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COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS

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Abstract. In this paper, we establish some common fixed point theorems for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a metric space. Our results unify and extend many results existing in the literature including those of Aliouche [3], Bouhadjera [4] and some results of Popa [19]-[23]. Also, we establish another common fixed point theorem for four occasionally weakly compatible single and set-valued maps of Greguš type which improves the results of Djoudi and Nisse [5], Pathak et al. [17] and others and we end our work by giving another theorem which generalizes the results given by Elamrani and Mehdaoui [6], Mbarki [14] and references therein.

1. Introduction and preliminaries

Throughout this paper, $(\mathcal{X}, d)$ denotes a metric space and $\mathcal{P}_{fb}(\mathcal{X})$ the class of all nonempty bounded closed subsets of $\mathcal{X}$. We recall these usual notations: for $x \in \mathcal{X}$ and $A \subseteq \mathcal{X}$,

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let H be the associated Hausdorff metric on $\mathcal{P}_{fb}(\mathcal{X})$: for every A and every B in $\mathcal{P}_{fb}(\mathcal{X})$,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\}$$

and

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

For simplicity, we write $\delta(a, B)$ in place of $\delta(\{a\}, B)$; as well as $\delta(A, b)$ in place of $\delta(A, \{b\})$.

In the following, we use small letters: f, g, ... to denote maps from $\mathcal{X}$ to $\mathcal{X}$ and capital letters: F, G, ... for set-valued maps; that is, maps from $\mathcal{X}$ to $\mathcal{P}_{fb}(\mathcal{X})$ and we write fx for $f(x)$ and Fx for $F(x)$.

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The concepts of weak commutativity, compatibility, noncompatibility and weak compatibility were frequently used to prove existence theorems on fixed and common fixed points for single and set-valued maps satisfying certain conditions in different spaces. The study of common fixed points on occasionally weakly compatible maps is new and also interesting. This notion, which is defined by Al-Thagafi and Shahzad [2] and which is published in 2008, has been used by Jungck and Rhoades [12] in 2006 and by Abbas and Rhoades [1] in 2007.

We begin by a short history of these different notions. Generalizing the concept of commuting maps, Sessa [26] introduced the concept of weakly commuting maps. f and g are weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx)$$

for all $x \in \mathcal{X}$, where f and g are two self-maps of $(\mathcal{X}, d)$.

In 1986, Jungck [8] made more generalized commuting and weakly commuting maps called compatible maps. f and g are said to be compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \tag{1.1}$$

whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$. This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible (see [8]).

Further, the same author with Murthy and Cho [10] gave another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A). f and g are said to be compatible of type (A) if in place of (1.1) we have the two equalities

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0.$$

Obviously, weakly commuting maps are compatible of type (A). From [10], it follows that the implication is not reversible.

In their paper [16], Pathak and Khan extended type (A) maps by introducing the concept of compatible maps of type (B) and compared these maps with compatible and compatible maps of type (A) in normed spaces. To be compatible of type (B), f and g above have to satisfy, instead of condition (1.1), the inequalities

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right].$$

It is clear that compatible maps of type (A) are compatible of type (B). The converse is not true ([16]).

In 1998, Pathak et al. [17] introduced an extension of compatibility of type (A) by giving the notion of compatible maps of type (C). f and g are compatible of type (C) if they satisfy the two inequalities

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)\right]$$
$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)\right].$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A) (resp., type (B)).

In [15], the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A). f and g are compatible of type (P) if instead of (1.1) we have

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0.$$

Note that compatibility, compatibility of type (A) (resp. (B), (C) and (P)) are equivalent if f and g are continuous.

Afterwards, Jungck [9] generalized the compatibility, the compatibility of type (A), (B), (C) and (P) by introducing the concept of weak compatibility. He defines f and g to be weakly compatible if $ft = gt$, $t \in \mathcal{X}$ implies $fgt = gft$.

It is known that all of the above compatibility notions imply weakly compatible notion, however, there exist weakly compatible maps which are neither compatible nor compatible of type (A), (B), (C) and (P) (see [3]).

Recently, in a paper submitted before 2006 but published only in 2008, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps. Two self-maps f and g of $\mathcal{X}$ are called occasionally weakly compatible maps (shortly owc) if there is a point x in $\mathcal{X}$ such that $fx = gx$ at which f and g commute. This notion is used in 2006 by Jungck and Rhoades [12] to prove some common fixed point theorems in symmetric spaces.

In their paper [13], Kaneko and Sessa extended the compatibility to the setting of single and set-valued maps as follows: $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ are said to be compatible if $fFx \in \mathcal{P}_{fb}(\mathcal{X})$ for all $x \in \mathcal{X}$ and

$$\lim_{n\to\infty} H(Ffx_n, fFx_n) = 0$$

whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{X}$ such that $fx_n \to t$, $Fx_n \to A \in \mathcal{P}_{fb}(\mathcal{X})$ and $t \in A$.

After, in [11] Jungck and Rhoades extend the concept of compatible single and set-valued maps by giving the concept of weak compatibility. Maps f and F are weakly compatible if they commute at their coincidence points; i.e., if $fFx = Ffx$ whenever $fx \in Fx$.

More recently, Abbas and Rhoades [1] extended the definition of owc maps to the setting of set-valued maps and they proved some common fixed point theorems satisfying generalized contractive condition of integral type. f and F are said to be owc if and only if there exists some point x in $\mathcal{X}$ such that $fx \in Fx$ and $fFx \subseteq Ffx$. Clearly, weakly compatible maps are occasionally weakly compatible. However, the converse is not true in general. The example below illustrates this fact.

