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REGULARIZATION OF A BACKWARD HEAT TRANSFER PROBLEM WITH A NONLINEAR SOURCE

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Dedicated to Professor Tran Duc Van on the occasion of his sixtieth birthday

Abstract. We consider the problem of finding, from the final data $u(x,T)$, the function u satisfying

$$u_t - u_{xx} = f(x,t,u(x,t),u_x(x,t)), \quad (x,t) \in \mathbb{R} \times (0,T).$$

The problem is ill-posed and we shall use the Fourier transform to get a nonlinear integral equation in the frequency space. By truncating high frequencies, we give a regularized solution. Error estimates are given.

1. Introduction

Let T be a positive number, we consider the problem of finding a solution $u(x,t), (x,t) \in \mathbb{R} \times [0,T]$ of the system

$$\begin{cases} u_t - u_{xx} = f(x,t,u(x,t),u_x(x,t)), \quad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,T) = \varphi(x), \end{cases} \tag{1.1}$$

where $\varphi(x), f(x,t,y,z)$ are given. The problem is called the nonlinear backward heat problem included the first-order derivative. From now on, we shall denote

$$F_{u,v}(x,t) := f(x,t,u(x,t),v(x,t)).$$

Using the Fourier transform, we can rewrite the above system in the following form

$$\widehat{u}(p,t) = e^{(T-t)p^2}\widehat{\varphi}(p) - \int_t^T e^{(s-t)p^2}\widehat{F_{u,u_x}}(p,s)\,ds, \tag{1.2}$$

where

$$\widehat{g}(p,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\xi,t)e^{-ip\xi}\,d\xi.$$

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As known, the problem is severely ill-posed. The solution does not always exist and in the case of existence, the solution can be non-unique. Moreover, in the case of existence and uniqueness, it may not depend continuously on the given data. Hence a regularization is in order. For forty years, many authors studied the linear backward problems. Lattes-Lions [6], Miller [7], and Trong-Tuan [12] studied a regularization method called the quasi-reversibility (QR for short) method by perturbing the main equation. Clark and Oppenheimer [3] gave another regularization method by perturbing the final value (quasi-boundary value (QBV) method). Recently, the problem was also studied in [4, 5, 10].

In the last nine years, we can find a few papers concerning the nonlinear backward heat transfer problem. In [1, 2], the authors gave a result for the structural stability for the Ginzburg-Landau equation. Quan and Dung, in [8], studied a regularization method by transforming the problem into the one of minimizing an appropriate functional. In [9], the authors used the Fourier transform to get an integral equation in the frequency space. By perturbing directly the integral equation, they constructed a regularization method. In [11], the authors mixed two methods QR and QBV to regularize the problem. And recently, in [14], the authors used the method of truncated Fourier series to regularize the problem. However, we did not find any papers dealing with the nonlinear problem included the first-order derivative u_x.

Noting that, in (1.2), the "bad" factors are

$$e^{(T-t)p^2},\ e^{(s-t)p^2},\quad 0<t<s<T.$$

Since $e^{(s-t)p^2}\to+\infty$ very fast when $p\to\infty$, the solution is unstable. To regularize the problem, we have to replace the factors by some appropriate ones. In fact, we can truncate high frequencies $|p|>c_\epsilon$ where $\lim_{\epsilon\to 0}c_\epsilon=\infty$. Letting $\alpha>0,\ 0<\epsilon<1$, in the present paper, we choose

$$c_\epsilon=\sqrt{\alpha\ln\left(\frac{1}{\epsilon}\right)}. \tag{1.3}$$

We put

$$A_\epsilon=[-c_\epsilon,c_\epsilon] \tag{1.4}$$

and

$$\chi_{A_\epsilon}(p)=\begin{cases}1 & if\quad p\in A_\epsilon,\\ 0 & if\quad p\notin A_\epsilon.\end{cases}$$

