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DETERMINATION OF THE INTERNAL HEAT SOURCE FOR A HALF-BURIED ROD

VU KIM TUAN AND FADHEL AL-MUSALLAM

Dedicated to Professor Tran Duc Van on the occasion of his sixtieth birthday

Abstract. The paper considers a one-dimensional heat transfer in a perfectly lateral insulated uniform rod, which is partially exposed for measurement at one end, and proposes an algorithm that uniquely recovers the heat source proportional coefficient, the coefficient of convective heat transfer from a countable set of temperature recordings at one end of the rod. The length of the rod can be recovered by just one single measurement.

1. Introduction

Consider a uniform and perfectly lateral insulated rod of unknown length $b \leq \infty$, and an unknown heat source (sink) where the heat source is known to be proportional to the temperature distribution. One end of the rod (say $x = 0$) is in open air, so different initial temperatures can be imposed at this end, and the rest of the rod is buried under earth, hence no measurement or observation is allowed at that part of the rod. At the end(s) of the rod, there is a convection process with unknown convective heat transfer constant(s). We vary the initial temperature on the exposed part of the rod, and measure the corresponding temperature at this end, $x = 0$, of the rod.

The problem is to determine the heat source (sink) proportionality coefficient, the coefficient(s) of convective heat transfer, and the length of the rod. Without loss of generality, we can assume that the left-end of the rod, $x = 0$, is in open air, and the exposed part of the rod has length > 1, so we can change the initial temperature on the interval $(0, 1)$. The heat transfer model can be described by the boundary value problem

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$$
(1.1) \quad \begin{cases} u_t\,(x,t) = u_{xx}(x,t) - q(x)u(x,t), \quad 0 < x < b \leq \infty, \quad t \geq 0, \quad b > 1, \\ u(0,t) - hu_x(0,t) = 0, \\ u(b,t) + Hu_x(b,t) = 0, \\ u(x,0) = f(x), \quad f(x) = 0 \quad \text{for} \quad x > 1. \end{cases}
$$

The constants h, H in the boundary conditions are the convective heat transfer coefficients. These parameters basically control the amount of heat released through radiation process at the end points. If the rod is infinite ($b = \infty$), we do not require any boundary condition at b. The initial temperature f in (1.1) was chosen so that no measurement was needed in the buried part of the rod. To express the dependence of u on the initial temperature f, we sometimes use the notation $u = u^f$, when needed.

We are interested in the determination of the heat source proportionality coefficient q, the length b, the convective heat transfer coefficients h and H (when $b < \infty$), from the measurements of $u^f(0,t)$, which is the temperature at the left-end of the rod.

The inverse heat equation is usually considered under the assumption that the full lateral Dirichlet-to-Neumann map, $u(0,t) \to u_x\,(0,t)$, is given. It is shown that q is uniquely determined by the Dirichlet-to-Neumann map [7, 8, 3, 1, 2]. In [10], the strategy is to introduce another family of source terms ψ_j, independent of the temperature distribution $u(x,t)$, and then solve the countable family of equations

$$
u_t = u_{xx}(x,t) - q(x)u(x,t) + \psi_j(x)
$$

with Dirichlet boundary and initial conditions, i.e. $u(x,0) = u(0,t) = u(\pi,t) = 0$. It is then shown that if the sequence $\{\psi_j\}_{j\geq 0}$ is complete in $L_2(0,\pi)$, then q can be uniquely determined by the sequence $\left\{\frac{\partial}{\partial x}u(0,t^*)\right\}_j$ when q is small enough and t^* is large enough.

In [4, 6], the inverse heat equation on $(0,\pi)$ is considered under the initial-to-boundary map. It is shown in [4] that for any $q \in L_1(0,\pi)$ there exists N such that q is uniquely determined by a finite number of measurements $f_i(x) \to u^{f_i}(0,t), t \in (0,T), i = 1, \cdots, N$, that are read for two different convective heat transfer coefficients $h_1 \neq h_2$ at $x = 0$. Furthermore, in case some extra information is known about q, then four measurements are sufficient to uniquely determine q [6]. The approach that [4] and [6] follow is out of the mainstream in the sense that it does not use the full Dirichlet-to-Neumann map $u(0,t) \to u_x\,(0,t)$, but bases the recovery algorithm of q solely on a finite number of initial-to-boundary measurements $\{u^{f_i}(0,t)\}$.

The present paper will consider the inverse heat problem in the singular case where the interval is infinite ($b = \infty$) as well as in the regular case, when the interval is finite ($b < \infty$), but no a priori information on the length or even the finiteness of the interval is given. We allow no change in the boundary condition, even in the regular case, as in [4, 6]. Thus, in the present setting, there is no way to determine two sets of eigenvalues, and therefore one is unable to determine the spectral function through two complete sets of eigenvalues.

Our approach rests on the asymptotics of the boundary temperature at infinity to determine the eigenvalues and decide whether the rod is of a finite length or infinite. Moreover, the finiteness and the length of the rod can be established by just one measurement.

By choosing a suitable set of initial temperatures for the boundary observations, we recover as an inverse Laplace transform, the product of an unknown entire function of exponential type with a spectral function associated with a Sturm-Liouville problem. The observations are made on discrete time intervals, so we are faced with the problem of recovering the inverse Laplace transforms from discrete data (Section 2). Changing the initial temperature in an appropriate way, we can determine all Fourier coefficients of the entire function, and therefore, the function itself. This would recover the spectral function, and by the Gelfand-Levitan theorem, the coefficient q and the boundary constants h, H can be determined.

2. An inverse formula for the Laplace transform

Let f be an arbitrary function defined on the interval $(0, \infty)$. Recall that the integral

$$g(s) = \int_0^\infty e^{-st} f(t) dt, \tag{2.1}$$

if it exists, is called the Laplace transform of f.

In this section, we derive an asymptotics formula for the Laplace transform and establish a formula for the inverse Laplace transform that we will need in the next section.