Example 1.1. Let $\mathcal{X} = [1, \infty[$ with the usual metric. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ by, for all $x \in \mathcal{X}$,

$$fx = 2x + 1, \; Fx = [1, 2x + 1].$$

$$fx = 2x + 1 \in Fx \text{ and } fFx = [3, 4x + 3] \subset Ffx = [1, 4x + 3].$$

Hence, f and F are occasionally weakly compatible but non weakly compatible.

2. General fixed point theorems

In this section, before giving our first main result, we recall this definition.

Definition 2.1. Let $F : \mathcal{X} \to 2^{\mathcal{X}}$ be a set-valued map on $\mathcal{X}$. $x \in \mathcal{X}$ is a fixed point of F if $x \in Fx$.

Theorem 2.2. *Let f, $g : \mathcal{X} \to \mathcal{X}$ be maps and F, $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Let $\varphi : (\mathbb{R}^+)^5 \to \mathbb{R}$ be a real map satisfying the following conditions:*
(φ_1) : *φ is nonincreasing in variables t_4 and t_5,*
(φ_2) : $\varphi(t, 0, 0, t, t) \geq 0 \; \forall t > 0$.
If, for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$,

$$\varphi(d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0 \tag{2.1}$$

then, f, g, F and G have a unique common fixed point.

Proof. i) We begin to show the existence of a common fixed point.

Since the pairs $\{f, F\}$ and $\{g, G\}$ are owc then, there exist u, v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$ and $gGv \subseteq Ggv$.

First, we show that $gv = fu$. Suppose that is not the case, then by (2.1), we have

$$\begin{aligned} &\varphi(d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ =&\varphi(d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0 \end{aligned}$$

and by (φ_1),

$$\varphi(d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from (φ_2) gives $d(fu, gv) = 0$. So $fu = gv$.

Next, we claim that $f^2u = fu$. If it is not, then condition (2.1) implies that

$$\begin{aligned} &\varphi(d(f^2u, gv), d(f^2u, Ffu), d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ =&\varphi(d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By (φ_1), we have

$$\varphi(d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from (φ_2), gives $d(f^2u, fu) = 0$. We have $f^2u = fu$.

Since (f, F) and (g, G) have the same role, we have $gv = g^2v$. Therefore, $ffu = fu = gv = ggv = gfu$, $fu = f^2u \in fFu \subset Ffu$, so $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is a common fixed point of f, g, F and G.

ii) Now, we show uniqueness of the common fixed point.

Put $fu = w$ and let w' be another common fixed point of the four maps such that $w \neq w'$, then, by (2.1), we get

$$\begin{aligned}&\varphi(d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw))\\ =&\varphi(d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0.\end{aligned}$$

By (φ_1), we get

$$\varphi(d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw')) < 0.$$

So, by (φ_2), $d(fw, gw') = 0$ and thus, $d(fw, gw') = d(w, w') = 0$. □

Now, in the following corollary, we give some examples of applications.

Corollary 2.3. *Let f, $g : \mathcal{X} \to \mathcal{X}$ be maps and F, $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. If we suppose one of the following inequalities is satisfied, for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$, then f, g, F and G have a unique common fixed point.*

(a) $$d(fx, gy) < k \max\{d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}$$

where $0 < k \leq 1$,

(b) $$\begin{aligned}d^p(fx, gy) &< \alpha d^{p-1}(fx, gy)d(fx, Fx) + \beta d^{p-1}(fx, gy)d(gy, Gy)\\ &+\gamma d(fx, Gy)d^{p-1}(gy, Fx) + \delta d(gy, Fx)d^{p-1}(fx, Gy)\end{aligned}$$

where $p \in \mathbb{N}^*$, $\gamma > 0$, $\delta > 0$ *and* $\gamma + \delta \leq 1$,

(c) $$d(fx, gy) < (\alpha d^p(fx, Fx) + \beta d^p(gy, Gy) + \gamma d^p(fx, Gy) + \delta d^p(gy, Fx))^{\frac{1}{p}}$$

where $p \in \mathbb{N}^*$, $\gamma > 0$, $\delta > 0$ *and* $\gamma + \delta \leq 1$,

(d) $$\begin{aligned}d^2(fx, gy) &< \alpha \max\{d^2(fx, Fx), d^2(gy, Gy)\}+\\ &+\beta \max\{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} + \gamma d(fx, Gy)d(gy, Fx)\end{aligned}$$

where $\beta > 0$, $\gamma > 0$ *and* $\alpha + \gamma \leq 1$,

(e) $$d(fx, gy) < c\frac{d(fx, Gy)d(gy, Fx)}{1 + d(fx, Fx) + d(gy, Gy)} \quad \text{where } c \in]0, \frac{1}{2}],$$

(f) $$d(fx, gy) < \alpha \max\{d(fx, Fx), d(gy, Gy)\} + \beta \max\{d(fx, Gy), d(gy, Fx)\}$$

where $\beta \in]0, 1]$.

Proof. For the proofs of (a), (b), (c), (d), (e) and (f), we use Theorem 2.2 with the following functions φ which satisfy, for every case, hypotheses (φ_1) and (φ_2).

For (a): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 - k \max\{t_2, t_3, t_4, t_5\}$ with $k \in]0, 1]$.
This function φ is that one of Example 1 of [25].

For (b): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1^p - \alpha t_1^{p-1} t_2 - \beta t_1^{p-1} t_3 - \gamma t_4 t_5^{p-1} - \delta t_5 t_4^{p-1}$
where p is an integer ≥ 2, $\gamma > 0$, $\delta > 0$ and $\gamma + \delta \leq 1$.