We shall approximate problem (1.2) by the following

Problem $\mathbf{P}_\varphi$: For $\varphi\in L^2(\mathbf{R})$, find $u^\epsilon\in C([0,T];H^1(\mathbf{R}))$ satisfying

$$\widehat{u^\epsilon}(p,t)=\chi_{A_\epsilon}(p)e^{(T-t)p^2}\widehat{\varphi}(p)-\chi_{A_\epsilon}(p)\int_t^T e^{(s-t)p^2}\widehat{F_{u^\epsilon,u_x^\epsilon}}(p,s)\,ds \tag{1.5}$$

or

$$u^\epsilon(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(T-t)p^2} \widehat{\varphi}(p) e^{ipx} \chi_{A_\epsilon}(p)\, dp$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_t^T e^{(s-t)p^2} \widehat{F_{u^\epsilon, u^\epsilon_x}}(p,s) e^{ipx} \chi_{A_\epsilon}(p)\, ds\, dp. \tag{1.6}$$

From now on, we shall denote by $\| \, . \, \|$ the norm of $L^2(\mathbf{R})$; $| \, . \, |_1$ the norm of $H^1(\mathbf{R})$ and $||| \, . \, |||$ the sup-norm of $C([0,T], H^1(\mathbf{R}))$.

The remains of our paper are divided into three sections. Section 2 gives some preliminary results. In Section 3, we investigate the well-posedness of Problem $(\mathbf{P}_\varphi)$. In Section 4, we give two regularization results in the exact and non-exact data cases.

2. Preliminary results

We first find some conditions of f such that (1.5) is defined. The integral in the right hand side of (1.5) is well-defined if F_{u,u_x} is in $L^\infty(0,T; L^2(\mathbf{R}))$. In fact, we have

Lemma 2.1. *Let $k > 0$, let $f : \mathbf{R} \times [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying*

$$|f(x,y,v,w) - f(x,y,v',w')| \le k(|v-v'| + |w-w'|),$$

where $x, v, w, v', w' \in \mathbf{R}$, $y \in [0,T]$.

If $F_{0,0} \in C([0,T], L^2(\mathbf{R}))$, $V, W \in C([0,T], L^2(\mathbf{R}))$, then

$$F_{V,W} \in L^\infty((0,T), L^2(\mathbf{R})).$$

Moreover, for $V, V_1 \in C([0,T], H^1(\mathbf{R}))$, $0 \le t \le T$, one has

$$\|\widehat{F_{V,V_x}}(.,t) - \widehat{F_{V_1,V_{1x}}}(.,t)\|^2 \le 2k^2 |V(.,t) - V_1(.,t)|_1^2.$$

Proof. For every $0 \le t \le T$, one has

$$|F_{V,W}(x,t) - F_{0,0}(x,t)| \le k(|V(x,t) - 0| + |W(x,t) - 0|).$$

It follows that

$$\begin{aligned} \|F_{V,W}(.,t)\| &\le \|F_{0,0}(.,t)\| + k\|V(.,t)\| + k\|W(.,t)\| \\ &\le \sup_{0\le t\le T} \|F_{0,0}(.,t)\| + k \sup_{0\le t\le T} \|V(.,t)\| + k \sup_{0\le t\le T} \|W(.,t)\|. \end{aligned}$$

Hence $F_{V,W} \in L^\infty((0,T), L^2(\mathbf{R}))$. The last inequality of Lemma 2.1 can be proved by the Plancherel theorem

$$\begin{aligned}
&\|\widehat{F_{V,V_x}}(.,t) - \widehat{F_{V_1,V_{1x}}}(.,t)\|^2 \\
=&\|F_{V,V_x}(.,t) - F_{V_1,V_{1x}}(.,t)\|^2 \\
\le&k^2 \int_{-\infty}^{\infty} \left(|V(x,t) - V_1(x,t)| + |V_x(x,t) - V_{1x}(x,t)|\right)^2 dx \\
\le&2k^2 \left(\|V(.,t) - V_1(.,t)\|^2 + \|V_x(.,t) - V_{1x}(.,t)\|^2\right).
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Now, we give some estimates used in next sections. Putting

$$b_\epsilon = 1 + \alpha \ln\left(\frac{1}{\epsilon}\right), \tag{2.1}$$

we get

Lemma 2.2. *Let $0 < \epsilon < 1$, $\alpha > 0$ and let $0 \le t \le s \le T$. We have*

$$e^{(s-t)p^2}\chi_{A_\epsilon}(p) \le \epsilon^{(t-s)\alpha}$$

and

$$\sqrt{1+p^2}e^{(s-t)p^2}\chi_{A_\epsilon}(p) \le \sqrt{b_\epsilon}\epsilon^{(t-s)\alpha}.$$

