-1}_V(\lambda - \epsilon)$ intersects $\mathbb{S}^{n-1}_{\|h(t)\|} \cap (C_{[i-1]}, C_{[i]})$ at a unique point, which will be denoted by $\widetilde{h}(t)$. Hence, we can define a curve

$$\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n, t \mapsto \widetilde{h}(t).$$

Clearly, $\widetilde{h}$ is continuous and $\|h(t)\| = \|\widetilde{h}(t)\|$ for all $t \in (\delta_-, t_\epsilon)$. On the other hand, it is easily seen that f/g is everywhere $> \lambda - \epsilon$ on $(C_{[i-1]}, C_{[i]})\backslash\mathbb{B}_{\|h(t)\|}$ and everywhere $< \lambda - \epsilon$ on $(C_{[i-1]}, C_{[i]}) \cap \mathbb{B}_{\|h(\delta_-)\|}$. Therefore, the intersection of $(f/g)^{-1}_V(\lambda - \epsilon)$ with the band $(C_{[i-1]}, C_{[i]})$ is a continuous curve $\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n$.

The same reasoning applies to the other cases. A sketch of Lemma 3.1 is found in Figure 3. □

Lemma 3.2. *Let $C_{[i]}$ be a half-branch at infinity of $\Gamma[(f/g)_V]$. We assume that $\lim_{C_{[i]}} f/g = \lambda \in \mathbb{R}$ and $\theta(C_{[i]}) = -1$. Then, for every $\epsilon > 0$ sufficiently small, there is a unique $t_\epsilon > R$ such that $(f/g)[h(t_\epsilon)] = \lambda + \epsilon$, where $h : (R, +\infty) \to \mathbb{R}^n$ is the parameterization of $C_{[i]}$. Moreover,*

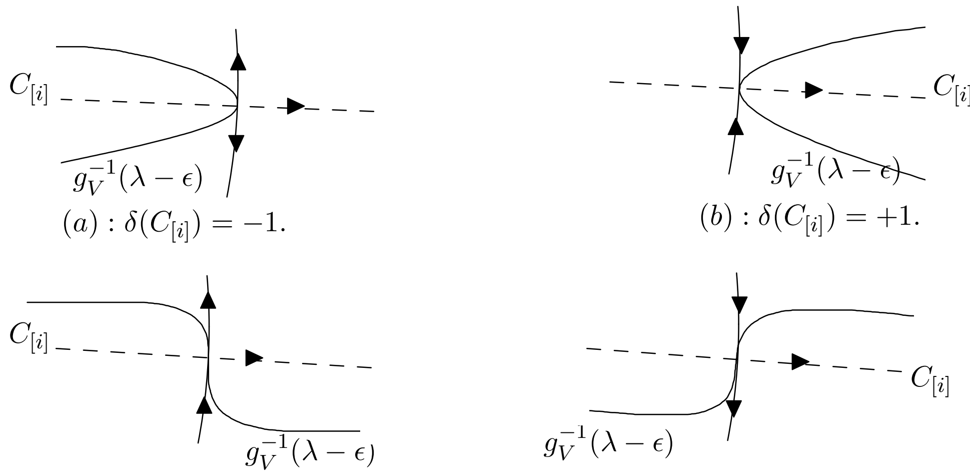


Figure 3:

(a) *if* $\widetilde{\theta}(C_{[i]}) = +1$*, there exist* δ_- *and* δ_+ *in* $[R, t_\epsilon)$ *such that the intersection of* $(f/g)_V^{-1}(\lambda+\epsilon)$ *with the band* $(C_{[i-1]}, C_{[i]})$ *(resp.* $(C_{[i]}, C_{[i+1]})$*) is a continuous curve* $\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n$ *(resp.* $(\delta_+, t_\epsilon) \to \mathbb{R}^n$*) with* $\|h(t)\| = \|\widetilde{h}(t)\|$ *for all* $t \in (\delta_-, t_\epsilon)$ *(resp.* (δ_+, t_ϵ)*).*