For (c): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 - (\alpha t_2^p + \beta t_3^p + \gamma t_4^p + \delta t_5^p)^{\frac{1}{p}}$,
where $p \in \mathbb{N}^*$, $\gamma > 0$, $\delta > 0$ and $\gamma + \delta \leq 1$.
For $p = 1$, examples (b) and (c) coincide and are used by many authors.

For (d): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1^2 - \alpha \max\{t_2^2, t_3^2\} - \beta \max\{t_2 t_4, t_3 t_5\} - \gamma t_4 t_5$ where $\beta > 0$, $\gamma > 0$ and $\alpha + \gamma \leq 1$.

For (e): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 - c\frac{t_4 + t_5}{1 + t_2 + t_3}$, where $c \in]0, \frac{1}{2}]$.

For (f): $\varphi(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha \max\{t_2, t_3\} - \beta \max\{t_4, t_5\}$ with $\beta \in]0, 1]$. □

Now, we can give two variants of Theorem 2.2:

Theorem 2.4. *Let f, $g : \mathcal{X} \to \mathcal{X}$ be maps and F, $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Let $\varphi : (\mathbb{R}^+)^6 \to \mathbb{R}$ be a real map satisfying the following conditions:*
(φ_1) : φ is nonincreasing in variables t_5 and t_6,
(φ_2) : for every $t' \in \mathbb{R}^+$, $\varphi(t', t, 0, 0, t, t) \geq 0$ $\forall t > 0$.
If, for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$,

$$\varphi(H(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0 \tag{2.2}$$

then, f, g, F and G have a unique common fixed point.

Proof. i) We begin to show the existence of a common fixed point in a similar proof of Theorem 2.2.

Since the pairs $\{f, F\}$ and $\{g, G\}$ are owc then, there exist u, v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$ and $gGv \subseteq Ggv$.

First, we show that $gv = fu$. Suppose that is not the case, then condition (2.2) implies that

$$\begin{aligned} &\varphi(H(Fu, Gv), d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu)) \\ =&\varphi(H(Fu, Gv), d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0. \end{aligned}$$

By (φ_1), we have

$$\varphi(H(Fu, Gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from (φ_2) gives $d(fu, gv) = 0$. So $fu = gv$.

Next, we claim that $f^2u = fu$. If it is not, then condition (2.2) implies that

$$\begin{aligned} &\varphi(H(Ffu, Gv), d(f^2u, gv), d(f^2u, Ffu), d(gv, Gv), d(f^2u, Gv), d(gv, Ffu)) \\ =&\varphi(H(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0. \end{aligned}$$

By (φ_1), we have

$$\varphi(H(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from (φ_2), gives $d(f^2u, fu) = 0$. We have $f^2u = fu$.

Since (f, F) and (g, G) have the same role, we have $gv = g^2v$. Therefore, $ffu = fu = gv = ggv = gfu$, $fu = f^2u \in fFu \subset Ffu$, so $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is a common fixed point of f, g, F and G.

ii) Now, we show uniqueness of the common fixed point.

Put $fu = w$ and let w' be another common fixed point of the four maps such that $w \neq w'$, then, by (2.2), we get

$$\begin{aligned}&\varphi(H(Fw, Gw'), d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw))\\ =&\varphi(H(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0.\end{aligned}$$

By (φ_1), we get

$$\varphi(H(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw')) < 0.$$

So, by (φ_2), $d(fw, gw') = 0$ and thus, $d(fw, gw') = d(w, w') = 0$. □

Example 2.5. As above, we give some examples of function $\varphi : (\mathbb{R}^+)^6 \to \mathbb{R}$ for which it is easy to verify that (φ_1) and (φ_2) are satisfied.

(1) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a < 0$, $d > 0$, $e > 0$ and $a + d + e \leq 0$.
This φ is the function of the Example 2.9 of [7].
(2) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 + t_2^2 - a\max\{t_3^2, t_5^2\} - b\max\{t_3t_5, t_4t_6\} - ct_5t_6$, where $a > 0$, $b > 0$, $c > 0$ and $a + c \leq 1$.
(3) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - at_2^2 - b\frac{t_5t_6}{1+t_3^2+t_4^2}$, where $b > 0$ and $a + b \geq 0$.
This function φ is that one of Example 6 of [19] or Example 2.6 of [7].
(4) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c\frac{t_3t_4}{1+t_5+t_6}$, where $c < 0$.
(5) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - c\frac{1+t_3^{p-1}t_4^2+t_4^{p-1}t_3^2}{1+t_5^p+t_6^p}$, where $c < 0$ and $p \in \mathbb{N}^*$.
(6) $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 + t_2 - (\alpha t_3^2 + \beta t_4^2)^{\frac{1}{2}} - \gamma(t_5t_6)^{\frac{1}{2}}$, where $\gamma \in]0, 1]$.

Theorem 2.6. *Let f, $g : \mathcal{X} \to \mathcal{X}$ be maps and F, $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Let $\varphi : (\mathbb{R}^+)^6 \to \mathbb{R}$ be a real map satisfying the following conditions:*

(φ_1) : *φ is nondecreasing in variable t_1 and nonincreasing in variables t_5 and t_6,*

(φ_2) : $\varphi(t, t, 0, 0, t, t) \geq 0 \ \forall \ \ t > 0$.

If, for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, gy), d(fx, Fx), d(gy, Gy)\} > 0$,

$$\varphi(\delta(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0 \quad (2.3)$$

then, f, g, F and G have a unique common fixed point.