Proof. We have

$$e^{(s-t)p^2}\chi_{A_\epsilon}(p) \le e^{(s-t)\alpha\ln\left(\frac{1}{\epsilon}\right)} = \epsilon^{(t-s)\alpha}.$$

Similarly, we have the second inequality. This completes the proof of Lemma 2.2. □

3. The well-posedness of problem (P_φ)

Now, we investigate the well-posedness of Problem (P_φ). The functions as in (1.5) are often called the band-limited ones. In the pioneering paper [15], Zimmerman studied a class of nonlinear PDE in the space of band-limited functions. Under the assumption

$$|f(u,w)| \le A_\alpha u^2 + A_\beta w^2,$$

he studied the local existence and the stability of the mentioned problem. In the present paper, we have a slightly different condition

$$|f(x,t,u,w)| \le |f(x,t,0,0)| + k(|u| + |w|).$$

However, the Zimmerman method can be applied to prove the global existence result for our problem. In fact, we have

Theorem 3.1. *Let $0 < \epsilon < 1$, $\varphi \in L^2(\mathbf{R})$ and let f be as in Lemma 2.1. Then Problem (P_φ) has a unique solution $u^\epsilon \in C([0,T]; H^1(\mathbf{R}))$.*

Proof. For $w \in C([0,T];H^1(\mathbf{R}))$, we put

$$Q(w)(x,t) = \frac{1}{\sqrt{2\pi}}\psi(x,t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_t^T e^{(s-t)p^2}\widehat{F_{w,w_x}}(p,s)e^{ipx}\chi_{A_\epsilon}(p)\,ds\,dp,$$

where

$$\psi(x,t) = \int_{-\infty}^{+\infty} e^{(T-t)p^2}\widehat{\varphi}(p)\chi_{A_\epsilon}(p)e^{ipx}dp.$$

We first prove that $Q(w) \in C([0,T];H^1(\mathbf{R}))$. In fact, one has

$$\widehat{Q(w)}(p,t) = \chi_{A_\epsilon}(p)e^{(T-t)p^2}\widehat{\varphi}(p) - \chi_{A_\epsilon}(p)\int_t^T e^{(s-t)p^2}\widehat{F_{w,w_x}}(p,s)\,ds.$$

Using Lemma 2.1, we can verify directly that $\widehat{Q(w)}(p,t); p\widehat{Q(w)}(p,t)$ are in $C([0,T];L^2(\mathbf{R}))$. Hence, the Plancherel theorem gives that $Q(w)$ is in $C([0,T];H^1(\mathbf{R}))$ for every $w \in C([0,T];H^1(\mathbf{R}))$.

For every $w, v \in C([0,T];H^1(\mathbf{R}))$, using the Zimmerman method, we shall get after some direct estimates

$$(3.1) \qquad |Q^m(v)(.,t) - Q^m(w)(.,t)|_1^2 \le \frac{2^m(T-t)^{2m}k^{2m}a_\epsilon^m}{(2m-1)!!}|||v-w|||^2,$$

where $(2m-1)!! = 1.3\ldots(2m-1)$ and $a_\epsilon = b_\epsilon\epsilon^{-2T\alpha}$.