The Abel theorem for the Laplace transform [16] says that

$$\lim_{s \to \infty} sg(s) = f(0+).$$

Suppose that f does not change sign on $(0, a)$. Without loss of generality, we may assume that $f(t) > 0$ on $(0, a)$. Let $0 < \epsilon < a$. Then

$$g(s) = \int_0^\epsilon e^{-st} f(t)\, dt + \int_\epsilon^\infty e^{-st} f(t)\, dt =: g_1(s) + g_2(s).$$

Apply the mean value theorem to the first integral to get

$$g_1(s) = e^{-st_0} \int_0^\epsilon f(t)\, dt \geq e^{-s\epsilon} \int_0^\epsilon f(t)\, dt > 0, \quad 0 < t_0 < \epsilon.$$

Hence

$$\underline{\lim}_{s \to \infty} e^{\epsilon s} g_1(s) > 0.$$

As for the second integral, one can see from its definition that $e^{\epsilon s}g_2(s)$ is the Laplace transform of $f(t+\epsilon)$, therefore, the use of the Abel theorem yields

$$\lim_{s\to\infty} se^{\epsilon s}g_2(s) = f(\epsilon+).$$

Consequently,

$$\underline{\lim}_{s\to\infty} e^{\epsilon s}g(s) > 0.$$

Hence if f does not change sign on $(0,a)$, then

$$\lim_{s\to\infty} e^{as}\,|g(s)| = \infty.$$

Using this and the fact that $\lim_{s\to\infty}|g(s)| = 0$ show that there is an $s_0 = s_0(a) > 0$ such that

$$1 > |g(s)| > e^{-as} \qquad \text{for } s > s_0.$$

Hence

$$-a \le \underline{\lim}_{s\to\infty}\frac{\ln|g(s)|}{s} \le \overline{\lim}_{s\to\infty}\frac{\ln|g(s)|}{s} \le 0.$$

Since a can be chosen as small as one would like it to be, it follows that

$$\lim_{s\to\infty}\frac{\ln|g(s)|}{s} = 0, \tag{2.2}$$

which means that $g(s)$ decays to 0 slower than any exponential function.

If the Laplace transform g is known in the complex plane, the original f can be recovered by the Bromwich contour integral [16]. However, if the Laplace transform g is given only on the positive real axis $(0,\infty)$, that is the case when the Laplace transform (2.1) is considered as an integral equation of the first kind, one should use an inverse formula involving only the values of g on $(0,\infty)$. A first formula of this kind was introduced by Post [11] and Widder [15, 16]

$$f(t) = \lim_{n\to\infty}\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} g^{(n)}\left(\frac{n}{t}\right). \tag{2.3}$$

It allows reconstruction of the original f by means of the values of the derivatives of high order of its Laplace image g at points of the real axis almost everywhere. Recently, real-variable inverse formulas for the Laplace transform without using derivatives of g have been considered ([13, 14]). All of these formulas require sampling g at non-integer points depending on t. The following theorem states a new real-variable inverse formula for the Laplace transform free of this dependency.

Theorem 1. *Let $f \in L_\infty(0,\infty)$. If f has a jump discontinuity at t, then*

$$\lim_{n\to\infty} n\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(j-1)!}e^{njt}g\,(nj) = (1-e^{-1})f(t+0) + e^{-1}f(t-0),$$

where g is as in (2.1)*. In particular, if f is continuous at t, then*

$$\lim_{n\to\infty} n\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(j-1)!}e^{njt}g\,(nj) = f(t). \tag{2.4}$$

In other words, f is completely determined by $\{g(n)\}_{n\ge1}$.

Proof. Define

$$f_n(t) = \frac{n}{1-e^{-e^{nt}}} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{njt} g\,(nj)\,.$$

The series defining $f_n(t)$ converges absolutely for every t. Indeed,

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{1}{(j-1)!} e^{njt} \left|g\,(nj)\right| &\leq \sum_{j=1}^{\infty} \frac{1}{(j-1)!} e^{njt} \int_0^{\infty} e^{-njx} |f(x)|\,dx \\
&\leq \|f\|_{\infty} \int_0^{\infty} e^{n(t-x)} \sum_{j=1}^{\infty} \frac{1}{(j-1)!} e^{n(j-1)(t-x)} dx \\
&= \|f\|_{\infty} \int_0^{\infty} e^{n(t-x)} e^{e^{n(t-x)}} dx = \frac{1}{n}\left(e^{e^{nt}} - 1\right) \|f\|_{\infty} < \infty.
\end{aligned}$$

Since

$$\left| \sum_{j=1}^{M} \frac{(-1)^{j-1}}{(j-1)!} e^{nj(t-x)} f(x) \right| \leq \|f\|_{\infty} e^{n(t-x)} \sum_{j=0}^{\infty} \frac{1}{j!} e^{njt} = \|f\|_{\infty} e^{n(t-x)} e^{e^{nt}}$$

and $\|f\|_{\infty} e^{-xn}$ is in $L_1(0,\infty)$, Lebesque's dominated convergence theorem justifies the interchange of the order of integration and summation in the following manipulation

$$\begin{aligned}
f_n(t) &= \frac{n}{1-e^{-e^{nt}}} \sum_{j=1}^{\infty} \int_0^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{nj(t-x)} f(x)\,dx \\
&= \frac{n}{1-e^{-e^{nt}}} \int_0^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{nj(t-x)} f(x)\,dx \\
&= \frac{n}{1-e^{-e^{nt}}} \int_0^{\infty} e^{n(t-x)} f(x) \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{n(j-1)(t-x)} dx \\
&= \frac{n}{1-e^{-e^{nt}}} \int_0^{\infty} e^{n(t-x)} e^{-e^{n(t-x)}}\, f(x) dx = \int_0^{\infty} K_n(t,x) f(x) dx,
\end{aligned}$$

where

$$K_n(t,x) = \frac{n}{1-e^{-e^{nt}}} e^{n(t-x)} e^{-e^{n(t-x)}}.$$

Clearly, $K_n(t,x) > 0$, and making the change of variables $z = e^{n(t-x)}$, we get

$$\int_a^b K_n(t,x) dx = \frac{1}{1-e^{-e^{nt}}} \int_{e^{n(t-b)}}^{e^{n(t-a)}} e^{-z}\,dz = \frac{e^{-e^{n(t-b)}} - e^{-e^{n(t-a)}}}{1-e^{-e^{nt}}}. \tag{2.5}$$