Proof. i) We begin to show the existence of a common fixed point. The beginning of the proof is similar to that of the previous theorem.

With the same notations, we suppose that $gv \neq fu$. Then condition (2.3) implies that

$$\begin{aligned}&\varphi(\delta(Fu, Gv), d(fu, gv), d(fu, Fu), d(gv, Gv), d(fu, Gv), d(gv, Fu))\\=&\varphi(\delta(Fu, Gv), d(fu, gv), 0, 0, d(fu, Gv), d(gv, Fu)) < 0.\end{aligned}$$

By (φ_1), we have

$$\varphi(d(fu, gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) < 0$$

which from (φ_2) gives $d(fu, gv) = 0$. So $fu = gv$. Next, we claim that $f^2u = fu$. If it is not, then condition (2.3) implies that

$$\begin{aligned}&\varphi(\delta(Ffu, Gv), d(f^2u, gv), d(f^2u, Ffu), d(gv, Gv), d(f^2u, Gv), d(gv, Ffu))\\=&\varphi(\delta(Ffu, Gv), d(f^2u, fu), 0, 0, d(f^2u, Gv), d(fu, Ffu)) < 0.\end{aligned}$$

By (φ_1), we have

$$\varphi(d(f^2u, fu), d(f^2u, fu), 0, 0, d(f^2u, fu), d(f^2u, fu)) < 0$$

which, from (φ_2), gives $d(f^2u, fu) = 0$ which implies that $f^2u = fu$.

Since (f, F) and (g, G) have the same role, we have: $g^2v = gv$. Therefore, $ffu = fu = gv = ggv = gfu$, $fu = f^2u \in fFu \subset Ffu$, so $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is a common fixed point of f, g, F and G.

ii) Now, we show uniqueness of the common fixed point.

Put $fu = w$ and let w' be another common fixed point of the four maps such that $w \neq w'$, by (2.3), we get

$$\begin{aligned}&\varphi(\delta(Fw, Gw'), d(fw, gw'), d(fw, Fw), d(gw', Gw'), d(fw, Gw'), d(gw', Fw))\\=&\varphi(\delta(Fw, Gw'), d(fw, gw'), 0, 0, d(fw, Gw'), d(gw', Fw)) < 0.\end{aligned}$$

By (φ_1), we get

$$\begin{aligned}&\varphi(d(fw, gw'), d(fw, gw'), 0, 0, d(fw, gw'), d(fw, gw'))\\=&\varphi(d(w, w'), d(w, w'), 0, 0, d(w, w'), d(w, w')) < 0.\end{aligned}$$

So, by (φ_2), $d(w, w') = 0$ and thus, $w = w'$. □

Remark 2.7. Truly, Theorems 2.4, 2.6 are generalizations of corresponding theorems of [3], [4], [19]-[24] and others since we extended the setting of single-valued maps to the one of single and set-valued maps, also, we deleted the compactness in [3], [22], we further add that we do not require the continuity, although we used the strict contractive conditions (2.2), (2.3) which are substantially more general than the inequalities in the cited papers, and we weakened the concepts of compatibility, compatibility of type (A), compatibility of type (C), compatibility of type (P) and weak compatibility to the more general one say occasional weak compatibility. Finally, we deleted some assumptions of functions φ which are superflous for us but are necessary in the papers [3], [4], [19]-[24].

If we let $f = g$ and $F = G$ in Theorem 2.6, we get the following corollaries:

Corollary 2.8. *Let $f : \mathcal{X} \to \mathcal{X}$ and let $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ such that the pair $\{f, F\}$ is owc. Let $\varphi : (\mathbb{R}^+)^6 \to \mathbb{R}$ be a real map satisfying conditions (φ_1) and (φ_2) of Theorem 2.6 and*

$$\varphi(\delta(Fx, Fy), d(fx, fy), d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx)) < 0$$

for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, fy), d(fx, Fx), d(fy, Fy)\} > 0$. Then f and F have a unique common fixed point.

Now, if we let $f = g$, we get the next result:

Corollary 2.9. *Let f be a self-map of a metric space $(\mathcal{X}, d)$ and let F, $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps. Suppose the pairs $\{f, F\}$ and $\{f, G\}$ are owc and $\varphi : (\mathbb{R}^+)^6 \to \mathbb{R}$ satisfies conditions (φ_1) and (φ_2) of Theorem 2.6 and*

$$\varphi(\delta(Fx, Gy), d(fx, fy), d(fx, Fx), d(fy, Gy), d(fx, Gy), d(fy, Fx)) < 0$$

for all x and $y \in \mathcal{X}$ for which $\max\{d(fx, fy), d(fx, Fx), d(fy, Gy)\} > 0$. Then f, F and G have a unique common fixed point.

With different choices of the real map φ, we obtain the following corollaries:

Corollary 2.10. *If in the hypotheses of Theorem 2.6, we have instead of (2.3) one of the following inequalities, for all x and $y \in \mathcal{X}$ whenever the right hand side of each inequality is not zero, then the four maps have a unique common fixed point.*