Since $\lim_{m\to\infty} T^m k^m\sqrt{\frac{2^m a_\epsilon^m}{(2m-1)!!}} = 0$, there exists a positive integer m_0 such that Q^{m_0} is a contraction in $C([0,T];H^1(\mathbf{R}))$. It follows that the equation $Q^{m_0}(w) = w$ has a unique solution $U \in C([0,T];H^1(\mathbf{R}))$. We prove that $Q(U) = U$. In fact, one has $Q(Q^{m_0})(U) = Q(U)$. Hence $Q^{m_0}(Q(U)) = Q(U)$. By the uniqueness of the fixed point of Q^{m_0}, one has $Q(U) = U$. This completes the proof of Theorem 3.1. □

To get a stability result for the solution of problem (P_φ), we consider

Theorem 3.2. *Let $0 < \epsilon < 1$, $\varphi, g \in L^2(\mathbf{R})$ and let f be as in Lemma 2.1. If $u,\ v \in C([0,T],H^1(\mathbf{R}))$ are solutions of Problem (P_φ), (P_g) respectively, then*

$$|||u-v||| \le \sqrt{2b_\epsilon}e^{2k^2T^2}\epsilon^{-\alpha T(1+2k^2T)}||\varphi - g||.$$

Proof. From (1.5) and (1.6), we have

$$\begin{aligned} |u(.,t)-v(.,t)|_1^2 &= ||u(.,t)-v(.,t)||^2 + ||u_x(.,t)-v_x(.,t)||^2 \\ &= ||\widehat{u}(.,t)-\widehat{v}(.,t)||^2 + ||\widehat{u}_x(.,t)-\widehat{v}_x(.,t)||^2 \\ &\le K_1 + K_2, \end{aligned}$$

where

$$\begin{aligned} K_1 &= 2\int_{-\infty}^{+\infty}(1+p^2)|e^{(T-t)p^2}\chi_{A_\epsilon}(p)(\widehat{\varphi}(p)-\widehat{g}(p))|^2\,dp, \\ K_2 &= 2\int_{-\infty}^{+\infty}(1+p^2)\left|\int_t^T e^{(s-t)p^2}\chi_{A_\epsilon}(p)\left(\widehat{F_{u,u_x}}(p,s)-\widehat{F_{v,v_x}}(p,s)\right)\,ds\right|^2\,dp. \end{aligned}$$

We estimate K_1. Lemma 2.2 gives

$$\begin{aligned} K_1 &\leq 2b_\epsilon\epsilon^{2(t-T)\alpha}\int_{-\infty}^{+\infty}|\widehat{\varphi}(p)-\widehat{g}(p)|^2\,dp \\ &\leq 2b_\epsilon\epsilon^{2(t-T)\alpha}||\varphi-g||^2. \end{aligned}$$

We estimate K_2. From Lemma 2.2, we have

$$\begin{aligned} K_2 &= 2\int_{-\infty}^{+\infty}\left|\int_t^T\sqrt{1+p^2}e^{(s-t)p^2}\chi_{A_\epsilon}(p)\left(\widehat{F_{u,u_x}}(p,s)-\widehat{F_{v,v_x}}(p,s)\right)\,ds\right|^2\,dp \\ &\leq 2b_\epsilon\epsilon^{2t\alpha}\int_{-\infty}^{+\infty}\left|\int_t^T\epsilon^{-s\alpha}\left(\widehat{F_{u,u_x}}(p,s)-\widehat{F_{v,v_x}}(p,s)\right)\,ds\right|^2\,dp \\ &\leq 2(T-t)b_\epsilon\epsilon^{2t\alpha}\int_{-\infty}^{+\infty}\int_t^T\epsilon^{-2s\alpha}\left|\widehat{F_{u,u_x}}(p,s)-\widehat{F_{v,v_x}}(p,s)\right|^2\,ds\,dp. \end{aligned}$$

Lemma 2.1 gives

$$\begin{aligned} K_2 &\leq 2(T-t)b_\epsilon\epsilon^{2t\alpha}\int_t^T\epsilon^{-2s\alpha}||\widehat{F_{u,u_x}}(.,s)-\widehat{F_{v,v_x}}(.,s)||^2\,ds \\ &\leq 4k^2(T-t)b_\epsilon\epsilon^{2t\alpha}\int_t^T\epsilon^{-2s\alpha}|u(.,s)-v(.,s)|_1^2\,ds. \end{aligned}$$