Assume that f has a jump discontinuity at t. Then for any $\epsilon > 0$ there exists a $\delta \in (0, t)$ such that $|f(x) - f(t+0)| < \epsilon$ for $x \in (t, t+\delta)$ and $|f(x) - f(t-0)| < \epsilon$ for $x \in (t-\delta, t)$. Let

$$J_n(t) = \frac{1 - e^{-1}}{1 - e^{-e^{nt}}} f(t+0) + \frac{e^{-1} - e^{-e^{nt}}}{1 - e^{-e^{nt}}} f(t-0) - f_n(t).$$

Formula (2.5) yields

$$\int_t^\infty K_n(t,x)dx = \frac{1 - e^{-1}}{1 - e^{-e^{nt}}},$$

$$\int_0^t K_n(t,x)dx = \frac{e^{-1} - e^{-e^{nt}}}{1 - e^{-e^{nt}}},$$

so that

$$\begin{aligned}
J_n(t) &= \int_0^t K_n(t,x)[f(t-0) - f(x)]\,dx + \int_t^\infty K_n(t,x)[f(t+0) - f(x)]\,dx \\
&= \left(\int_0^{t-\delta} + \int_{t-\delta}^t\right) K_n(t,x)[f(t-0) - f(x)]\,dx \\
&+ \left(\int_t^{t+\delta} + \int_{t+\delta}^\infty\right) K_n(t,x)[f(t+0) - f(x)]\,dx \\
&= (I_1 + I_2) + (I_3 + I_4).
\end{aligned}$$

Let us estimate $I_i, i = 1, 2, 3, 4$. We have

$$\begin{aligned}
|I_1| &\le \int_0^{t-\delta} K_n(t,x)|f(t-0) - f(x)|\,dx \\
&\le 2\|f\|_\infty \int_0^{t-\delta} K_n(t,x)dx = 2\|f\|_\infty \frac{e^{-e^{n\delta}} - e^{-e^{nt}}}{1 - e^{-e^{nt}}}
\end{aligned}$$

and

$$\begin{aligned}
|I_4| &\le \int_{t+\delta}^\infty K_n(t,x)|f(t+0) - f(x)|\,dx \\
&\le 2\|f\|_\infty \int_{t+\delta}^\infty K_n(t,x)dx = 2\|f\|_\infty \frac{1 - e^{-e^{-n\delta}}}{1 - e^{-e^{nt}}},
\end{aligned}$$

where the last equality in each estimate comes from (2.5).

As for I_2 and I_3, we have

$$|I_2| \le \int_{t-\delta}^{t} K_n(t,x)|f(t-0)-f(x)|\,dx \le \epsilon \int_{t-\delta}^{t} K_n(t,x)dx = \epsilon \frac{e^{-1}-e^{-e^{n\delta}}}{1-e^{-e^{nt}}}$$

and

$$|I_3| \le \int_{t}^{t+\delta} K_n(t,x)|f(t+0)-f(x)|\,dx \le \epsilon \int_{t}^{t+\delta} K_n(t,x)dx = \epsilon \frac{e^{-e^{-n\delta}}-e^{-1}}{1-e^{-e^{nt}}}.$$

Thus, for a fixed t, I_1 and I_4 tend to 0 while I_2 and I_3 become smaller than ϵ as $n \to \infty$, and since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} J_n(t) = 0.$$

Since $\lim_{n\to\infty} e^{-e^{nt}} = 0$, the proof of the theorem is complete. □

3. Determination of b, q, h, H

We start with the direct problem to characterize the solution u of (1.1). First consider the case $b = \infty$.

Let $\varphi(x,\lambda)$ be the solution of the associated initial-value problem

$$\begin{cases} -\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda), & 0 < x < \infty, \\ \varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h. \end{cases}$$

It is known [12] that there exists a non-decreasing function $\rho(\lambda)$ such that if $f \in L_2(R^+)$, then

$$F(\lambda) = \int_0^\infty f(x)\varphi(x,\lambda)\,dx \tag{3.1}$$

is well-defined and belongs to $L_2(R, d\rho)$, and

$$f(x) = \int_{-\infty}^\infty F(\lambda)\varphi(x,\lambda)\,d\rho(\lambda), \tag{3.2}$$

with

$$\|f\|_{L_2(R^+)} = \|F\|_{L_2(R,d\rho)}. \tag{3.3}$$

The function ρ is the spectral function for the Sturm-Liouville operator

$$Ly = -\frac{d^2y}{dx^2} + qy, \qquad y'(0) - hy(0) = 0.$$

We shall assume as in many cases of interest that $q \in L_1(0,\infty)$. In this case, the continuous part of the spectrum, $\sigma(L)$, of the operator L is $[0,\infty)$ and there might be a point spectrum (discrete) in $(-\infty, 0)$, which is bounded below. In fact, if

$$\|q\|_{loc} = \sup_{x \geq 0} \int_x^{x+1} |q(t)|\, dt \leq \|q\|_1$$

and

$$E_0 = -\max\left\{2\,\|q\|_{loc}\,, e^2\,\|q\|_{loc}^2\right\},$$

then $\inf \sigma(L) > E_0$. Thus, if $\{\lambda_n\}_{n\geq 1}$ is the point spectrum of L, arranged in increasing order, then $E_0 < \lambda_1 < \lambda_2 < \ldots < 0$. Thus, the spectral function ρ is absolutely continuous and strictly increasing on $(0, \infty)$ and has the jump

$$\rho_n = \rho(\lambda_n^+) - \rho(\lambda_n^-)$$

at λ_n.