(a) $\delta(Fx, Gy) < k \max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}$

where $0 < k \leq 1$,

(b)
$$\delta^2(Fx, Gy) < c_1 \max\{d^2(fx, gy), d^2(fx, Fx), d^2(gy, Gy)\}$$
$$+ c_2 \max\{d(fx, Fx)d(fx, Gy), d(gy, Gy)d(gy, Fx)\} + c_3 d(fx, Gy)d(gy, Fx),$$

where $c_1 > 0$, c_2, $c_3 \geq 0$ and $c_1 + c_3 \leq 1$,

(c)
$$\delta(Fx, Gy) < [\alpha\delta^{p-1}(Fx, Gy)d(fx, gy) + \beta\delta^{p-2}(Fx, Gy)d(fx, Fx)d(gy, Gy)$$
$$+ \gamma d^{p-1}(fx, Gy)d(gy, Fx) + \delta d(fx, Gy)d^{p-1}(gy, Fx)]^{\frac{1}{p}},$$

where $\alpha > 0$, β, γ, $\delta \geq 0$, $\alpha + \gamma + \delta \leq 1$ and $p \geq 2$,

(d)
$$\delta^2(Fx, Gy) < \frac{1}{\alpha}\left[\beta d^2(fx, gy) + \frac{\gamma d(fx, Gy)d(gy, Fx)}{1 + \delta d^2(fx, Fx) + \epsilon d^2(gy, Gy)}\right],$$

where $\alpha > 0$, $\beta, \gamma, \delta, \epsilon \geq 0$ and $\beta + \gamma \leq \alpha$,

(e)
$$\delta(Fx, Gy) < [\alpha d^p(fx, gy) + \beta d^p(fx, Fx) + \gamma d^p(gy, Gy)]^{\frac{1}{p}}$$
$$+ \delta\,[d(fx, Gy)d(gy, Fx)]^{\frac{1}{2}},$$

where $0 < \alpha \leq (1 - \delta)^p$, β, γ, $\delta \geq 0$ and $p \in \mathbb{N}^* = \{1, 2, \ldots\}$.

Proof. For proofs of (a), (b), (c), (d) and (e), we use Theorem 2.6 with the following functions φ which satisfy, for every case, hypotheses (φ_1) and (φ_2).

For (a):

$$\begin{aligned}&\varphi(\delta(Fx,Gy),d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))\\=&\delta(Fx,Gy)-k\max\{d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx)\}.\end{aligned}$$

This function φ is used by many authors with single maps, for example: [12] in Theorem 1, Example 3.4 in [18].

For (b):

$$\begin{aligned}&\varphi(\delta(Fx,Gy),d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))\\=&\delta^2(Fx,Gy)-c_1\max\{d^2(fx,gy),d^2(fx,Fx),d^2(gy,Gy)\}\\&-c_2\max\{d(fx,Fx)d(fx,Gy),d(gy,Gy)d(gy,Fx)\}\\&-c_3d(fx,Gy)d(gy,Fx).\end{aligned}$$

This function φ is Example 2 of [22].

For (c):

$$\begin{aligned}&\varphi(\delta(Fx,Gy),d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))\\=&\delta(Fx,Gy)-\Big[\alpha\delta^{p-1}(Fx,Gy)d(fx,gy)+\beta\delta^{p-2}(Fx,Gy)d(fx,Fx)d(gy,Gy)\\&+\gamma d^{p-1}(fx,Gy)d(gy,Fx)+\delta d(fx,Gy)d^{p-1}(gy,Fx)\Big]^{\frac{1}{p}}.\end{aligned}$$

For $p=3$, we have Example 3.4 of [4] and Example 3 of [23]. If we take $p=2$, φ is Example 1 of [20].

For (d):

$$\begin{aligned}&\varphi(\delta(Fx,Gy),d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))\\=&\delta^2(Fx,Gy)-\frac{1}{\alpha}\left[\beta d^2(fx,gy)+\frac{\gamma d(fx,Gy)d(gy,Fx)}{1+\delta d^2(fx,Fx)+\epsilon d^2(gy,Gy)}\right].\end{aligned}$$

This function φ is that one of Example 6 of [19].

And for (e):

$$\begin{aligned}&\varphi(\delta(Fx,Gy),d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx))\\=&\delta(Fx,Gy)-[\alpha d^p(fx,gy)+\beta d^p(fx,Fx)+\gamma d^p(gy,Gy)]^{\frac{1}{p}}\\&-\delta\,[d(fx,Gy)d(gy,Fx)]^{\frac{1}{2}}.\end{aligned}$$

□

Corollary 2.11. *Let f, g be two self-maps of a metric space $(\mathcal{X},d)$ and let F and $G:\mathcal{X}\to\mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f,F\}$ and $\{g,G\}$ are owc. Suppose that, for all x, $y\in\mathcal{X}$, we have the inequality*

(f) $\delta^p(Fx,Gy)<\alpha d^p(fx,gy)+\beta d^p(fx,Fx)+\gamma d^p(gy,Gy)$

such that $0 < \alpha \leq 1$*,* β *and* $\gamma \geq 0$ *and* $p \in \mathbb{N}^* = \{1, 2, \ldots\}$ *whenever the right hand side of the above inequality is positive. Then* f*,* g*,* F *and* G *have a unique common fixed point.*

Proof. We give this corollary because it is an interesting particular case of the previous corollary. We obtain the result by using (e) in Corollary 2.10 with $\delta = 0$. □

3. Two other type common fixed point theorems

We begin by a Greguš type common fixed point theorem. As we already said, in 1998, Pathak et al. [17] gave an extension of compatibility of type (A) by introducing the concept of compatibility of type (C) and they proved a common fixed point theorem of Greguš type for four compatible maps of type (C) in a Banach space. Further, Djoudi and Nisse [5] extended the result of [17] by weakening compatibility of type (C) to the weak one without continuity.

Our objective here is to establish a common fixed point theorem for four occasionally weakly compatible single and set-valued maps of Greguš type in a metric space which improves the results of [5], [17] and others.