So, we have

$$\begin{aligned} \epsilon^{-2t\alpha}|u(.,t)-v(.,t)|_1^2 &\leq 2b_\epsilon\epsilon^{-2T\alpha}||\varphi-g||^2 \\ &\quad +4k^2Tb_\epsilon\int_t^T\epsilon^{-2s\alpha}|u(.,s)-v(.,s)|_1^2\,ds. \end{aligned}$$

Using the Gronwall inequality, we have

$$\begin{aligned} |u(.,) - v(.,t)|_1 &\leq \sqrt{2b_\epsilon}\epsilon^{(t-T)\alpha} \exp\left(2b_\epsilon k^2 T(T-t)\right) ||\varphi - g|| \\ &= \sqrt{2b_\epsilon} e^{2k^2T(T-t)} \epsilon^{-\alpha(T-t)(1+2k^2T)} ||\varphi - g|| \\ &\leq \sqrt{2b_\epsilon} e^{2k^2T^2} \epsilon^{-\alpha T(1+2k^2T)} ||\varphi - g||. \end{aligned}$$

It follows that

$$|||u - v||| \leq \sqrt{2b_\epsilon} e^{2k^2T^2} \epsilon^{-\alpha T(1+2k^2T)} ||\varphi - g||.$$

This completes the proof of Theorem 3.2. □

4. Regularization and error estimates

In this section, we shall state and prove some regularization results under many preassumed conditions on the exact solution u of problem (1.2). We first have

Theorem 4.1. *Let $\beta \geq 0$, let φ, f be as in Theorem 3.1. Assume that problem* (1.2) *has a solution*

$$u \in C([0,T]; H^1(\mathbf{R}))$$

satisfying

$$A := \sup_{0 \leq t \leq T} \left\{ \int_{-\infty}^{+\infty} (1+p^2) e^{2(\beta+t)p^2} |\widehat{u}(p,t)|^2 \, dp \right\} < +\infty. \tag{4.1}$$

Then, for every $t \in [0,T]$, we have

$$|u(.,t) - u^\epsilon(.,t)|_1 \leq \sqrt{A} \exp(k^2T(T-t)) \epsilon^{\alpha(\beta+t-k^2T(T-t))},$$

where we denote by u^ϵ the unique solution of Problem (P_φ).

Remarks. 1. If $\beta = 0$ in (4.1), $f \equiv 0$ and if we have the preassumption $u(\cdot,0) \in H^1(\mathbf{R})$, then (4.1) holds. In fact, in this case, we have $e^{tp^2}\widehat{u}(p,t) = \widehat{u}(p,0)$. Since $u(x,0)$ is in $H^1(\mathbf{R})$ one has

$$\int_{-\infty}^{+\infty} (1+p^2) e^{2tp^2} |\widehat{u}(p,t)|^2 \, dp = ||\sqrt{1+p^2}\widehat{u}(.,0)||^2 = |u(.,0)|_1^2.$$

Hence the condition (4.1) is reasonable.

2. If $\beta > k^2T^2$, then $\lim_{\epsilon \to 0} |u(.,0) - u^\epsilon(.,0)|_1 = 0$.

3. If $\beta = 0$, then $u^\epsilon(x,t)$ is a good approximation of $u(x,t)$ when $t - k^2T(T - t) > 0$, i.e. $\frac{k^2T^2}{1+k^2T} < t \leq T$.

4. If $f = f(x,t,u)$ does not depend on u_x, using the technique of the proof of Theorem 4.1 (but easier), we can prove that

$$||u(.,0) - u^\epsilon(.,0)|| \leq M\epsilon^{\alpha(\beta+t)}.$$

Proof of Theorem 4.1. We have

$$\begin{aligned} |u(.,t)-u^\epsilon(.,t)|_1^2 &= ||u(.,t)-u^\epsilon(.,t)||^2 + ||u_x(.,t)-u_x^\epsilon(.,t)||^2 \\ &= ||\widehat{u}(.,t)-\widehat{u^\epsilon}(.,t)||^2 + ||\widehat{u}_x(.,t)-\widehat{u^\epsilon}_x(.,t)||^2 \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} (1+p^2)\left|(1-\chi_{A_\epsilon}(p))\widehat{u}(p,t)\right|^2 dp, \\ I_2 &= \int_{-\infty}^{+\infty} (1+p^2)\left|\int_t^T e^{(s-t)p^2}\chi_{A_\epsilon}(p)\left(\widehat{F_{u^\epsilon,u_x^\epsilon}}(p,s)-\widehat{F_{u,u_x}}(p,s)\right) ds\right|^2 dp. \end{aligned}$$