(b) *if* $\widetilde{\theta}(C_{[i]}) = -1$*, there exist* δ_- *and* δ_+ *in* $[t_\epsilon, +\infty)$ *such that the intersection of* $(f/g)_V^{-1}(\lambda+\epsilon)$ *with the band* $(C_{[i-1]}, C_{[i]})$ *(resp.* $(C_{[i]}, C_{[i+1]})$*) is a continuous curve* $\widetilde{h} : (t_\epsilon, \delta_-) \to \mathbb{R}^n$ *(resp.* $(t_\epsilon, \delta_+) \to \mathbb{R}^n$*) with* $\|h(t)\| = \|\widetilde{h}(t)\|$ *for all* $t \in (t_\epsilon, \delta_-)$ *(resp.* (t_ϵ, δ_+)*).*

(c) *if* $\widetilde{\theta}(C_{[i]}) = 0$*, there exist* δ_+ *in* $[R, t_\epsilon)$ *and* δ_- *in* $(t_\epsilon, +\infty)$ *such that the intersection of* $(f/g)_V^{-1}(\lambda+\epsilon)$ *with the band* $(C_{[i-1]}, C_{[i]})$ *(resp.* $(C_{[i]}, C_{[i+1]})$*) is a continuous curve* $\widetilde{h} : (\delta_-, t_\epsilon) \to \mathbb{R}^n$ *(resp.* $(t_\epsilon, \delta_+) \to \mathbb{R}^n$*) with* $\|h(t)\| = \|\widetilde{h}(t)\|$ *for all* $t \in (\delta_-, t_\epsilon)$ *(resp.* (t_ϵ, δ_+)*), or a continuous curve* $\widetilde{h} : (t_\epsilon, \delta_+) \to \mathbb{R}^n$ *(resp.* $(\delta_-, t_\epsilon) \to \mathbb{R}^n$*) with* $\|h(t)\| = \|\widetilde{h}(t)\|$ *for all* $t \in (t_\epsilon, \delta_+)$ *(resp.* (δ_-, t_ϵ)*).*

Proof. The proof of this lemma is similar to that of Lemma 3.1. □

Lemma 3.3. *Let* $C_{[i]} < C_{[i+1]}$ *be consecutive half-branches of* $\Gamma[(f/g)_V]$ *starting from* K*. Suppose that* $\lim_{C_{[i]}} f/g = \lim_{C_{[i+1]}} f/g = \lambda \in \mathbb{R}$*.*

(a) *If* $\theta(C_{[i]}) = +1$ *or* $\theta(C_{[i]}) = 2$*, and* $\theta(C_{[i+1]}) = -1$ *or* $\theta(C_{[i+1]}) = 2$*, then* $H > 0$ *on* $(C_{[i]}, C_{[i+1]})$*.*

(b) *If* $\theta(C_{[i]}) = -1$ *or* $\theta(C_{[i]}) = 2$*, and* $\theta(C_{[i+1]}) = +1$ *or* $\theta(C_{[i+1]}) = 2$*, then* $H < 0$ *on* $(C_{[i]}, C_{[i+1]})$*.*

Moreover, in cases either (a) or (b), $\theta(C_{[i]}) = 2$ *and* $\theta(C_{[i+1]}) = 2$ *cannot hold simultaneously; and if neither* $\theta(C_{[i]}) = 2$ *nor* $\theta(C_{[i+1]}) = 2$*, then a half-branch at infinity of* $(f/g)_V^{-1}(\lambda)$ *is contained in the band* $(C_{[i]}, C_{[i+1]})$*.*

Proof. Let us consider Case (a): Assume that $\theta(C_{[i]}) = +1$ and $\theta(C_{[i+1]}) = -1$. By contradiction, assume that $H < 0$ on $(C_{[i]}, C_{[i+1]})$. Then, for $t \in (\tau_i, \tau_{i+1})$, we have $\frac{d}{d\tau}(f/g)[\rho(\tau)] = H[\rho(\tau)] < 0$. Hence, $(f/g)[\rho(\tau)]$ is strictly decreasing on (τ_i, τ_{i+1}) and we obtain the situation described in Figure 4. But this situation cannot happen, because f/g increases along the circuit indicated by the arrows and it has the same limit λ at both ends.