Theorem 3.1. *Let* f *and* $g : \mathcal{X} \to \mathcal{X}$ *be maps,* F *and* $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ *be set-valued maps such that the pairs* $\{f, F\}$ *and* $\{g, G\}$ *are owc. Let* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nondecreasing map such that, for every* $t > 0$*,* $\Psi(t) < t$ *and satisfying the following condition:*

$$\delta^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)] \tag{3.1}$$

for all x *and* $y \in \mathcal{X}$*, where* $0 < a \leq 1$ *and* $p \geq 1$*. Then* f*,* g*,* F *and* G *have a unique common fixed point.*

Proof. Since f, F and g, G are owc, as in the proof of Theorem 2.4, there exist u, v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$.

i) As in the proof of Theorem 2.4, we begin to show the existence of a common fixed point. We have

$$\delta^p(Fu, Gv) \leq \Psi(ad^p(fu, gv) + (1-a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv))$$

and by the properties of δ and Ψ, we get

$$d^p(fu, gv) \leq \delta^p(Fu, Gv) \leq \Psi(d^p(fu, gv)).$$

So, if $d(fu, gv) > 0$, $\Psi(t) < t$ for $t > 0$, we obtain

$$d^p(fu, gv) \leq \delta^p(Fu, Gv) \leq \Psi(d^p(fu, gv)) < d^p(fu, gv)$$

which is a contradiction, thus, we have $d(fu, gv) = 0$, hence $fu = gv$.

Again, if $d(f^2u, fu) > 0$, then by (3.1), we have

$$\delta^p(Ffu, Gv) \leq \Psi[ad^p(f^2u, gv) + (1-a)d^{\frac{p}{2}}(gv, Ffu)d^{\frac{p}{2}}(f^2u, Gv)]$$

and hence,

$$d^p(f^2u, fu) \leq \delta^p(Ffu, Gv) \leq \Psi[d^p(f^2u, fu)].$$

Since $d(f^2u, fu) > 0$, we obtain

$$d^p(f^2u, fu) \leq \delta^p(Ffu, Gv) \leq \Psi[d^p(f^2u, fu)] < d^p(f^2u, fu)$$

which is impossible. Then we have $d(f^2u, fu) = 0$; i.e., $f^2u = fu$. Similarly, we can prove that $g^2v = gv$, let $fu = w$ then, $fw = w = gw$, $w \in Fw$ and $w \in Gw$, this completes the proof of the existence.

ii) For the uniqueness, let w' be a second common fixed point of f, g, F and G with $w' \neq w$. Then, $d(w, w') = d(fw, gw') \leq \delta(Fw, Gw')$ and, by assumption (3.1), we obtain

$$\delta^p(Fw, Gw') \leq \Psi[ad^p(fw, gw') + (1-a)d^{\frac{p}{2}}(fw, Gw')d^{\frac{p}{2}}(gw', Fw)]$$

and thus,

$$d^p(w, w') = d^p(fw, gw') \leq \delta^p(Fw, Gw') \leq \Psi[d^p(w, w')] < d^p(w, w').$$

Since $d(w, w') > 0$, we have a contradiction. So, $w = w'$. □

Theorem 3.2. *Let f and $g : \mathcal{X} \to \mathcal{X}$ be maps, F and $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing map such that, for every $t > 0$, $\Psi(t) < t$ and satisfying the following condition:*

$$\begin{aligned}\delta^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)\max\{&\alpha d^p(fx, Fx), \beta d^p(gy, Gy),\\ &d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy),\\ &\frac{1}{2}(d^p(fx, Fx) + d^p(gy, Gy))\}]\end{aligned}$$

for all x and $y \in \mathcal{X}$, where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1$ and $p \geq 1$. Then f, g, F and G have a unique common fixed point.

Proof. Since f, F and g, G are owc, as in the proof of Theorem 2.2, there exist u, v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$. Since Ψ is a nondecreasing function and since for any real numbers c and d, $\frac{c+d}{2} \leq \max\{c, d\}$ we have, for all x, $y \in \mathcal{X}$,

$$\begin{aligned}\delta^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)\max\{&d^p(fx, Fx), d^p(gy, Gy)\\ &d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(gy, Fx), d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)\}]\end{aligned}$$

and, for u and v,

$$\delta^p(Fu, Gv) \leq \Psi[ad^p(fu, gv) + (1-a)d^{\frac{p}{2}}(gv, Fu)d^{\frac{p}{2}}(fu, Gv)].$$

The continuation of the proof is identical with that of Theorem 2.4. □

Remark 3.3. Theorems 3.1 and 3.2 can be deduced from Theorem 2.6 thanks to an appropriate choice of the function φ.
For Theorem 3.1, we choose $\varphi(t_1, \cdots, t_6) = t_1^p - \Psi[at_2^p + (1-a)t_5^{\frac{p}{2}}t_6^{\frac{p}{2}}]$ and, for Theorem 3.2, $\varphi(t_1, \cdots, t_6) = t_1^p - \Psi[at_2^p + (1-a)\max\{\alpha t_3^p, \beta t_4^p, t_3^{\frac{p}{2}}t_6^{\frac{p}{2}}, t_5^{\frac{p}{2}}t_6^{\frac{p}{2}}, \frac{1}{2}(t_3^p + t_4^p)\}]$.