If the initial condition $u(x, 0) = f(x) \in L_2(0, \infty)$ is given, then the general solution $u^f = u$ of (1.1) can be obtained by the integral transform

$$u(x, t) = \int_{-\infty}^{\infty} e^{-\lambda t} \varphi(x, \lambda) F^f(\lambda) d\rho(\lambda), \tag{3.4}$$

where

$$F^f(\lambda) = \int_0^{\infty} f(x) \varphi(x, \lambda)\, dx. \tag{3.5}$$

Thus, the observation $u^f(0, t) = u(0, t)$ has the integral representation

$$u^f(0, t) = \int_{-\infty}^{\infty} e^{-\lambda t} \varphi(0, \lambda) F^f(\lambda) d\rho(\lambda) = G^f(t) + \sum_{n=1} \rho_n F^f(\lambda_n) e^{-\lambda_n t}, \tag{3.6}$$

where

$$G^f(t) = \int_0^{\infty} e^{-\lambda t} F^f(\lambda) d\rho(\lambda) = \int_0^{\infty} e^{-\lambda t} F^f(\lambda) \rho'(\lambda) d\lambda$$

is the Laplace transform of $F^f(\lambda)\rho'(\lambda)$. The formula for $G^f(t)$ and (3.6) make it clear that $u^f(0, t)$ is an analytic function in the right-half plane $\mathrm{Re}\, t > 0$.

Now assume $b < \infty$.

Let $0 < \mu_1 < \mu_2 < \cdots$ be the complete set of eigenvalues and $\varphi_n(x)$ be the eigenfunction associated with the eigenvalue μ_n for the Sturm-Liouville problem

$$\begin{cases} -\varphi_n''(x) + q(x)\varphi_n(x) = \mu_n \varphi_n(x), \quad 0 < x < b < \infty, \\ \varphi_n'(0) - h\varphi_n(0) = 0, \\ \varphi_n'(b) + H\varphi_n(b) = 0, \end{cases}$$

normalized by $\varphi_n(0) = 1$. The spectral function $\rho(\lambda)$ in this case is a step function, with the jump $\alpha_n = \frac{1}{\int_0^b \varphi_n^2(x)\,dx}$ at $\lambda = \mu_n$. Any function $f \in L_2(0,b)$ can be expanded into the generalized Fourier series

$$f(x) = \sum_{n=1}^{\infty} c_n^f \varphi_n(x),$$

with

$$c_n^f = \frac{(f,\varphi_n)}{(\varphi_n,\varphi_n)} = \frac{\int_0^b f(x)\varphi_n(x)\,dx}{\int_0^b \varphi_n^2(x)\,dx}.$$

The solution of (1.1) can be obtained by the series

$$u^f(x,t) = \sum_{n=1}^{\infty} c_n^f e^{-\mu_n t}\varphi_n(x).$$

Thus, the observation $u^f(0,t) = u(0,t)$ has the series representation

$$u^f(0,t) = \sum_{n=1}^{\infty} c_n^f e^{-\mu_n t}\varphi_n(0) = \sum_{n=1}^{\infty} c_n^f e^{-\mu_n t}, \tag{3.7}$$

which clearly reveals that $u^f(0,t)$ is an analytic function in the right-half plane $\mathrm{Re}\, t > 0$.

We now state our main theorem.

Theorem 2. *Assume a priori $q \in L_1(0,b)$. Let the initial temperatures $\{f_j\}_{j\geq 1}$ be a sequence such that $f_j\chi_{[1,\infty)} = 0$, $(j \geq 1)$ and $\{f_j\chi_{(0,1)}\}_{j\geq 1}$ is an orthonormal basis of $L_2(0,1)$. Then the observations $\{u^{f_j}(0,k)\}_{j\geq 1}$, $k = 1,2,\ldots,$ determine $q, h, H, b,$ uniquely.*

Proof. Clearly, $f_j \in L_2(0,\infty)$, $(j \geq 1)$. If we choose the initial temperature to be f_j, then

$$F^{f_j}(\lambda) = \int_0^1 f_j(x)\varphi(x,\lambda)dx$$

is an entire function, which is real-valued on the real line.

We shall break the remainder of the proof into 8 steps.

(i) Determining the nature of b, is it finite or infinite

Let $b = \infty$. Fix λ. Observe that there is a j_n such that $F^{f_{j_n}}(\lambda) \neq 0$, for otherwise

$$0 = F^{f_i}(\lambda) = \int_0^1 f_i(x)\varphi(x,\lambda)dx, \quad i = 1,2,\cdots,$$

and from the fact that $\{f_i\}_{i\geq 1}$ is a basis of $L_2(0,1)$ it would follow that $\varphi(x,\lambda)=0$ for $x\in(0,1)$. Furthermore,

$$u^{f_i}(0,t)=G^{f_i}(t)+\sum_{n=1}^{\infty}\rho_n F^{f_i}(\lambda_n)e^{-\lambda_n t}.$$

As $F^{f_i}(\lambda)$ is a nontrivial entire function, its zeros are isolated, and therefore $F^{f_i}(\lambda)$ does not change sign on some interval $(0,a_i)$. Since $\rho'(\lambda)>0$ on $(0,\infty)$, we have $F^{f_i}(\lambda)\rho'(\lambda)\neq 0$ on $(0,a_i)$. Thus, from the previous section it follows that $G^{f_i}(t)$ as the Laplace transform of $F^{f_i}(\lambda)\rho'(\lambda)$ decays to 0 slower than any exponential function. In other words, $\lim\limits_{t\to\infty} e^{\epsilon t}\left|G^{f_i}(t)\right|=\infty$ for any $\epsilon>0$. Since $\lim\limits_{t\to\infty} e^{-\lambda_n t}=\infty$ for all n, it follows that

$$\lim_{k\to\infty}\frac{\ln\left|u^{f_i}(0,k)\right|}{k}\geq 0.$$

If $b<\infty$, then

$$u^{f_i}(0,t)=\sum_{n=1}^{\infty}c_n^{f_i}e^{-\mu_n t},$$

and since $c_n^{f_i}\neq 0$ for at least one n, it follows that

$$\lim_{t\to\infty}e^{\epsilon t}u^{f_i}(0,t)=0$$

for any $0<\epsilon<\mu_1$. Hence

$$\underline{\lim}_{k\to\infty}\frac{\ln\left|u^{f_i}(0,k)\right|}{k}<0.$$

Hence we arrive at

$$\lim_{k\to\infty}\frac{\ln\left|u^{f_1}(0,k)\right|}{k}<0 \quad\Rightarrow\quad b<\infty,$$

$$\lim_{k\to\infty}\frac{\ln\left|u^{f_1}(0,k)\right|}{k}\geq 0 \quad\Rightarrow\quad b=\infty.$$

Thus, from the observations one can determine whether the interval is finite or not.