The same reasoning applies to the other cases. □

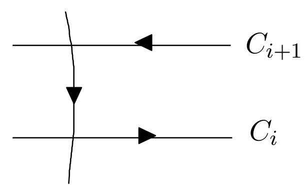


Figure 4: Impossible situation.

Proof of Theorem 2.1. We decompose the proof into several claims, which are direct consequences of Lemmas 3.1 and 3.3, and we omit their proofs.

Claim 3.1. *If there exists a critical cluster* $\mathfrak{C}$ *belonging to* λ *such that* $\Delta(\mathfrak{C}) = \pm 2$*, then* λ *is an atypical value at infinity of* $(f/g)_V$*.*

Note that in this case, the critical cluster $\mathfrak{C}$ consists of only one half-branch at infinity of $\Gamma[(f/g)_V]$. Moreover, this half-branch is a curve of critical points of $(f/g)_V$.

Claim 3.2. *If there exists a critical cluster* $\mathfrak{C}$ *belonging to* λ *such that* $\Delta(\mathfrak{C}) = -1$*, then* λ *is an atypical value at infinity of* $(f/g)_V$*.*

A sketch of Claim 3.2 is found in Figure 5. In this case, in the terminology of [7], a connected component of the fiber $(f/g)_V^{-1}(t)$ is vanishing at infinity as t tends to λ, $t < 0$ or $t > 0$.

Claim 3.3. *If there exists a critical cluster* $\mathfrak{C}$ *belonging to* λ *such that* $\Delta(\mathfrak{C}) = +1$*, then* λ *is an atypical value at infinity of* $(f/g)_V$*.*

A sketch of Claim 3.3 is found in Figure 6. In this case, we can say that a connected component of the fiber $(f/g)_V^{-1}(t)$ which is cleaving at infinity as t tends to λ, $t < 0$ or $t > 0$.

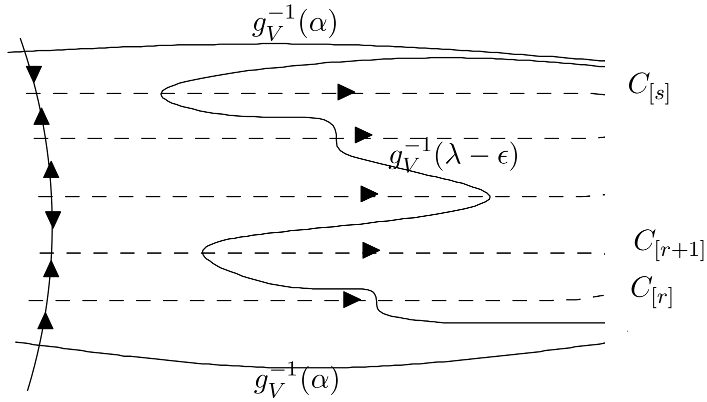


Figure 5: Vanishing at infinity. Another possibility is obtained by reversing the arrows and replacing $\lambda - \epsilon$ with $\lambda + \epsilon$.