If in (3.1), we replace δ with H and $\Psi(t) < t$ with $\Psi(t) \leq t$, we can prove

Theorem 3.4. *Let f and $g : \mathcal{X} \to \mathcal{X}$ be maps, F and $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc. Let u and v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing map such that, for every $t > 0$, $\Psi(t) \leq t$ and satisfying the following condition:*

$$H^p(Fx, Gy) \leq \Psi[ad^p(fx, gy) + (1-a)d^{\frac{p}{2}}(gy, Fx)d^{\frac{p}{2}}(fx, Gy)] \tag{3.2}$$

for all x and $y \in \mathcal{X}$, where $0 < a \leq 1$ and $p \geq 1$.
If $fu = gv$ is a common fixed point of f and g, then fu is a common fixed point of f, g, F and G and $Fu = Gv$.

Proof. Since $gv \in Gv$, $fu \in Fu$ and $f^2u \in fFu \subseteq Ffu$, we have $d(gv, Fu) \leq H(Fu, Gv)$, $d(fu, Gv) \leq H(Fu, Gv)$, $d(gv, Ffu) \leq H(Ffu, Gv)$ and $d(f^2u, Gv) \leq H(Ffu, Gv)$.

Since Ψ is nondecreasing, we obtain

$$\begin{aligned} H^p(Ffu, Gv) &\leq \Psi[ad^p(f^2u, gv) + (1-a)d^{\frac{p}{2}}(gv, Ffu)d^{\frac{p}{2}}(f^2u, Gv)] \\ &\leq \Psi[ad^p(f^2u, gv) + (1-a)H^p(Ffu, Gv)] \\ H^p(Fu, Gv) &\leq \Psi[ad^p(fu, gv) + (1-a)H^p(Fu, Gv)] \end{aligned}$$

and

$$H^p(Fu, Ggv) \leq \Psi[ad^p(fu, g^2v) + (1-a)H^p(Fu, Ggv)].$$

Now, if $Fu \neq Gv$, since, for every $t > 0$, $\Psi(t) \leq t$,

$$H^p(Fu, Gv) \leq ad^p(fu, gv) + (1-a)H^p(Fu, Gv).$$

Consequently, $H(Fu, Gv) \leq d(fu, gv)$ and $fu \neq gv$. We have shown that if $fu = gv$, then $Fu = Gv$. By similar proofs, if $f^2u = gv$, then $Gv = Ffu$ and if $fu = g^2v$, then $Fu = Ggv$. The proof is finished. □

Remark 3.5. Obviously, Theorems 3.1 and 3.2 extend the results of [5] and [17] to the class of four single and set-valued maps. In particular, Theorem 3.2 improves the cited results since we do not require the closedness of the sets $F(\mathcal{X})$ and $G(\mathcal{X})$, also, we deleted the inclusions $F(\mathcal{X}) \subset f(\mathcal{X})$ and $G(\mathcal{X}) \subset g(\mathcal{X})$ in [5], we weakened the weak compatibility in [5] and the compatibility of type (C) in [17] to the wider one cited occasional weak compatibility and we deleted the continuity which is indispensable in [17] and the upper semicontinuity imposed on Ψ in [5].

If we put $f = g$ in Theorem 3.1, then we get the corollary:

Corollary 3.6. *Let $f : \mathcal{X} \to \mathcal{X}$ be a map and let F and $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing map such that, for every $t > 0$, $\Psi(t) < t$. Suppose the pairs $\{f, F\}$ and $\{f, G\}$ are owc and satisfy the inequality*

$$\delta^p(Fx, Gy) \leq \Psi[ad^p(fx, fy) + (1-a)d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Gy)]$$

for all x, $y \in \mathcal{X}$, where $0 < a \leq 1$ and $p \geq 1$. Then f, F and G have a unique common fixed point.

If we put $f = g$ and $F = G$ in Theorem 3.2, then we obtain the following result:

Corollary 3.7. *Let $f : \mathcal{X} \to \mathcal{X}$ be a map and let $F : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be a set-valued map such that f and F are owc . Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing map such that, for every $t > 0$, $\Psi(t) < t$. If*

$$\begin{aligned}\delta^p(Fx, Fy) \quad \leq \quad & \Psi[ad^p(fx, fy) + (1-a)\max\{\alpha d^p(fx, Fx), \beta d^p(fy, Fy),\\ & d^{\frac{p}{2}}(fx, Fx)d^{\frac{p}{2}}(fy, Fx), d^{\frac{p}{2}}(fy, Fx)d^{\frac{p}{2}}(fx, Fy),\\ & \frac{1}{2}(d^p(fx, Fx) + d^p(fy, Fy))\}]\end{aligned}$$

for all x, $y \in \mathcal{X}$, where $0 < a \leq 1$, $\{\alpha, \beta\} \subset]0, 1]$ and $p \geq 1$, then f and F have a unique common fixed point.

Now, we end our work by establishing a near-contractive common fixed point theorem which improves those given by Elamrani and Mehdaoui [6], Mbarki [14] and others since our version does not impose continuity and we use occasional weak compatibility which is more general than compatibility and weak compatibility; also, we delete, on Φ, some strong conditions which are necessary in papers [6] and [14] on a metric space instead of a complete metric space.

Theorem 3.8. *Let f and $g : \mathcal{X} \to \mathcal{X}$ be maps, F and $G : \mathcal{X} \to \mathcal{P}_{fb}(\mathcal{X})$ be set-valued maps and Φ be a nondecreasing function of $[0, \infty[$ into itself such that $\Phi(t) = 0$ if and only if $t = 0$ and satisfying inequality*

$$\begin{aligned}\Phi(\delta(Fx, Gy)) \leq\ & \alpha(d(fx, gy))\Phi(d(fx, gy))\\ & + \beta(d(fx, gy))[\Phi(d(fx, Gy)) + \Phi(d(gy, Gy))]\\ & + \gamma(d(fx, gy))[\Phi(d(fx, Fx)) + \Phi(d(gy, Fx))] \qquad (3.3)\end{aligned}$$

for all x, $y \in \mathcal{X}$ and α, β, $\gamma : [0, \infty[\to [0, 1[$ satisfying condition

$$\alpha(t) + \beta(t) + \gamma(t) < 1 \quad \forall t > 0. \qquad (3.4)$$

If the pairs $\{f, F\}$ and $\{g, G\}$ are owc, then f, g, F and G have a unique common fixed point in $\mathcal{X}$.