(ii) Determination of μ_n and $c_n^{f_j}$

Let $b<\infty$, so the spectrum of L is discrete. There exists a j_1 such that $c_1^{f_{j_1}}\neq 0$. Then

$$\begin{aligned}\lim_{k\to\infty}\frac{u^{f_{j_1}}(0,k+1)}{u^{f_{j_1}}(0,k)} &= \lim_{k\to\infty}\frac{c_1^{f_{j_1}}e^{-\mu_1}+\sum_{n=l}c_n^{f_{j_1}}e^{-\mu_n}e^{(\mu_1-\mu_n)k}}{c_1^{f_{j_1}}+\sum_{n=l}c_n^{f_{j_1}}e^{(\mu_1-\mu_n)k}}\\ &= \frac{c_1^{f_{j_1}}e^{-\mu_1}}{c_1^{f_{j_1}}}=e^{-\mu_1}\end{aligned}$$

so that

$$\mu_1 = -\lim_{k\to\infty} \ln \left| \frac{u^{f_{j_1}}(0,k+1)}{u^{f_{j_1}}(0,k)} \right|.$$

Now, for any f_j, $c_n^{f_j} \neq 0$ for at least one n. Assume l is the smallest index such that $c_l^{f_j} \neq 0$. Then

$$\begin{aligned} \lim_{k\to\infty} \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} &= \lim_{k\to\infty} \frac{c_l^{f_j} e^{-\mu_l} + \sum_{n>l} c_n^{f_j} e^{-\mu_n} e^{(\mu_l-\mu_n)k}}{c_l^{f_j} + \sum_{n>l} c_n^{f_j} e^{(\mu_l-\mu_n)k}} \\ &= \frac{c_l^{f_j} e^{-\mu_l}}{c_l^{f_j}} = e^{-\mu_l}. \end{aligned}$$

Hence

$$-\lim_{k\to\infty} \ln \left| \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \right| = \mu_l \geq \mu_1.$$

Consequently, the first eigenvalue can be determined uniquely from the observations by the formula

$$\mu_1 = \inf_{j\geq 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \right| \right\}.$$

Once the first eigenvalue μ_1 is determined, $c_1^{f_j}$ can be obtained from the formula

$$c_1^{f_j} = \lim_{k\to\infty} \left[e^{\mu_1 k} u^{f_j}(0,k) \right], \quad j = 1, 2, \cdots.$$

Applying the above argument to

$$U_2^{f_j}(t) = u^{f_j}(0,t) - c_1^{f_j} e^{-\mu_1 t} = \sum_{n=2} c_n^{f_j} e^{-\mu_n t} \qquad (j = 1, 2, 3, \ldots)$$

shows that the second eigenvalue can be determined uniquely by the formula

$$\mu_2 = \inf_{j\geq 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{U_2^{f_j}(k+1)}{U_2^{f_j}(k)} \right| \right\},$$

and $c_2^{f_j} = \lim_{k\to\infty} \left[e^{\mu_2 k} U_2^{f_j}(k) \right]$, $j = 1, 2, 3, \ldots$. It is now clear how one can determine recursively $\{\mu_3, c_3^{f_j}\}, \{\mu_4, c_4^{f_j}\}, \ldots$

(iii) Determination of b, ρ when $b < \infty$

Let $b < \infty$. From the previous step we have found $\mu_1, \mu_2, \cdots$. The asymptotics formula for μ_n

$$\mu_n = \left(\frac{n\pi}{b} + O(1) \right)^2, \quad n \to \infty,$$

yields a formula for b

$$b = \lim_{n\to\infty} \frac{n\pi}{\sqrt{\mu_n}}.$$

Since $\{f_{j|(0,1)}\}_{j\geq 1}$ is an orthonormal basis of $L_2(0,1)$, we have

$$\varphi_n(x) = \sum_{j=1}^{\infty} (\varphi_n, f_j)_{L_2(0,1)} \, f_j(x), \quad 0 < x < 1,$$

where

$$(\varphi_n, f_j)_{L_2(0,1)} = \int_0^1 \varphi_n(x) \, f_j(x) \, dx = \int_0^b \varphi_n(x) \, f_j(x) \, dx = (\varphi_n, f_j) = (\varphi_n, \varphi_n) c_n^{f_j}.$$

Therefore,

$$\frac{\varphi_n(x)}{(\varphi_n, \varphi_n)} = \sum_{j=1}^{\infty} c_n^{f_j} \, f_j(x), \quad 0 < x < 1.$$

Consequently, the eigenfunctions $\varphi_n(x)$ can be determined on $(0,1)$. The restriction $\varphi_n(0) = 1$ easily yields the jump $\alpha_n = \frac{1}{(\varphi_n, \varphi_n)}$ at $\lambda = \mu_n$. Hence the spectral function ρ is determined.

Once the first eigenfunction is known on $(0,1)$, the heat convective constant h can be easily obtained

$$h = \varphi_1'(0).$$

(iv) Existence of the discrete spectrum for the infinite interval case

Let from Step (i) we have found that $b = \infty$. Since $q \in L_1(0,\infty)$ the spectrum of L consists of the continuous part $[0,\infty)$, and a (possibly empty) discrete part on $(-\infty, 0)$.

We have

$$u^{f_i}(0,t) = G^{f_i}(t) + \sum_{n=1} \rho_n F^{f_i}(\lambda_n) e^{-\lambda_n t},$$

and $\lim_{t\to\infty} G^{f_i}(t) = 0$. Thus, if the discrete spectrum of L is empty, so that $u^{f_i}(0,t) = G^{f_i}(t)$, then

$$\left\{ \lim_{k\to\infty} \left| u^{f_i}(0,k) \right| \right\}_{i\geq 1} = \{0\}.$$

On the other hand, if the discrete spectrum of L is nonempty, then there is a j_1 such that $F^{f_{j_1}}(\lambda_1) \neq 0$. Since $\lim_{t\to\infty} e^{-\lambda_1 t} = \infty$, it follows that $\lim_{k\to\infty} \left| u^{f_{j_1}}(0,k) \right| = \infty$. Hence

$$\infty \in \left\{ \lim_{k\to\infty} \left| u^{f_i}(0,k) \right| \right\}_{i\geq 1}.$$

Thus, in case $b = \infty$, one can determine from the observations whether the discrete spectrum of L is empty or not.