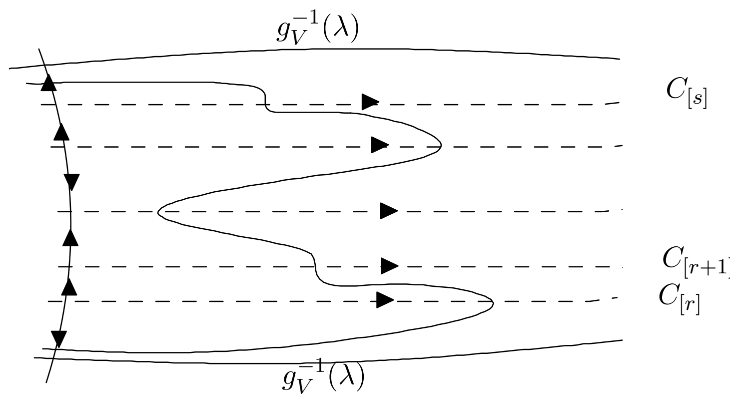


Figure 6: Cleaving at infinity. Another possibility is obtained by reversing the arrows and replacing $\lambda - \epsilon$ with $\lambda + \epsilon$.

Claim 3.4. *If for every critical cluster $\mathfrak{C}$ belonging to λ, $\Delta(\mathfrak{C}) = 0$, then λ is a typical value at infinity of $(f/g)_V$.*

A sketch of Claim 3.4 is found in Figure 7. $\square$

4. Proof of Theorem 2.2

We will denote by $P_{\mathbb{C}}$ the complexification of the polynomial map P. Put $V_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$, $V_{\mathbb{C}}^0 = V_{\mathbb{C}} \backslash (g_{\mathbb{C}} = 0)$. Denote by $\mathbb{S}_R^{n-1}(\mathbb{C})$ the sphere in $\mathbb{C}^n$ of radius R centered at the origin. Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\pi_{1,\mathbb{C}} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the projections on the first n coordinates.

By Theorem 2.1, the set $A_\infty[(f/g)_V)]$ has at most j points, where j is the number of half-branches at infinity of $\Gamma[(f/g)_V]$. Clearly, j is equal to the number

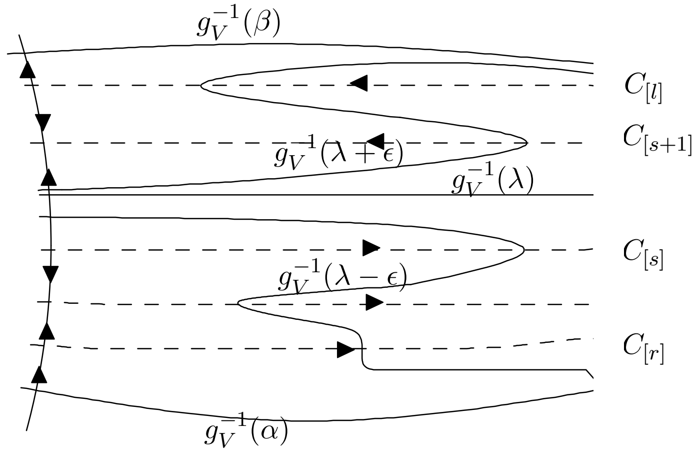


Figure 7: Local triviality at infinity.

of isolated real solutions of the system of polynomial equations

$$\begin{cases} f_1(x) = 0, \\ \dots \\ f_{n-2}(x) = 0, \\ H(x) = 0, \\ x_1^2 + \cdots + x_n^2 = R^2. \end{cases} \qquad (*)$$

Claim: For $R > 0$ large enough, each real solution of $(*)$ is an isolated complex solution.

In fact, put

$$X(\mathbb{C}) = \{(x, a) \in \mathbb{C}^n \times \mathbb{C}^n : F_{\mathbb{C}}(x) = 0, H_{\mathbb{C}}(x, a) = 0\}.$$

By the same argument as in the proof of Lemma 2.1, we see that $\dim_{\mathbb{C}} X(\mathbb{C}) = n+1$. Let $X^r(\mathbb{C})$ be the set of regular points of $X(\mathbb{C})$. Let $\pi_{\mathbb{C}} : X^r(\mathbb{C}) \to \mathbb{C}^n$, $\pi_{\mathbb{C}}(x, a) = a$. We denote by $S(\pi_{\mathbb{C}})$ the set of singular points of $\pi_{\mathbb{C}}$. It is easily seen that $\pi_{\mathbb{C}}^{-1}(0)\backslash S(\pi_{\mathbb{C}})$ is a smooth set of complex dimension 1. Hence,

$$\dim_{\mathbb{C}} \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\backslash S(\pi_{\mathbb{C}})\big] = 1.$$