Proof. Since f, F and g, G are owc, as in the proof of Theorem 2.2, there exist u, v in $\mathcal{X}$ such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$.

i) First, we prove that $fu = gv$. By (3.3), we have

$$\begin{aligned}\Phi(\delta(Fu, Gv)) &\leq \alpha(d(fu, gv))\Phi(d(fu, gv))\\ &\quad + \beta(d(fu, gv))[\Phi(d(fu, Gv)) + \Phi(d(gv, Gv))]\\ &\quad + \gamma(d(fu, gv))[\Phi(d(fu, Fu)) + \Phi(d(gv, Fu))]\\ &= \alpha(d(fu, gv))\Phi(d(fu, gv)) + \beta(d(fu, gv))\Phi(d(fu, Gv))\\ &\quad + \gamma(d(fu, gv))\Phi(d(gv, Fu)).\end{aligned}$$

If $d(fu,gv)>0$, since Φ is nondecreasing and $\Phi(t)=0$ if and only if $t=0$, from inequalities (3.3) and (3.4) we get

$$\begin{aligned}\Phi(d(fu,gv)) &\leq \Phi(\delta(Fu,Gv))\\ &\leq \alpha(d(fu,gv))\Phi(d(fu,gv))+\beta(d(fu,gv))\Phi(d(fu,Gv))\\ &\quad+\gamma(d(fu,gv))\Phi(d(gv,Fu))\\ &\leq [\alpha(d(fu,gv))+\beta(d(fu,gv))+\gamma(d(fu,gv))]\Phi(d(fu,gv))\\ &< \Phi(d(fu,gv))\end{aligned}$$

which is a contradiction. Hence, $d(fu,gv)=0$ and thus, $fu=gv$.

Now we claim that $f^2u=fu$. Suppose not, since Φ is nondecreasing and $\Phi(t)=0$ if and only if $t=0$, the use of (3.3) and (3.4) gives

$$\begin{aligned}\Phi(d(f^2u,fu)) &\leq \Phi(\delta(Ffu,Gv))\\ &\leq \alpha(d(f^2u,gv))\Phi(d(f^2u,gv))\\ &\quad+\beta(d(f^2u,gv))[\Phi(d(f^2u,Gv))+\Phi(d(gv,Gv))]\\ &\quad+\gamma(d(f^2u,gv))[\Phi(d(f^2u,Ffu))+\Phi(d(gv,Ffu))]\\ &= \alpha(d(f^2u,fu))\Phi(d(f^2u,fu))+\beta(d(f^2u,fu))\Phi(d(f^2u,Gv))\\ &\quad+\gamma(d(f^2u,fu))\Phi(d(fu,Ffu))\\ &\leq [\alpha(d(f^2u,fu))+\beta(d(f^2u,fu))\\ &\quad+\gamma(d(f^2u,fu))]\Phi(d(f^2u,fu))\\ &< \Phi(d(f^2u,fu)).\end{aligned}$$

This contradiction implies that $\Phi(d(f^2u,fu))=0$ and hence, $f^2u=fu$. Similarly, we can prove that $g^2v=gv$. So, if $w=fu=gv$ therefore, $fw=w=gw$, $w\in Fw$ and $w\in Gw$. The existence of a common fixed point is proved.

ii) Assume that there exists a second common fixed point w' of f, g, F and G such that $w'\neq w$. We have $d(w,w')=d(fw,gw')\leq\delta(Fw,Gw')$. Since $d(w,w')>0$, by inequality (3.3) and properties of functions Φ, α and β, we obtain

$$\begin{aligned}\Phi(d(w,w')) &\leq \Phi(\delta(Fw,Gw'))\\ &\leq \alpha(d(fw,gw'))\Phi(d(fw,gw'))\\ &\quad+\beta(d(fw,gw'))[\Phi(d(fw,Gw'))+\Phi(d(gw',Gw'))]\\ &\quad+\gamma(d(fw,gw'))[\Phi(d(fw,Fw))+\Phi(d(gw',Fw))]\\ &= \alpha(d(w,w'))\Phi(d(w,w'))+\beta(d(w,w'))\Phi(d(w,Gw'))\\ &\quad+\gamma(d(w,w'))\Phi(d(w',Fw))\\ &\leq [\alpha(d(w,w'))+\beta(d(w,w'))+\gamma(d(w,w'))]\Phi(d(w,w'))\\ &< \Phi(d(w,w')).\end{aligned}$$

This contradiction implies that $\Phi(d(w,w'))=0$, hence, $w'=w$. □

Remark 3.9. The above theorem remains valid if we replace inequality (3.3) by the following one:

$$\begin{aligned}\Phi(\delta(Fx,Gy)) \leq \alpha(d(fx,gy))\Phi(d(fx,gy)) \\ + \beta(d(fx,gy))\max\{\Phi(d(fx,Gy)),\Phi(d(gy,Gy))\} \\ + \gamma(d(fx,gy))[\Phi(d(fx,Fx))+\Phi(d(gy,Fx))].\end{aligned}$$

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