(v) Determination of λ_n and $\rho_n F^{f_j}(\lambda_n)$

Let $b = \infty$ and the discrete spectrum of L be nonempty. There exists a j_1 such that $F^{f_{j_1}}(\lambda_1) \neq 0$. Then

$$\begin{aligned}
&\lim_{k\to\infty} \frac{u^{f_{j_1}}(0,k+1)}{u^{f_{j_1}}(0,k)} \\
&= \lim_{k\to\infty} \frac{G^{f_{j_1}}(k+1)e^{\lambda_1 k} + \rho_1 F^{f_{j_1}}(\lambda_1)e^{-\lambda_1} + \sum_{n=2} \rho_n F^{f_{j_1}}(\lambda_n)e^{-\lambda_n} e^{(\lambda_1-\lambda_n)k}}{G^{f_{j_1}}(k)e^{\lambda_1 k} + \rho_1 F^{f_{j_1}}(\lambda_1) + \sum_{n=2} \rho_n F^{f_{j_1}}(\lambda_n)e^{(\lambda_1-\lambda_n)k}} \\
&= \frac{\rho_1 F^{f_{j_1}}(\lambda_1)e^{-\lambda_1}}{\rho_1 F^{f_{j_1}}(\lambda_1)} = e^{-\lambda_1}
\end{aligned}$$

so that

$$\lambda_1 = -\lim_{k\to\infty} \ln \left| \frac{u^{f_{j_1}}(0,k+1)}{u^{f_{j_1}}(0,k)} \right| .$$

Now, for any f_j, it is either $F^{f_j}(\lambda_n) = 0$ for all n, so that

$$u^{f_j}(0,t) = G^{f_j}(t)$$

and

$$\lim_{k\to\infty} u^{f_j}(0,k) = 0,$$

or $F^{f_j}(\lambda_n) \neq 0$ for at least one n. Assume the latter case and λ_l is the smallest eigenvalue such that $F^{f_j}(\lambda_l) \neq 0$. Then

$$\begin{aligned}
&\lim_{k\to\infty} \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \\
&= \lim_{k\to\infty} \frac{G^{f_j}(k+1)e^{\lambda_l k} + \rho_l F^{f_j}(\lambda_l)e^{-\lambda_l} + \sum_{n>l} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n} e^{(\lambda_l-\lambda_n)k}}{G^{f_j}(k)e^{\lambda_l k} + \rho_l F^{f_j}(\lambda_l) + \sum_{n>l} \rho_n F^{f_j}(\lambda_n)e^{(\lambda_l-\lambda_n)k}} \\
&= \frac{\rho_l F^{f_j}(\lambda_l)e^{-\lambda_l}}{\rho_l F^{f_j}(\lambda_l)} = e^{-\lambda_l}.
\end{aligned}$$

Hence

$$-\lim_{k\to\infty} \ln \left| \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \right| = \lambda_l \geq \lambda_1.$$

But $F^{f_j}(\lambda_l) \neq 0$ for at least one l if and only if

$$\lim_{k\to\infty} \left| u^{f_j}(0,k) \right| = \infty.$$

Consequently, the first eigenvalue can be determined uniquely from the observations by the formula

$$\lambda_1 = \inf_{j\geq 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \right| : \lim_{k\to\infty} \left| u^{f_j}(0,k) \right| = \infty \right\}.$$

Once the first eigenvalue λ_1 is determined, $\rho_1 F^{f_j}(\lambda_1)$ can be obtained from the formula

$$\rho_1 F^{f_j}(\lambda_1) = \lim_{k\to\infty} \left[e^{\lambda_1 k} u^{f_j}(0,k) \right], \quad j = 1, 2, \cdots .$$

Applying the above argument to

$$U_2^{f_j}(t) = u^{f_j}(0,t) - \rho_1 F^{f_j}(\lambda_1)e^{-\lambda_1 t} = G^{f_j}(t) + \sum_{n=2}^{\infty} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n t}, \ \ (j = 1,2,3,\ldots)$$

shows that if

$$\lim_{k\to\infty} U_2^{f_j}(k) = 0,$$

for any j, then there is no other eigenvalue of L, and the discrete spectrum of L consists of only one eigenvalue λ_1. Otherwise, a second eigenvalue exists, and can be determined uniquely by the formula

$$\lambda_2 = \inf_{j\ge 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{U_2^{f_j}(k+1)}{U_2^{f_j}(k)} \right| : \lim_{k\to\infty} \left| U_2^{f_j}(k) \right| = \infty \right\},$$

and $\rho_2 F^{f_j}(\lambda_2) = \lim_{k\to\infty} \left[e^{\lambda_2 k} U_2^{f_j}(k) \right]$, $j = 1,2,3,\ldots$ It is now clear how one can determine recursively pairs $\{\lambda_3, \rho_3 F^{f_j}(\lambda_3)\}, \{\lambda_4, \rho_4 F^{f_j}(\lambda_4)\}, \ldots$, if they exist. This process yields $\sum_{n=1}^{\infty} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n t}$, and consequently, one would determine

$$G^{f_j}(k) = \int_0^{\infty} e^{-\lambda k} F^{f_j}(\lambda)\rho'(\lambda)\, d\lambda = u^{f_j}(0,k) - \sum_{n=1}^{\infty} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n k}.$$

(vi) Determination of $F^{f_j}(\lambda)\rho'(\lambda)$ for $\lambda > 0$

Let $b = \infty$. Since $q \in L_1(0,\infty)$, the solution $\varphi(x,\lambda)$ satisfies $\left|\varphi(x,\lambda^2)\right| \le Ce^{x|\operatorname{Im}\lambda|}$ for some independent constant. This shows that $\varphi(x,\lambda)$ is bounded by C for $\lambda > 0$, and consequently, $\left|F^{f_j}(\lambda)\right| \le C\left\|f_j\right\|_2$ for $\lambda > 0$. Since $\rho'(\lambda) \sim \frac{1}{\pi\sqrt{\lambda}}$ as $\lambda \to \infty$, it follows that $F^{f_j}(\lambda)\rho'(\lambda) \in L_\infty(0,\infty)$.