Since the set of critical values of the restriction to $\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\backslash S(\pi_{\mathbb{C}})\big]$ of the function $Q_{\mathbb{C}}(x) = \sum_{i=1}^n x_i^2$ is finite, we have

$$\dim_{\mathbb{C}} \Big(\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\backslash S(\pi_{\mathbb{C}})\big] \cap \mathbb{S}_R^{2n-1}(\mathbb{C})\Big) \le 0.$$

Moreover, it is easily seen that

$$H_{\mathbb{C}}^{-1}(0) \cap V_{\mathbb{C}} = \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\backslash S(\pi_{\mathbb{C}})\big] \cup \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0) \cap S(\pi_{\mathbb{C}})\big] \cup S\big[(f/g)_{V_{\mathbb{C}}}\big],$$

where $S\big[(f/g)_{V_{\mathbb{C}}}\big]$ consists of the set of singular points $S(V^0_{\mathbb{C}})$ of $V^0_{\mathbb{C}}$ and the set of singular points $S\big[(f/g)_{V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})}\big]$ of $(f/g)_{V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})}$. Also, for every $x \in \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})\big]$, we see that

(i) $x \in V^0_{\mathbb{C}}$, but since $S(V^0_{\mathbb{C}})$ is closed in $V^0_{\mathbb{C}}$, $x \notin \overline{S(V^0_{\mathbb{C}})} \subset V^0_{\mathbb{C}}$;
(ii) $x \in V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})$, but since $S\big[(f/g)_{V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})}\big]$ is closed in $V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})$, $x \notin \overline{S\big[(f/g)_{V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})}\big]} \subset V^0_{\mathbb{C}}\setminus S(V^0_{\mathbb{C}})$;
(iii) $x \in V^0_{\mathbb{C}}\setminus S\big[(f/g)_{V_{\mathbb{C}}}\big]$, but since the set $\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0) \cap S(\pi_{\mathbb{C}})\big]$ is closed in $V^0_{\mathbb{C}}\setminus S\big[(f/g)_{V_{\mathbb{C}}}\big]$, we have $x \notin \overline{\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0) \cap S(\pi_{\mathbb{C}})\big]} \subset V^0_{\mathbb{C}}\setminus S\big[(f/g)_{V_{\mathbb{C}}}\big]$.

Thus, each $x \in \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})\big] \cap \mathbb{S}_R^{2n-1}(\mathbb{C})$ is an isolated point of $H_{\mathbb{C}}^{-1}(0) \cap V_{\mathbb{C}} \cap \mathbb{S}_R^{2n-1}(\mathbb{C})$. In other words, every point of $\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})\big] \cap \mathbb{S}_R^{2n-1}(\mathbb{C})$ is an isolated complex solution of $(*)$.

Since, by assumption, $0 \in \mathbb{R}^n$ is a regular value of π, $\pi^{-1}(0) \subset \pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})$. Hence

$$\pi_1[\pi^{-1}(0)] \cap \mathbb{S}_R^{n-1} \subset \pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})\big] \cap \mathbb{S}_R^{2n-1}(\mathbb{C}).$$

Moreover, since $\Gamma[(f/g)_V] = \pi_1[\pi^{-1}(0)] \cup S[(f/g)_V]$ and the set $S[(f/g)_V]$ of singular points of $(f/g)_V$ is bounded, we have

$$\Gamma[(f/g)_V] \cap \mathbb{S}_R^{n-1} = \pi_1\big[\pi^{-1}(0)\big] \cap \mathbb{S}_R^{n-1}.$$

Thus, $\Gamma[(f/g)_V] \cap \mathbb{S}_R^{n-1}$ of real solutions of $(*)$ is contained in $\pi_{1,\mathbb{C}}\big[\pi_{\mathbb{C}}^{-1}(0)\setminus S(\pi_{\mathbb{C}})\big] \cap \mathbb{S}_R^{2n-1}(\mathbb{C})$, and therefore, each real solution of $(*)$ is an isolated complex solution.