We estimate I_1. We have

$$\begin{aligned} I_1 &= \int_{|p|\geq c_\epsilon} (1+p^2)|\widehat{u}(p,t)|^2\, dp \\ &= \int_{|p|\geq c_\epsilon} e^{-2(\beta+t)p^2}(1+p^2)e^{2(\beta+t)p^2}|\widehat{u}(p,t)|^2\, dp, \end{aligned}$$

where we recall that c_ϵ is defined in (1.3). For $|p|>c_\epsilon$, we have

$$e^{-2(\beta+t)p^2} \leq \exp\left(-2(\beta+t)c_\epsilon^2\right) = \epsilon^{2\alpha(\beta+t)}.$$

It follows that

$$\begin{aligned} I_1 &\leq \epsilon^{2\alpha(\beta+t)}\int_{-\infty}^{+\infty}(1+p^2)e^{2(\beta+t)p^2}|\widehat{u}(p,t)|^2\, dp \\ &\leq \epsilon^{2\alpha(\beta+t)}A. \end{aligned}$$

We estimate I_2. We first note that

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty}(1+p^2)\left|\int_t^T e^{(s-t)p^2}\chi_{A_\epsilon}(p)\left(\widehat{F_{u^\epsilon,u_x^\epsilon}}(p,s)-\widehat{F_{u,u_x}}(p,s)\right) ds\right|^2 dp \\ &= \int_{-\infty}^{+\infty}\left|\int_t^T \sqrt{1+p^2}e^{(s-t)p^2}\chi_{A_\epsilon}(p)\left(\widehat{F_{u^\epsilon,u_x^\epsilon}}(p,s)-\widehat{F_{u,u_x}}(p,s)\right) ds\right|^2 dp. \end{aligned}$$

Using Lemma 2.2, we have

$$\begin{aligned} I_2 &\leq b_\epsilon \epsilon^{2t\alpha} \int_{-\infty}^{+\infty} \left| \int_t^T \epsilon^{-s\alpha} \left(\widehat{F_{u^\epsilon, u_x^\epsilon}}(p,s) - \widehat{F_{u,u_x}}(p,s) \right) ds \right|^2 dp \\ &\leq (T-t) b_\epsilon \epsilon^{2t\alpha} \int_{-\infty}^{+\infty} \int_t^T \epsilon^{-2s\alpha} \left| \widehat{F_{u^\epsilon, u_x^\epsilon}}(p,s) - \widehat{F_{u,u_x}}(p,s) \right|^2 ds\, dp \\ &\leq (T-t) b_\epsilon \epsilon^{2t\alpha} \int_t^T \epsilon^{-2s\alpha} ||\widehat{F_{u^\epsilon, u_x^\epsilon}}(.,s) - \widehat{F_{u,u_x}}(.,s)||^2\, ds. \end{aligned}$$

Lemma 2.1 gives

$$I_2 \leq 2k^2 (T-t) b_\epsilon \epsilon^{2t\alpha} \int_t^T \epsilon^{-2s\alpha} |u^\epsilon(.,s)) - u(.,s)|_1^2\, ds.$$

It follows that

$$\epsilon^{-2t\alpha} |u^\epsilon(.,t)) - u(.,t)|_1^2 \leq \epsilon^{2\alpha\beta} A + 2k^2 T b_\epsilon \int_t^T \epsilon^{-2s\alpha} |u^\epsilon(.,s)) - u(.,s)|_1^2\, ds.$$