As $F^{f_j}(\lambda)\rho'(\lambda)$ $(\lambda > 0)$ is the inverse Laplace transform of $G^{f_j}(t) = u^{f_j}(0,t) - \sum_{n=1}^{\infty} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n t}$ that is known at $t = 1,2,3,\ldots$, the inversion formula for the Laplace transform (2.4) will then recover

$$F^{f_j}(\lambda)\rho'(\lambda) = \lim_{n\to\infty} n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{nkt} \left[u^{f_j}(0,nk) - \sum_{n=1}^{\infty} \rho_n F^{f_j}(\lambda_n)e^{-\lambda_n nk} \right], \lambda > 0.$$

Thus, $F^{f_j}(\lambda)\rho'(\lambda)$ for $\lambda > 0$ is obtained uniquely from the observations $u^{f_j}(0,k)$, $k = 1,2,3,\ldots$

(vii) Determination of $\rho(\lambda)$

The relation

$$\int_0^1 f_j(x)\varphi(x,\lambda)\, dx = F^{f_j}(\lambda)$$

shows that for each $\lambda > 0$, the constans $F^{f_j}(\lambda)$, $(j \geq 1)$, are the Fourier coefficients in the expansion of $\varphi(x,\lambda)$ into its generalized Fourier series relative to the basis $\{f_j\}_{j\geq 1}$ on $(0,1)$, i.e.

$$\varphi(x,\lambda) = \sum_{j=1}^{\infty}(\varphi, f_j)_{L_2(0,1)} f_j(x) = \sum_{j=1}^{\infty} f_j(x) \int_0^1 f_j(x)\varphi(x,\lambda)\, dx = \sum_{j=1}^{\infty} F^{f_j}(\lambda) f_j(x).$$

Hence

$$\varphi(x,\lambda)\rho'(\lambda) = \sum_{j=1}^{\infty} F^{f_j}(\lambda)\rho'(\lambda) f_j(x), \quad 0 < x < 1, \qquad \lambda > 0.$$

Consequently, once $F^{f_j}(\lambda)\rho'(\lambda)$ has been determined for all $\lambda > 0$ and for any j, we can recover $\varphi(x,\lambda)\rho'(\lambda)$ on $(0,1)$, and because $\varphi(0,\lambda) = 1$ one can determine $\rho'(\lambda)$. Of course, this along with the determination of λ_n, ρ_n, in Step (iii) completely determines $\rho(\lambda) = H(\lambda)\int\limits_0^{\lambda}\rho'(t)dt + \sum\limits_{n=1}^{\infty}\rho_n H(\lambda - \lambda_n)$ on the whole real line, where $H(t)$ is the Heaviside step function.

(viii) Determination of q and h, H

Now that $\rho(\lambda)$ has been determined (in Step (v) for $b < \infty$, and in Step (vii) for $b = \infty$), we can apply the Gelfand-Levitan inverse spectral theory [9] to recover q and h, H. To this end, we define

$$T(x) = \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda})d\left(\rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda_+}\right), \tag{3.8}$$

where $\lambda_+ = \max(\lambda, 0)$, and

$$L(x,y) = \frac{1}{2}T(x+y) + \frac{1}{2}T(x-y),$$

and solve the Fredholm integral equation

$$L(x,y) + K(x,y) + \int_0^x K(x,s)L(s,y)ds = 0, \qquad 0 \leq y \leq x,$$

to find $K(x,y)$, which is also differentiable. The heat source proportionality coefficient q is then given by $q(x) = 2\frac{d}{dx}K(x,x)$. The boundary constant h follows from $h = K(0,0)$. For $b < \infty$, solving the initial value problem

$$\varphi_1''(x) - q(x)\varphi_1(x) = \mu_1\varphi_1(x),$$
$$\varphi_1(0) = 1, \quad \varphi_1'(0) = h,$$

will yield

$$H = -\frac{\varphi_1'(b)}{\varphi_1(b)}.$$

This completes the proof of the theorem. □

Remark 1. The observations $\{u^{f_j}(0,t)\}_{j\geq 1},\ t\in(0,T)$, determine q,h,H,b, uniquely.

In fact, $u^{f_j}(0,t)$ is an analytic function in $\operatorname{Re} t>0$, therefore, from $u^{f_j}(0,t)$ on $(0,T)$ by analytic continuation one can determine $u^{f_j}(0,t)$ for any $t>0$, and Theorem 2 can be applied.

4. Algorithm

Let $\{f_j\}_{j\geq 1}$ be an orthonormal basis of $L_2(0,1)$ and $\{u^{f_j}(0,n)\}_{j\geq 1},\ n=1,2,\cdots$, be the observations.

(1) If
$$\lim_{k\to\infty}\frac{\ln\left|u^{f_1}(0,k)\right|}{k}<0,$$
then $b<\infty$. Go to Step 2

Otherwise, $b=\infty$. Go to Step 4.

(2) The first eigenvalue can be determined by the formula
$$\mu_1=\inf_{j\geq 1}\left\{-\lim_{k\to\infty}\ln\left|\frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)}\right|\right\}.$$
And $c_1^{f_j}$ can be obtained from the formula
$$c_1^{f_j}=\lim_{k\to\infty}\left[e^{\mu_1 k}u^{f_j}(0,k)\right],\quad j=1,2,\cdots.$$
Let
$$U_2^{f_j}(t)=u^{f_j}(0,t)-c_1^{f_j}e^{-\mu_1 t}\qquad (j=1,2,3,\ldots),$$
then the second eigenvalue can be determined by the formula
$$\mu_2=\inf_{j\geq 1}\left\{-\lim_{k\to\infty}\ln\left|\frac{U_2^{f_j}(k+1)}{U_2^{f_j}(k)}\right|\right\},$$
and $c_2^{f_j}=\lim_{k\to\infty}\left[e^{\mu_2 k}U_2^{f_j}(k)\right]$, $j=1,2,3,\ldots$. Determine recursively $\{\mu_3, c_3^{f_j}\}$, $\{\mu_4,c_4^{f_j}\},\ldots$ and go to Step 3 below.