Now, by Bezout's theorem, $j \le 2d_1 \dots d_{n-2}[c_1 + c_2 + d_1 + \dots + d_{n-2} + 2 - n]$.

5. Remarks

The following example shows that the use of the polar curve (as in [6], [4] and [3]) does not work for functions on a surface. In fact, for using the polar curve, it is necessary to consider a linear function $L : \mathbb{R}^2 \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}$, the restriction function $L : f_V^{-1}(D_\delta) \to \mathbb{R}$ is proper for $\delta > 0$ sufficiently small. However, in general, such a linear function does not exist if $\mathbb{R}^2$ is replaced by an algebraic surface.

Put

$$V = \{(x, y, z) \in \mathbb{R}^3 : x(xyz - 1) = 0)\},$$

and

$$f : V \to \mathbb{R}, \quad (x, y, z) \mapsto z.$$

Let us consider the value $\lambda = 0$ and a linear function $L(x, y, z) = Ax + By + Cz$. Let δ be a small positive number. It is seen that the restriction function $L : f_V^{-1}(D_\delta) \to \mathbb{R}$ is not proper. In fact, let us consider the closed interval $[-1, 1] \subset \mathbb{R}$. Then, there exists a sequence of points $\{(x_k, y_k, z_k)\}_{k \ge k_0} \subset f_V^{-1}(D_\delta)$ such that $\|(x_k, y_k, z_k)\|$ tends to $+\infty$ as k tends to $+\infty$ and $L(x_k, y_k, z_k) \in [-1, 1]$:

- If $A = 0$, take $\{(x_k = k^2, y_k = \frac{1}{k}, z_k = \frac{1}{k})\}_{k \geq k_0}$. Clearly, $f_V(x_k, y_k, z_k) = \frac{1}{k}$, $\|(x_k, y_k, z_k)\| \to +\infty$ as $k \to +\infty$, and $L(x_k, y_k, z_k) = (B + C).\frac{1}{k} \in [-1, 1]$.
- If $B = 0$, take $\{(x_k = 0, y_k = k, z_k = \frac{1}{k})\}_{k \geq k_0}$. Clearly, $f_V(x_k, y_k, z_k) = \frac{1}{k}$, $\|(x_k, y_k, z_k)\| \to +\infty$ as $k \to +\infty$, $L(x_k, y_k, z_k) = C.\frac{1}{k} \in [-1, 1]$.
- Assume that $A \neq 0$ and $B \neq 0$. Set
$$\epsilon = \begin{cases} +1 & \text{if } A.B > 0, \\ -1 & \text{if } A.B < 0. \end{cases}$$
Then, take
$$\{(x_k = \sqrt{\left|\frac{B}{A}\right| k}, y_k = -\frac{A}{B}\sqrt{\left|\frac{B}{A}\right| k}, z_k = \epsilon\frac{1}{k})\}_{k \geq k_0}.$$
Clearly, $f_V(x_k, y_k, z_k) = \epsilon\frac{1}{k}$, $\|(x_k, y_k, z_k)\| \to +\infty$ as $k \to +\infty$, $L(x_k, y_k, z_k) = \epsilon C.\frac{1}{k} \in [-1, 1]$.

Therefore, the set $L^{-1}([-1, 1]) \cap f_V^{-1}(D_\delta)$ is non-compact, and $L : f_V^{-1}(D_\delta) \to \mathbb{R}$ is not proper.