Using the Gronwall inequality, we have

$$|u^\epsilon(.,t)) - u(.,t)|_1^2 \leq A\epsilon^{2\alpha(\beta+t)} \exp\left(2b_\epsilon k^2 T(T-t)\right).$$

On the other hand,

$$\begin{aligned} \exp\left(2b_\epsilon k^2 T(T-t)\right) &= \exp\left(\left(2 + 2\alpha \ln\left(\frac{1}{\epsilon}\right)\right) k^2 T(T-t) \right) \\ &= e^{2k^2T(T-t)} \epsilon^{-2\alpha k^2 T(T-t)}. \end{aligned}$$

This completes the proof of Theorem 4.1. □

In the case of non-exact data, one has

Theorem 4.2. *Let φ, f be as in Theorem 3.1 and let $\beta > k^2T^2$. Assume that problem* (1.2) *has a solution*

$$u \in C([0,T]; H^1(\mathbf{R}))$$

satisfying

$$A := \sup_{0\leq t\leq T} \left\{ \int_{-\infty}^{+\infty} (1+p^2) e^{(\beta+t)p^2} |\widehat{u}(p,t)|^2\, dp \right\} < +\infty.$$

Let $\delta \in (0,1)$ and let $\varphi_\delta \in L^2(\mathbf{R})$ be a measured data such that

$$||\varphi_\delta - \varphi|| \leq \delta.$$

Then from φ_δ, we can construct a function $z^\delta \in C([0,T];H^1(\mathbf{R}))$ satisfying

$$|z^\delta(.,t) - u(.,t)|_1 \le \left(\sqrt{A}e^{k^2T^2} + \sqrt{2(1+\mu)\ln\left(\frac{1}{\delta}\right)}e^{2k^2T^2}\right)\delta^\nu \tag{4.2}$$

for every $t \in [0,T]$, where

$$\mu = \alpha^2(\beta + T + k^2T^2),\ \nu = \frac{\beta - k^2T^2}{\beta + T + k^2T^2}.$$

Proof. Let u^ϵ be the solution of Problem (P_φ) and let $u^{\epsilon,\delta}$ be the solution of problem (P_{φ_δ}).

From Theorem 4.1, we have

$$|u(.,t) - u^\epsilon(.,t)|_1 \le \sqrt{A}e^{k^2T^2}\epsilon^{\alpha(\beta - k^2T^2)} \tag{4.3}$$

for every $t \in [0,T]$.

From Theorem 3.2, we have

$$\begin{aligned} |u^\epsilon(.,t) - u^{\epsilon,\delta}(.,t)|_1 &\le \sqrt{2b_\epsilon}e^{2k^2T^2}\epsilon^{-\alpha(T+2k^2T^2)}||\varphi - \varphi_\delta|| \\ &\le \delta\sqrt{2b_\epsilon}e^{2k^2T^2}\epsilon^{-\alpha(T+2k^2T^2)} \end{aligned} \tag{4.4}$$

where b_ϵ is defined in (2.1). So we have

$$|u(.,t) - u^{\epsilon,\delta}(.,t)|_1 \le \sqrt{A}e^{k^2T^2}\epsilon^{\alpha(\beta-k^2T^2)} + \delta\sqrt{2b_\epsilon}e^{2k^2T^2}\epsilon^{-\alpha(T+2k^2T^2)}.$$

Choosing

$$\epsilon = \epsilon(\delta) = \delta^{\frac{1}{\alpha(\beta+T+k^2T^2)}},$$

we get

$$|u(.,t) - u^{\epsilon(\delta),\delta}(.,t)|_1 \le \left(\sqrt{A}e^{k^2T^2} + \sqrt{2(1+\mu)\ln\left(\frac{1}{\delta}\right)}e^{2k^2T^2}\right)\delta^\nu.$$

Put $z^\delta(x,t) = u^{\epsilon(\delta),\delta}(x,t)$, for every $x \in \mathbf{R}$, $t \in [0,T]$. Then from (4.3) and (4.4), we have the inequality (4.2).

This completes the proof of the theorem. □

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