(3) The length b
$$b=\lim_{n\to\infty}\frac{n\pi}{\sqrt{\mu_n}}.$$
The jump α_n
$$\alpha_n=\lim_{x\to 0+}\sum_{j=1}^{\infty}c_n^{f_j}\,f_j(x),\quad 0<x<1.$$
The spectral function
$$\rho(\lambda)=\sum_{n=1}^{\infty}\alpha_n H(\lambda-\mu_n).$$
Go to Step 8.

(4) If

$$\left\{ \lim_{k\to\infty} \left| u^{f_i}(0,k) \right| \right\}_{i\geq 1} = \{0\},$$

then the discrete part of the spectrum of L is empty. Define $G^{f_j}(k) = u^{f_j}(0,k)$ and go to Step 6.

Otherwise, the discrete part of the spectrum of L is nonempty. Go to Step 5 below.

(5) The first eigenvalue can be determined by the formula

$$\lambda_1 = \inf_{j\geq 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{u^{f_j}(0,k+1)}{u^{f_j}(0,k)} \right| : \lim_{k\to\infty} \left| u^{f_j}(0,k) \right| \neq 0 \right\},$$

and $\rho_1 F^{f_j}(\lambda_1)$ can be obtained from the formula

$$\rho_1 F^{f_j}(\lambda_1) = \lim_{k\to\infty} \left[e^{\lambda_1 k} u^{f_j}(0,k) \right], \quad j = 1, 2, \cdots .$$

Let

$$U_2^{f_j}(t) = u^{f_j}(0,t) - \rho_1 F^{f_j}(\lambda_1) e^{-\lambda_1 t} \qquad (j = 1, 2, 3, \ldots).$$

If

$$\left\{ \lim_{k\to\infty} U_2^{f_j}(k) \right\}_{j\geq 1} = \{0\},$$

then there is no other eigenvalue of L, and the discrete spectrum of L consists of only one eigenvalue λ_1. Otherwise, the second eigenvalue exists, and can be determined by the formula

$$\lambda_2 = \inf_{j\geq 1} \left\{ -\lim_{k\to\infty} \ln \left| \frac{U_2^{f_j}(k+1)}{U_2^{f_j}(k)} \right| : \lim_{k\to\infty} \left| U_2^{f_j}(k) \right| \neq 0 \right\},$$

and $\rho_2 F^{f_j}(\lambda_2) = \lim_{k\to\infty} \left[e^{\lambda_2 k} U_2^{f_j}(k) \right]$, $j = 1, 2, 3, \ldots$ Recursively, one can determine the pairs $\{\lambda_3, \rho_3 F^{f_j}(\lambda_3)\}, \{\lambda_4, \rho_4 F^{f_j}(\lambda_4)\}, \ldots$, if they exist. Define

$$G^{f_j}(k) = u^{f_j}(0,k) - \sum_{n=1} \rho_n F^{f_j}(\lambda_n) e^{-\lambda_n k}$$

and go to Step 6 below.

(6)

$$F^{f_j}(\lambda)\rho'(\lambda) = \lim_{n\to\infty} n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{nkt} G^{f_j}(nk), \qquad \lambda > 0, \quad j = 1, 2, \cdots$$

Go to Step 7 below.

(7)

$$\rho'(\lambda) = \lim_{x\to 0+} \sum_{j=1}^{\infty} F^{f_j}(\lambda)\rho'(\lambda) f_j(x), \quad \lambda > 0.$$

$$\rho(\lambda) = H(\lambda) \int_0^\lambda \rho'(t)dt + \sum_{n=1} \rho_n H(\lambda - \lambda_n), \quad -\infty < \lambda < \infty.$$

Go to Step 8 below.

(8) Define

$$T(x) = \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda}) d\left(\rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda_+}\right),$$

and

$$L(x,y) = \frac{1}{2}T(x+y) + \frac{1}{2}T(x-y).$$

Solve the Fredholm integral equation

$$L(x,y) + K(x,y) + \int_0^x K(x,s)L(s,y)ds = 0, \qquad 0 \le y \le x,$$

to find $K(x,y)$.

$$q(x) = 2\frac{d}{dx}K(x,x), \quad h = K(0,0).$$

For $b < \infty$ solving the initial value problem

$$\varphi_1''(x) - q(x)\varphi_1(x) = \mu_1\varphi_1(x),$$

$$\varphi_1(0) = 1, \quad \varphi_1'(0) = h,$$

to find

$$H = -\frac{\varphi_1'(b)}{\varphi_1(b)}.$$

5. Determination of the length b by one measurement

Suppose that we are interested only in the question of whether the half-buried rod is finite or infinite, and if it is finite, then what is the length of the rod? Then, we need only one measurement $\{u^{f_1}(0,k)\}_{k>0}$ for a suitable initial temperature f_1. In fact, from (i) Section 3, we have that

$$\lim_{k\to\infty} \frac{\ln|u^{f_1}(0,k)|}{k} < 0 \quad \Rightarrow \quad b < \infty,$$

$$\lim_{k\to\infty} \frac{\ln|u^{f_1}(0,k)|}{k} \ge 0 \quad \Rightarrow \quad b = \infty.$$

In [4, 6], it was shown that for a finite rod, with the initial temperature $f_1(x) = x^\alpha, -\frac{1}{2} < \alpha < 0$, the Fourier coefficients $c_n^{f_1} \neq 0$ for all n large enough. Then, by (ii) Section 3, one can find the eigenvalues μ_n for those $c_n^{f_1}$. Number the eigenvalues in an increasing order as $\nu_1, \nu_2, \ldots$. Since only a finite number of eigenvalues were missing, $\nu_n = \mu_{n-K}$ for large n and fixed K (K is actually the number of missing eigenvalues). The length b can then be found by the formula

$$b = \lim_{n\to\infty} \frac{(n+K)\pi}{\mu_{n+K}} = \lim_{n\to\infty} \frac{n\pi}{\nu_n}.$$

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Department of Mathematics
University of West Georgia, GA 30118, USA

E-mail address: vu@westga.edu

Department of Mathematics
Kuwait University, Kuwait

E-mail address: musallam@sci.kuniv.edu.kw