6. Examples

Example 6.1. Put
$$V = \{(x, y, z) \in \mathbb{R}^3 : x(xy - 1) - z = 0\},$$
and
$$f : V \to \mathbb{R},\ (x, y, z) \mapsto z.$$
We have

- $H(x, y, z) = y(2xy - 1) - x^3$.
- $\Gamma(f_V) = \{(x, y, z) \in \mathbb{R}^3 : 2xy^2 - y - x^3 = 0,\ z = x(xy - 1)\}$.
- $\mathbb{S}^2_R \cap V$ contains a unique compact connected component K.
- The half-branches at infinity of $\Gamma(f_V)$, starting from K, are
$$C_{[1]} < C_{[2]} < C_{[3]} < C_{[4]} < C_{[5]} < C_{[6]},$$
and there are two critical clusters $\mathfrak{C} = \{C_{[1]}\}$, $\mathfrak{C}' = \{C_{[4]}\}$ belonging to $\lambda = 0$, where $C_{[1]}$, $C_{[4]}$ are germs at infinity of the curves
$$\begin{cases} y = (1 + \sqrt{1 + 8x^4})/4x \\ z = x(xy - 1) \\ x \to 0\,(x > 0) \end{cases} \quad \text{and} \quad \begin{cases} y = (1 - \sqrt{1 + 8x^4})/4x \\ z = x(xy - 1) \\ x \to 0\,(x < 0), \end{cases}$$
respectively. We have $\theta(C_{[1]}) = +1$ and $\widetilde{\theta}(C_{[1]}) = +1$, hence $\Delta(\mathfrak{C}) = +1$. Thus, there is a connected component of $f_V^{-1}(t)$ which is cleaving at infinity as $t \searrow 0$. For the critical cluster $\mathfrak{C}' = \{C_{[4]}\}$, a similar argument as in the above case shows that there is a connected component of $f_V^{-1}(t)$ which is cleaving at infinity as $t \nearrow 0$.

Therefore, the set of atypical values at infinity of f_V, $A_\infty(f_V) = \{0\}$. Moreover, one connected component of $f_V^{-1}(t)$ is cleaving at infinity as $t \nearrow 0$, and one connected component of $f_V^{-1}(t)$ is cleaving at infinity as $t \searrow 0$.

Example 6.2. Put $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, and $f : V \to \mathbb{R}$, $(x, y, z) \mapsto x^2 z - z + y$. We have

- $H(x, y, z) = 2xz(2yz - 1)$.
- $\Gamma(f_V) = \{(x, y, z) : x = 0, y = \pm 1\} \cup \{(x, y, z) : x^2 + y^2 = 1, z = 0\} \cup \{(x, y, z) : x^2 + y^2 = 1, 2yz - 1 = 0\}$.
- $\mathbb{S}^2_R \cap V$ is the union of two compact connected components K_1, K_2, where

$$K_1 = \{(x, y, z) | x^2 + y^2 = 1, z = \sqrt{R^2 - 1}\} \text{ and}$$

$$K_2 = \{(x, y, z) | x^2 + y^2 = 1, z = -\sqrt{R^2 - 1}\}.$$

- The half-branches at infinity of $\Gamma(f_V)$, starting from K_1 are $C_{[1]} < C_{[2]} < C_{[3]} < C_{[4]}$, and there are two critical clusters $\mathfrak{C} = \{C_{[1]}\}$, $\mathfrak{C}' = \{C_{[3]}\}$ belonging to $\lambda = 0$, where $C_{[1]}$, $C_{[3]}$ are germs at infinity of the curves

$$\begin{cases} x = \sqrt{1 - y^2}) \\ z = \frac{1}{2y} \\ y \to 0\,(y > 0) \end{cases} \quad \text{and} \quad \begin{cases} x = -\sqrt{1 - y^2}) \\ z = \frac{1}{2y} \\ y \to 0\,(y > 0) \end{cases}$$

respectively. We have $\theta(C_{[1]}) = +1$ and $\widetilde{\theta}(C_{[1]}) = +1$, hence $\Delta(\mathfrak{C}) = +1$. Thus, there is a connected component of $f_V^{-1}(t)$ which is cleaving at infinity as $t \searrow 0$. For the critical cluster $\mathfrak{C}' = \{C_{[3]}\}$, a similar argument as in the above case shows that there is a connected component of $f_V^{-1}(t)$ which is cleaving at infinity as $t \searrow 0$.

Similar arguments as in the above case for K_2 show that the set of atypical values at infinity of f_V, $A_\infty(f_V) = \{0\}$. Moreover, two connected components of $f_V^{-1}(t)$ are cleaving at infinity as $t \nearrow 0$, and two connected components of $f_V^{-1}(t)$ are cleaving at infinity as $t \searrow 0$.

Example 6.3. Put $V = \{(x, y, z) \in \mathbb{R}^3 : x(xyz - 1) = 0\}$, and $f : V \to \mathbb{R}$, $(x, y, z) \mapsto z$. We have

- $H(x, y, z) = y(2xyz - 1) - x^3 z$.
- $\Gamma(f_V) = \{(x, y, z) : x = y = 0\} \cup \{(x, y, z) : y = x, z = \frac{1}{x^2}\} \cup \{(x, y, z) : y = -x, z = \frac{-1}{x^2}\}$.
- $\mathbb{S}^2_R \cap V$ is the union of five compact connected components K_1, K_2, K_3, K_4, K_5, where

$$\mathbb{S}^2_R \cap \{(x, y, z) \in \mathbb{R}^3 | x = 0\} = K_1 \text{ and}$$

$$\mathbb{S}^2_R \cap \{(x, y, z) \in \mathbb{R}^3 : xyz - 1 = 0\} = K_2 \cup K_3 \cup K_4 \cup K_5.$$

- The half-branches at infinity of $\Gamma(f_V)$, starting from K_1, are $\Gamma_{[1]} < \Gamma_{[2]}$, where $\Gamma_{[1]}$, $\Gamma_{[2]}$ are germs at infinity of the curves
$$\begin{cases} x = 0 \\ y = 0 \\ z \to +\infty \end{cases} \quad \text{and} \quad \begin{cases} x = 0 \\ y = 0 \\ z \to -\infty, \end{cases}$$
respectively, with $\lim_{\Gamma_{[1]}} f_V = +\infty$, $\lim_{\Gamma_{[2]}} f_V = -\infty$.
- The half-branches at infinity of $\Gamma(f_V)$, starting from K_2 are $C_{[1]} < C_{[2]}$, where $C_{[1]}$, $C_{[2]}$ are germs at infinity of the curves
$$\begin{cases} y = x \\ z = \frac{1}{x^2} \\ x \to +\infty \end{cases} \quad \text{and} \quad \begin{cases} y = x \\ z = \frac{1}{x^2} \\ x \to 0\,(x > 0), \end{cases}$$
respectively. Since $\lim_{C_{[1]}} f_V = 0$, $\lim_{C_{[2]}} f_V = +\infty$, there is the critical cluster $\mathfrak{C} = \{C_{[1]}\}$ belonging to $\lambda = 0$. We have $\theta(C_{[1]}) = +1$ and $\widetilde{\theta}(C_{[1]}) = -1$, hence $\Delta(\mathfrak{C}) = -1$. Thus, there is a connected component of $f_V^{-1}(t)$ which is vanishing at infinity as $t \searrow 0$.

Similar arguments as in the above case for K_3, K_4, K_5 show that the set of atypical values at infinity of f_V, $A_\infty(f_V) = \{0\}$. Moreover, two connected components of $f_V^{-1}(t)$ are vanishing at infinity as $t \nearrow 0$, and two connected components of $f_V^{-1}(t)$ are vanishing at infinity as $t \searrow 0$.

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